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Pure Strategy Saddle Points in the Generalized Progressive Discrete Silent Duel with Identical Linear Accuracy Functions

Vadim Romanuke *v.romanuke@amw.gdynia.pl*
Faculty of Mechanical and Electrical Engineering
Polish Naval Academy, Gdynia, Poland

Abstract

A finite zero-sum game defined on a subset of the unit square is considered. The game is a generalized progressive discrete silent duel, in which the kernel is skew-symmetric, and the players, referred to as duelists, have identical linear accuracy functions featured with an accuracy proportionality factor. As the duel starts, time moments of possible shooting become denser by a geometric progression. Apart from the duel beginning and end time moments, every following time moment is the partial sum of the respective geometric series. Due to the skew-symmetry, both the duelists have the same optimal strategies and the game optimal value is 0. If the accuracy factor is not less than 1, the duelist's optimal strategy is the middle of the duel time span. If the factor is less than 1, the duel solution is not always a pure strategy saddle point. In a boundary case, when the accuracy factor is equal to the inverse numerator of the ratio that is the time moment preceding the duel end moment, the duel has four pure strategy saddle points which are of the mentioned time moments. For a trivial game, where the duelist possesses just one moment of possible shooting between the duel beginning and end moments, and the accuracy factor is 1, any pure strategy situation, not containing the duel beginning moment, is optimal.

Keywords: game of timing, silent duel, accuracy function, linear accuracy, matrix game, pure strategy saddle point

1. Introduction

Games of timing represent a wide class of competitive interaction models. These models consider a time span during which the player must make a decision of acting [1], [2], [3], [4]. If a decision is made in a two-person game commonly referred to as a duel, the other player either learns it or does not learn it until the duel ends. The latter is the case of the silent duel [5], [6], [7], [8].

Silent duels, unlike noisy duels, are more complicated to study [9], [10], [5]. In one of a few types of the silent duel, each of two players (duelists) has exactly one bullet [11], [12], [13]. The bullet is an abstraction implying an implementation of the decision of acting. This also includes issuing information about the state of a system or object modeled by the duel [3], [12], [14]. The duelist is featured with an accuracy function which, generally speaking, is a nondecreasing function of time [5], [15], [16]. It is unknown to the duelist whether a bullet was fired by the other duelist or not until the end of the duel time span [6], [8]. The game builds on the fact that the duelist may obtain a greater payoff by firing as late as possible, but then the loss, when the other duelist shoots first, becomes more likely. If both the duelists shoot simultaneously, the payoff of each of them is 0 [9], [10], [5], [17].

In general, such a zero-sum game is

$$\langle X, Y, K(x, y) \rangle \tag{1}$$

whose kernel is defined on unit square

$$X \times Y = [0; 1] \times [0; 1] \tag{2}$$

being the Cartesian product of the duel time spans (the product is the square of the span). When the accuracy functions are linear and identical, the kernel is

$$K(x, y) = ax - ay + a^2 xy \operatorname{sign}(y - x) \text{ by } a > 0 . \tag{3}$$

Game (1) by (2) and (3) is a silent duel with linear accuracy functions of the duelists, which are allowed to shoot at any moment during the duel time span, and the duelist's accuracy is proportional to the time moment with factor a . Due to kernel (3) is skew-symmetric, i. e.

$$K(y, x) = ay - ax + a^2 yx \operatorname{sign}(x - y) = -K(x, y) ,$$

both the duelists have the same optimal strategies and the game optimal value is 0 [5], [8], [12]. However, even in this case of the identical linear accuracy functions, the duelist's optimal strategy is a non-continuous probability distribution as a mixed strategy with an uncountably infinite support whose measure is less than the duel time span length [5], [18], [19]. Although the duel can be a repeated game, any sequence of real-world actions cannot be unlimited (or cannot last forever), and thus practical implementation of such a solution cannot be complete [20], [21], [13], [22], [23]. This urges to consider a discrete version of the silent duel.

In a discrete silent duel, both the players can shoot only at specified time moments whose number is finite. Therefore, the kernel of the discrete silent duel is defined on a finite set of the pairs of pure strategies of the duelists. The moments (as pure strategies) of the duel beginning $x = y = 0$ and duel end $x = y = 1$ are included [24], [8], [12]:

$$X = \{x_i\}_{i=1}^{N} = Y = \{y_j\}_{j=1}^{N} = T = \{t_q\}_{q=1}^{N} \subset [0; 1]$$
$$\text{by } t_q < t_{q+1} \;\; \forall\, q = \overline{1, N-1} \text{ and } t_1 = 0 ,\; t_N = 1 \text{ for } N \in \mathbb{N} \setminus \{1\} . \tag{4}$$

Consequently, this finite symmetric game is a matrix game whose solution is of finite supports only [25], [26], [13], [18]. The solution is computed far easier than that in the case of infinite game (1).

In the general case, moments $\{t_q\}_{q=1}^{N}$ of possible shooting are not specified but only obey (4). As a particular case, these moments can be defined by uniformly breaking the time span. In practice, nevertheless, as the duelist approaches to the end moment $t_N = 1$, the space between consecutive moments t_q and t_{q+1}, $q = \overline{1, N-1}$, may shorten due to the growing tension and responsibility. This is equivalent to the weight (importance) of the duelist's pure strategy with approaching to the end moment grows. One of the patterns of such growth is a geometrical progression. In this case, the density of pure strategies of the duelist grows in the geometrical progression as the duelist approaches to the duel end [8], [12], [27]. Thus, if

$$t_q = \sum_{l=1}^{q-1} 2^{-l} = \frac{2^{q-1} - 1}{2^{q-1}} \text{ for } q = \overline{2, N-1} \tag{5}$$

then, apart from the duel beginning and end moments, every following moment is the partial sum of the respective geometric series. Subsequently, game (1) by (4), (5) with kernel (3), where $x \in X$, $y \in Y$, becomes a generalized progressive discrete silent duel with identical linear accuracy functions. It is obvious that this duel solution depends on N and a. Besides, the kernel herein is a skew-symmetric payoff matrix [12], [5]. The goal is to study its saddle points that are pure strategy solutions of the progressive discrete silent duel with identical linear accuracy functions.

To achieve the goal, additional preliminaries of the progressive discrete silent duel are first stated in Section 2. Then pure strategy solutions for various cases of N and a are presented in Section 3. Section 4 summarizes the solution results and recapitulates their peculiarities. The study is discussed and concluded in Section 5.

2. Additional preliminaries

In fact, the progressive discrete silent duel with identical linear accuracy functions is a matrix game

$$\left\langle \{x_i\}_{i=1}^{N}, \{y_j\}_{j=1}^{N}, \mathbf{K}_N \right\rangle \tag{6}$$

by (4), (5), and payoff matrix

$$\mathbf{K}_N = \left[k_{ij}\right]_{N \times N}$$

$$\text{by } k_{ij} = K\left(x_i, y_j\right) = ax_i - ay_j + a^2 x_i y_j \operatorname{sign}\left(y_j - x_i\right) \text{ by } a > 0 . \tag{7}$$

If $a = 1$ then the accuracy is exactly equal to the time moment at which the bullet is fired [9], [10], [5], [8]. Although by $a \neq 1$ the accuracy is just scaled by $a > 0$, it influences the solution by depending on whether $a > 1$ or $0 < a < 1$. This is yet to be shown below.

Formally, the most trivial case of $N=2$, where the duelist is allowed to shoot at either the very beginning or end, is not excluded. As the shooting is allowed only at moments $t_1=0$ and $t_2=1$, the respective payoff matrix is

$$\mathbf{K}_2=\left[k_{ij}\right]_{2\times 2}=\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \tag{8}$$

whence situation

$$\left\{x_2,\, y_2\right\}=\left\{1,1\right\} \tag{9}$$

is the single saddle point here.

In general,

$$K\left(x_1,\, y_N\right)=K\left(0,1\right)=-a=-K\left(1,0\right). \tag{10}$$

Therefore, whichever number N is (the case of $N=1$ is naturally excluded), the first row of matrix (7) contains a negative entry and so this row does not contain saddle points (because the minimum of the first row does not exceed $-a<0$ and thus the game optimal value $v_{\text{opt}}=0$ cannot be reached in this row). Due to the skew-symmetry of matrix (7), the stated inference is immediately followed by that the first column does not contain saddle points either. In the further consideration, only the inferences on saddle points in definite rows of matrix (7), which imply the same inferences on saddle points in respective columns, will be stated. It is clear that only a zero entry of matrix (7) can be a saddle point. If a row contains a negative entry, this row and respective column do not contain saddle points. If a nonnegative row contains a zero entry off the main diagonal, and this entry is a saddle point, both the zero entries are saddle points [2], [5], [18].

3. Existence of pure strategy saddle points

It is needful to get started with the most trivial case, apart from (8). This is the case of $N=3$, where the shooting, apart from the very beginning and end moments $t_1=0$, $t_3=1$, is also allowed at moment $t_2=\dfrac{1}{2}$.

Theorem 1. In a progressive discrete silent duel (6) by (4), (5), (7) for $N=3$, situation

$$\left\{x_2,\, y_2\right\}=\left\{\frac{1}{2},\frac{1}{2}\right\} \tag{11}$$

is optimal only by $a\geqslant 1$ but it is never optimal by $0<a<1$. Besides, saddle point (11) is single by $a>1$. Any pure strategy situation in the 3×3 duel, not containing the duel beginning moment, is optimal by $a=1$, whereas situation

$$\{x_3, y_3\} = \{1, 1\} \tag{12}$$

is the single saddle point by $0 < a < 1$.

Proof. Due to (10), situation

$$\{x_1, y_1\} = \{0, 0\}$$

is never optimal in the duel. The respective payoff matrix is

$$\mathbf{K}_3 = \left[k_{ij}\right]_{3\times 3} = \begin{bmatrix} 0 & -\frac{a}{2} & -a \\ \frac{a}{2} & 0 & \frac{a}{2}(a-1) \\ a & -\frac{a}{2}(a-1) & 0 \end{bmatrix}. \tag{13}$$

If $a > 1$ then the second row of matrix (13) is nonnegative and the third row contains a negative entry. The only zero entry in the second row is k_{22}. So, situation (11) is optimal and it is the single saddle point for (13) by $a > 1$. If $a = 1$ then the respective payoff matrix is

$$\mathbf{K}_3 = \left[k_{ij}\right]_{3\times 3} = \begin{bmatrix} 0 & -\frac{1}{2} & -1 \\ \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{14}$$

This matrix game has four saddle points: situation (11), situation (12), and non-symmetric situations

$$\{x_2, y_3\} = \left\{\frac{1}{2}, 1\right\}, \tag{15}$$

$$\{x_3, y_2\} = \left\{1, \frac{1}{2}\right\}. \tag{16}$$

If $0 < a < 1$ then the second row of matrix (13) contains a negative entry and the third row is nonnegative. The only zero entry in the third row is k_{33}. So, situation (12) is the single saddle point for (13) by $0 < a < 1$. □

Now, the general case is to be considered. Let it be divided into the subcases of $0 < a < 1$ and $a \geqslant 1$.

Theorem 2. In a progressive discrete silent duel (6) by (4), (5), (7) for $N \in \mathbb{N} \setminus \{1, 2, 3\}$, there exists only one $n \in \{\overline{3, N-1}\}$ by $0 < a < 1$ such that situation

$$\{x_n, y_n\} = \left\{ \frac{2^{n-1}-1}{2^{n-1}}, \frac{2^{n-1}-1}{2^{n-1}} \right\} \tag{17}$$

is a single saddle point by

$$a \in \left[\frac{1}{2^{n-1}-1}; \frac{2^{n-2}}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)} \right] \subset (0; 1) \tag{18}$$

and situation

$$\{x_N, y_N\} = \{1, 1\} \tag{19}$$

is a saddle point by

$$a \in \left(0; \frac{1}{2^{N-2}-1} \right] \subset (0; 1). \tag{20}$$

Besides, situation (11) is never optimal by $0 < a < 1$.

Proof. Due to

$$K(x_2, y_N) = K\left(\frac{1}{2}, 1\right) = \frac{a}{2}(a-1) = -K\left(1, \frac{1}{2}\right) < 0 \text{ by } 0 < a < 1, \tag{21}$$

situation (11) is never optimal for $N \in \mathbb{N} \setminus \{1, 2, 3\}$. Consider entry k_{nn} for $n \in \{\overline{3, N-1}\}$ and $N \in \mathbb{N} \setminus \{1, 2, 3\}$. This entry is the result of when both the duelists simultaneously shoot at moment

$$t_n = \sum_{l=1}^{n-1} 2^{-l} = \frac{2^{n-1}-1}{2^{n-1}} \tag{22}$$

corresponding to situation (17). If situation (17) is a saddle point, then, in the n-th row of matrix (7), inequalities

$$K(x_n, y_j) = ax_n - ay_j - a^2 x_n y_j \geqslant 0 \ \ \forall\, y_j < x_n \tag{23}$$

and

$$K(x_n, y_j) = ax_n - ay_j + a^2 x_n y_j \geqslant 0 \ \ \forall\, y_j > x_n \tag{24}$$

must hold. From inequality (23) it follows that

$$\frac{x_n}{1+ax_n} \geqslant y_j \quad \forall\, y_j < x_n . \tag{25}$$

As

$$y_j \leqslant \frac{2^{n-2}-1}{2^{n-2}} < \frac{2^{n-1}-1}{2^{n-1}} = x_n \tag{26}$$

then inequality (25) is transformed into

$$\frac{2^{n-1}-1}{2^{n-1}} \cdot \frac{1}{1+a\cdot\frac{2^{n-1}-1}{2^{n-1}}} \geqslant \frac{2^{n-2}-1}{2^{n-2}},$$

$$\frac{2^{n-1}-1}{2^{n-1}+a\cdot\left(2^{n-1}-1\right)} \geqslant \frac{2^{n-2}-1}{2^{n-2}},$$

$$2^{n-2}\cdot\left(2^{n-1}-1\right) \geqslant 2^{n-1}\cdot\left(2^{n-2}-1\right)+a\cdot\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right),$$

whence

$$\frac{2^{n-2}\cdot\left(2^{n-1}-1\right)-2^{n-1}\cdot\left(2^{n-2}-1\right)}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)} =$$

$$= \frac{2^{n-1}-2^{n-2}}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)} = \frac{2^{n-2}}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)} \geqslant a . \tag{27}$$

From inequality (24) it follows that

$$\frac{x_n}{1-ax_n} \geqslant y_j \quad \forall\, y_j > x_n . \tag{28}$$

As

$$1 \geqslant y_j > \frac{2^{n-1}-1}{2^{n-1}} = x_n \tag{29}$$

then inequality (28) is transformed into

$$\frac{2^{n-1}-1}{2^{n-1}} \cdot \frac{1}{1-a\cdot\frac{2^{n-1}-1}{2^{n-1}}} \geqslant 1,$$

$$\frac{2^{n-1}-1}{2^{n-1}-a\cdot\left(2^{n-1}-1\right)} \geqslant 1 . \tag{30}$$

If $0 < a < 1$ then

$$\frac{2^{n-1}}{2^{n-1}-1} > 1 > a\,, \tag{31}$$

whence

$$2^{n-1} - a \cdot \left(2^{n-1} - 1\right) > 0$$

and inequality (30) is written as

$$2^{n-1} - 1 \geqslant 2^{n-1} - a \cdot \left(2^{n-1} - 1\right). \tag{32}$$

Thus, inequality (32) is followed by

$$a \geqslant \frac{1}{2^{n-1}-1}\,. \tag{33}$$

Consequently, if the membership in (18) is true, then the n-th row of matrix (7) is nonnegative. The n-th column is nonpositive, whereas $k_{nn} = 0$ is a maximin and minimax entry, and situation (17) is indeed a saddle point.

Inasmuch as

$$\begin{aligned}
&\frac{2^{n-2}}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)} - \frac{1}{2^{n-1}-1} = \\
&= \frac{2^{n-2} - 2^{n-2} + 1}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)} = \\
&= \frac{1}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)} > 0 \quad \forall\, n = \overline{3, N-1}
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
&\frac{2^{n-2}}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)} - 1 = \frac{2^{n-2} - \left(2^{n-1}\cdot 2^{n-2} - 2^{n-2} - 2^{n-1} + 1\right)}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)} = \\
&= \frac{2^{n} - 2^{2n-3} - 1}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)} = \\
&= \frac{2^{n}\cdot\left(1 - 2^{n-3}\right) - 1}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)} < 0 \quad \forall\, n = \overline{3, N-1}
\end{aligned} \tag{35}$$

then there always $\exists\, n \in \left\{\overline{3, N-1}\right\}$ such that the inclusion in (18) is true, and the interval in (18) is nonempty, and situation (17) is a saddle point. Suppose that another entry on the main diagonal exists, k_{mm} by $m \in \left\{\overline{3, N-1}\right\}$ and $m \neq n$, which is a saddle point also. This can be possible if both the membership in (18) and membership

$$a \in \left[\frac{1}{2^{m-1}-1}; \frac{2^{m-2}}{\left(2^{m-1}-1\right)\cdot\left(2^{m-2}-1\right)} \right] \tag{36}$$

are true, i. e. intersection

$$\left[\frac{1}{2^{n-1}-1}; \frac{2^{n-2}}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)} \right] \cap \left[\frac{1}{2^{m-1}-1}; \frac{2^{m-2}}{\left(2^{m-1}-1\right)\cdot\left(2^{m-2}-1\right)} \right] \neq \varnothing . \tag{37}$$

If $m < n$ then

$$\frac{1}{2^{m-1}-1} > \frac{1}{2^{n-1}-1}$$

and so inequality

$$\frac{2^{n-2}}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)} > \frac{1}{2^{m-1}-1} \tag{38}$$

must hold for (37). However,

$$\begin{aligned} &\frac{2^{n-2}}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)} - \frac{1}{2^{m-1}-1} = \\ &= \frac{2^{n-2}\cdot 2^{m-1} - 2^{n-2} - \left(2^{n-1}\cdot 2^{n-2} - 2^{n-2} - 2^{n-1} + 1\right)}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)\cdot\left(2^{m-1}-1\right)} = \\ &= \frac{2^{n-2}\cdot 2^{m-1} + 2^{n-1} - 2^{2n-3} - 1}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)\cdot\left(2^{m-1}-1\right)} = \\ &= \frac{2^{n-1}\cdot\left(2^{m-2} + 1 - 2^{n-2}\right) - 1}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)\cdot\left(2^{m-1}-1\right)} . \end{aligned} \tag{39}$$

As $3 \leqslant m < n$ then

$$2^{m-2} + 1 < 2^{n-2}$$

and thus

$$2^{m-2} + 1 - 2^{n-2} < 0 ,$$

whence

$$2^{n-1}\cdot\left(2^{m-2} + 1 - 2^{n-2}\right) - 1 < 0 .$$

So, ratio (39) is negative and inequality (38) does not hold. If $m > n$ then

$$\frac{1}{2^{m-1}-1} < \frac{1}{2^{n-1}-1}$$

and so inequality

$$\frac{2^{m-2}}{\left(2^{m-1}-1\right)\cdot\left(2^{m-2}-1\right)} > \frac{1}{2^{n-1}-1} \tag{40}$$

must hold for (37). By analogy to (39), just by interchanging m and n,

$$\frac{2^{m-2}}{\left(2^{m-1}-1\right)\cdot\left(2^{m-2}-1\right)} - \frac{1}{2^{n-1}-1} =$$

$$= \frac{2^{m-1}\cdot\left(2^{n-2}+1-2^{m-2}\right)-1}{\left(2^{m-1}-1\right)\cdot\left(2^{m-2}-1\right)\cdot\left(2^{n-1}-1\right)}. \tag{41}$$

As $3 \leqslant n < m$ then

$$2^{n-2}+1-2^{m-2}<0,$$

whence

$$2^{m-1}\cdot\left(2^{n-2}+1-2^{m-2}\right)-1<0.$$

So, ratio (41) is negative and inequality (40) does not hold. Therefore, saddle point (17) existing for some $n \in \left\{\overline{3, N-1}\right\}$ is single for any $N \in \mathbb{N} \setminus \{1, 2, 3\}$.

Now, consider entry k_{NN} for $N \in \mathbb{N} \setminus \{1, 2, 3\}$. This entry is the result of when both the duelists simultaneously shoot at the duel end corresponding to situation (19). If situation (19) is a saddle point, then, in the N-th row of matrix (7), inequality

$$K\left(x_N, y_j\right) = ax_N - ay_j - a^2 x_N y_j = a - ay_j - a^2 y_j \geqslant 0 \quad \forall\, y_j < x_N = 1 \tag{42}$$

must hold. From inequality (42) it follows that

$$\frac{1}{1+a} \geqslant y_j \quad \forall\, y_j < 1. \tag{43}$$

As

$$y_j \leqslant y_{N-1} = \frac{2^{N-2}-1}{2^{N-2}} < 1 = x_N \tag{44}$$

then inequality (43) is transformed into

$$\frac{1}{1+a} \geqslant \frac{2^{N-2}-1}{2^{N-2}},$$

$$2^{N-2} \geqslant 2^{N-2} - 1 + a \cdot \left(2^{N-2} - 1\right),$$

$$1 \geqslant a \cdot \left(2^{N-2} - 1\right),$$

whence

$$1 > \frac{1}{2^{N-2} - 1} \geqslant a, \tag{45}$$

which is true due to the membership in (20) is true, i. e. inequality (42) holds by (45). Consequently, if the membership in (20) is true, then situation (19) is a saddle point. □

Theorem 3. In a progressive discrete silent duel (6) by (4), (5), (7) for $N \in \mathbb{N} \setminus \{1, 2\}$, situation (11) is optimal for $a \geqslant 1$. For $N \in \mathbb{N} \setminus \{1, 2, 3\}$, this saddle point is the single one.

Proof. For $N = 3$, owing to *Theorem 1*, situation (11) is the single saddle point for (13) by $a > 1$ and it is one of the four saddle points for (13) by $a = 1$. For $N \in \mathbb{N} \setminus \{1, 2, 3\}$ consider entry k_{22} that is the result of when both the duelists simultaneously shoot at the middle of the duel time span corresponding to situation (11). This entry is in the second row of matrix (7), where

$$K\left(x_2, y_1\right) = K\left(\frac{1}{2}, 0\right) = \frac{a}{2} > 0 \tag{46}$$

and

$$K\left(\frac{1}{2}, y_j\right) = \frac{a}{2} - ay_j + \frac{a^2}{2} y_j = \frac{a}{2} \cdot \left(1 - 2y_j + ay_j\right) \ \forall \ y_j > \frac{1}{2}. \tag{47}$$

If $a = 1$ then

$$1 - 2y_j + ay_j = 1 - y_j \geqslant 0$$

and (47) is nonnegative:

$$K\left(\frac{1}{2}, y_j\right) = \frac{a}{2} \cdot \left(1 - 2y_j + ay_j\right) = \frac{1 - y_j}{2} \geqslant 0 \ \forall \ y_j > \frac{1}{2}, \tag{48}$$

where

$$K\left(\frac{1}{2}, y_N\right) = K\left(\frac{1}{2}, 1\right) = k_{2N} = 0. \tag{49}$$

Inequality (46) with inequality (48) and (49) imply that the second row by $a = 1$, except for entries $k_{22} = 0$ and $k_{2N} = 0$, contains only positive entries, whence situation (11) is a saddle point. Owing to (45) is false, inequality (42) is impossible, the N-th column cannot contain saddle points, and so saddle point (11) is the single one in the second row. Inequalities (46) and (48) also imply that entries $k_{i2} < 0$ $\forall\, i = \overline{3, N-1}$ in the second column, so saddle point (11) is the single one by $a = 1$.

If $a > 1$ then, as $\frac{1}{2} < y_j \leqslant 1$,

$$\left(1 - y_j\right)^2 \geqslant 0\,,$$

$$1 - 2y_j + y_j^2 \geqslant 0\,,$$

$$2y_j - 1 \leqslant y_j^2\,,$$

$$\frac{2y_j - 1}{y_j} \leqslant y_j\,,$$

and

$$\frac{2y_j - 1}{y_j} \leqslant y_j \leqslant 1 < a \text{ by } y_j \in \left(\frac{1}{2}; 1\right]. \tag{50}$$

Inequality (50) is followed by inequality

$$1 - 2y_j + ay_j > 0 \tag{51}$$

whence (47) is positive:

$$K\left(\frac{1}{2}, y_j\right) = \frac{a}{2} \cdot \left(1 - 2y_j + ay_j\right) > 0 \;\; \forall\, y_j > \frac{1}{2}\,. \tag{52}$$

Inequality (46) with inequality (52) imply that the second row by $a > 1$, except for entry $k_{22} = 0$, contains only positive entries, whence situation (11) is a saddle point. Inequalities (46) and (52) also imply that situation (11) is the single saddle point in the second row, and entries $k_{i2} < 0$ $\forall\, i = \overline{3, N-1}$ in the second column. The latter implies that the second row is the single nonnegative row by $a > 1$ and saddle point (11) is the single one also. □

Strictly speaking, *Theorem 2* does not claim that pure strategy solutions (17) and (19) by respective conditions (18) and (20) are the only possible pure strategy saddle points of matrix (7). Nevertheless, by *Theorem 3*, pure strategy saddle point (11) is single for $a \geqslant 1$ and $N \in \mathbb{N} \setminus \{1, 2, 3\}$.

4. Peculiarities of the solutions

A peculiarity in *Theorem 2* is that the left endpoint in the membership (18) closed interval by $n = N-1$ and the right endpoint in the membership (20) half-interval are the same. This peculiar case of the duelist's accuracy factor a is considered separately.

Theorem 4. In the progressive discrete silent duel (6) by (4), (5), (7) for $N \in \mathbb{N} \setminus \{1, 2, 3\}$, situations (19),

$$\{x_{N-1}, y_{N-1}\} = \left\{\frac{2^{N-2}-1}{2^{N-2}}, \frac{2^{N-2}-1}{2^{N-2}}\right\} \tag{53}$$

and

$$\{x_{N-1}, y_N\} = \left\{\frac{2^{N-2}-1}{2^{N-2}}, 1\right\} \tag{54}$$

and

$$\{x_N, y_{N-1}\} = \left\{1, \frac{2^{N-2}-1}{2^{N-2}}\right\} \tag{55}$$

are saddle points by

$$a = \frac{1}{2^{N-2}-1}. \tag{56}$$

Apart from situations (19), (53) — (56), there are no other saddle points in this duel.

Proof. Owing to *Theorem 2*, here $n = N-1$ and (56) satisfies both (20) and (18). Thus, situations (19) and (53) are saddle points. Besides,

$$K\left(\frac{2^{N-2}-1}{2^{N-2}}, 1\right) = \frac{1}{2^{N-2}-1} \cdot \frac{2^{N-2}-1}{2^{N-2}} - \frac{1}{2^{N-2}-1} + \frac{1}{\left(2^{N-2}-1\right)^2} \cdot \frac{2^{N-2}-1}{2^{N-2}} =$$

$$= \frac{1}{2^{N-2}} - \frac{1}{2^{N-2}-1} + \frac{1}{2^{N-2}-1} \cdot \frac{1}{2^{N-2}} =$$

$$= \frac{2^{N-2}-1-2^{N-2}+1}{\left(2^{N-2}-1\right) \cdot 2^{N-2}} = 0 = K\left(1, \frac{2^{N-2}-1}{2^{N-2}}\right) \tag{57}$$

that implies the optimality of situations (54) and (55).

The first and second rows do not contain saddle points, so it remains to consider the rows whose index is $m \in \left\{\overline{3, N-2}\right\}$. Owing to *Theorem 2*, situation (53) is the single saddle point on the main diagonal, not considering the last row. This implies that each of the $N-4$ rows whose index is $m \in \left\{\overline{3, N-2}\right\}$ contains at least one negative entry, i. e. these rows do not contain saddle points. □

If the duelist's accuracy factor satisfying inequality $0 < a < 1$ is not (56), this is another supplement to *Theorem 2*.

Theorem 5. A progressive discrete silent duel (6) by (4), (5), (7) for $N \in \mathbb{N} \setminus \{1, 2, 3\}$ by (18) and

$$a \neq \frac{1}{2^{N-2}-1} \tag{58}$$

has a single saddle point (17).

Proof. Owing to *Theorem 2*, while situation (17) is a saddle point, the membership in (20) is impossible due to (58). Therefore, situation (19) is not a saddle point, and saddle point (17) is the single one on the main diagonal. Analogously to that in *Theorem 4*, this implies that each of the $N-4$ rows whose index is $m \in \left\{\overline{3, N-1}\right\} \setminus \{n\}$ contains at least one negative entry, i. e. these rows do not contain saddle points. □

So, pure strategy saddle point (17) is indeed single by (18) and (58). The next assertion is for the case when the membership in (18) is false.

Theorem 6. A progressive discrete silent duel (6) by (4), (5), (7) for $N \in \mathbb{N} \setminus \{1, 2, 3\}$ by

$$a \in \left(0; \frac{1}{2^{N-2}-1}\right) \tag{59}$$

has the single saddle point (19).

Proof. Owing to *Theorem 2*, situation (19) is a saddle point by (59). Analogously to (42) — (45), when (59) is true,

$$1 > \frac{1}{2^{N-2}-1} > a\,,$$

$$1 > a \cdot \left(2^{N-2}-1\right),$$

$$2^{N-2} > 2^{N-2} - 1 + a \cdot \left(2^{N-2}-1\right),$$

$$\frac{1}{1+a} > \frac{2^{N-2}-1}{2^{N-2}} = y_{N-1} \geqslant y_j\,,$$

$$\frac{1}{1+a} > y_j\,,$$

$$1 - y_j - ay_j > 0 \quad \forall\ y_j < 1. \tag{60}$$

Using inequality (60), there is inequality

$$K\left(x_N, y_j\right) = ax_N - ay_j - a^2 x_N y_j = a\left(1 - y_j - ay_j\right) > 0 \quad \forall\ y_j < x_N = 1 \tag{61}$$

in the N-th row of matrix (7). Inequality (61) implies that situation (19) is the single saddle point in the N-th row, and entries $k_{iN} < 0 \quad \forall\ i = \overline{1, N-1}$ in the N-th column. The latter implies that the N-th row is the single nonnegative row by (59) and saddle point (19) is the single one also. □

Consequently, by *Theorems 5* and *6*, a progressive discrete silent duel (6) by (4), (5), (7) for $N \in \mathbb{N} \setminus \{1, 2, 3\}$ and $0 < a < 1$ by either (58) or (59) has a single pure strategy saddle point. If (59) holds, then every duel has the same pure strategy solution, by which the optimal behavior of the duelist is to act (shoot) at the duel end moment. Otherwise, excluding the boundary case of *Theorem 4*, pure strategy saddle point (17) is single by

$$a \in \left(\frac{1}{2^{n-1} - 1}; \frac{2^{n-2}}{\left(2^{n-1} - 1\right) \cdot \left(2^{n-2} - 1\right)} \right] \tag{62}$$

and, if (62) does not hold, the duel is not solved in pure strategies.

5. Discussion and conclusion

Duels are game-theoretic patterns used to model timing competitive interactions between two sides personified as players or duelists. The solutions to duels allow players holding at the most reasonable strategies and develop rationalized processes of sharing resources for which the players compete [3], [24], [17], [25], [22], [12]. Particular examples of the interactions are auctioning, advertising, market control struggle, product placement, retailing, and many other socioeconomic competitive processes between two sides [13], [15], [23], [12], [14].

Although there are almost no linearly developing real-time processes, the linear accuracy with a proportionality factor is a generalization that allows to make an initial approximation of the interaction model. Besides, the accuracy factor allows to consider various patterns of the duelists, which may be worse or better shooters (i. e., be worse or better at making smarter decisions). In addition, specifying locations of moments (4) of possible shooting as (5) is natural as well. As the duelist approaches to the duel end, the tension and responsibility grow, and so time moments of possible shooting must be located progressively denser. The geometrical progression is a quite natural pattern of the growth, which still can be made more complicated by adding some noise.

The proved assertions contribute to the games of timing a few specificities of the discrete silent duel. Firstly, a specificity consists in that the solution of a generalized progressive discrete silent duel, with identical linear accuracy functions of the duelists, is not always a pure strategy saddle point if the duelist's accuracy factor is less than 1 (as a corollary from *Theorems 2*, *5* and *6*). Secondly, if the factor is not less than 1, the duelist's optimal strategy is the middle of the duel time span (*Theorem 3*). For a trivial game, where the duelist possesses just one moment of possible shooting between the duel beginning and end moments, and the accuracy factor is 1, any pure strategy situation in this 3×3 duel, not containing the duel beginning moment, is optimal (*Theorem 1*). As the accuracy factor becomes less than 1, this 3×3 duel has only one pure strategy saddle point which is of the duel end moments (*Theorem 1*). In the boundary case, when the accuracy factor is equal to the inverse numerator of the ratio that is the time moment preceding the duel end moment, the duel has four pure strategy saddle points which are of the mentioned time moments (*Theorem 4*).

Nonlinearity in the accuracy function can be a matter of further research on progressive discrete silent duels. For instance, it can be the quadratic and cubic accuracy. A special case is the square-root accuracy implying that the duelist is better at shooting (compared, e. g., to the linear accuracy by $a=1$ as $\sqrt{t}>t$ for $0<t<1$).

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