GLASNIK MATEMATIČKI
Vol. 59(79)(2024), 125 – 145

SOME RESULTS IN ASYMPTOTIC ANALYSIS OF FINITE-ENERGY SEQUENCES OF ONE-DIMENSIONAL CAHN–HILLIARD FUNCTIONAL WITH NON-STANDARD TWO-WELL POTENTIAL

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Abstract. In this paper we extend the consideration of G. Leoni pertaining to the finite-energy sequences of the one-dimensional Cahn-Hilliard functional

$$I_0^\varepsilon(u) = \int_0^1 \Big(\varepsilon^2 u'^2(s) + W(u(s))\Big) ds,$$

where $u \in \mathrm{H}^1(0,1)$ and where W is a two-well potential with symmetrically placed wells endowed with a non-standard integrability condition. We introduce several new classes of finite-energy sequences, we recover their underlying geometric properties as $\varepsilon \longrightarrow 0$, and we prove the related compactness result.

1. Introduction

We study asymptotic behavior of the functional $I_0^\varepsilon : \mathrm{H}^1(0,1) \longrightarrow \mathbf{R}$ defined by

$$I_0^\varepsilon(u) := \int_0^1 \Big(\varepsilon^2 u'^2(s) + W(u(s))\Big) ds, \tag{1.1}$$

as a small parameter ε tends to zero, where W is a non-negative continuous function with the suitable behavior at infinity such that $W(\zeta) = 0$ if and only if $\zeta \in \{-1, 1\}$ holds true (in short, the two-well potential with symmetrically placed wells). The functional (1.1) (see [2, 10, 20, 23]) is known as the Cahn-Hilliard functional (or as the Modica-Mortola functional). To simplify the notation, we often omit to relabel subsequences, and by "a sequence (x_ε)" we

2020 *Mathematics Subject Classification.* 34E15, 49J45.
Key words and phrases. Asymptotic analysis, singular perturbation, Young measures, Cahn-Hilliard functional, compactness.

mean a sequence defined only for countably many $\varepsilon = \varepsilon_n$ such that $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow +\infty$. We recall that the following definitions:

Definition 1.1. *We say that* (x_ε) *is a pre-compact sequence in a metric space* X *if every subsequence of* (x_ε) *admits a further subsequence which converges in* X. *We say that a sequence* (z_ε) *in a metric space* Z *is a finite-energy sequence (or an FE sequence for short) for a sequence of functionals* $F^\varepsilon : Z \longrightarrow [0, +\infty]$ *if it holds that* $\limsup_{\varepsilon \longrightarrow 0} F^\varepsilon(z_\varepsilon) < +\infty$.

In this paper, by (u_ε) we always denote an arbitrary FE sequence for $(\varepsilon^{-1} I_0^\varepsilon)$ in $\mathrm{H}^1(0,1)$. Accordingly, throughout Sections 2-5, the expression "an FE sequence" is reserved for an FE sequence (u_ε) for $(\varepsilon^{-1} I_0^\varepsilon)$. As a consequence of the definitions above, FE sequences for the rescaled functional (1.1) do not develop internally created small scale oscillations, which makes them easier to handle (compare [29–32]) for the study of the second-order variants of (1.1) with internally created small oscillatory scale). Such singular perturbation problems are studied within the framework of the gradient theory of phase transitions (cf. [1] or [18]). In the case of the functional (1.1), the term $\int_0^1 \varepsilon^2 |u'|^2$ penalizes rapid changes of the density u and it plays the role of an interfacial energy. The small positive quantity ε is the thickness of the transition layer separating two different phases or states of u within the domain $(0,1)$. Different phases develop as the result of the minimization process subject to a given mass constraint. As we pass to the limit as $\varepsilon \longrightarrow 0$, optimal configurations described by geometric properties of the minimizing sequence (u_ε) of (1.1) resemble more and more the optimal configuration of the system subject to classical assumptions in the theory of phase transitions, where it is assumed that the contact area between different phases of u is concentrated on the interfacial surface of thickness zero. Similar types of functionals appear in studying coherent solid-solid phase transformations and can be understood as a simplified one-dimensional model for a phase transition at a martensite-austenite interface (cf. [3, 24] and references therein). Extensive literature is available on a wider subject, and our list of references is by no means complete, nor does it attempt to cite the most important contributions (a more complete list is available in, for instance, [7, 25]). Although many authors studied asymptotic behavior of the functionals similar to (1.1), the analysis is usually done under rather strong growth conditions on W (cf. [4, 11, 19, 28, 34]). In particular, the commonly used classical Fonseca-Tartar assumption (cf. [15]) requires that W grows at least linearly at infinity (in such a case we say that W is coercive). For such a choice of W, we immediately deduce equi-integrability of (u_ε) in $\mathrm{L}^1(0,1)$ of an arbitrary FE sequence (u_ε), which (by the Vitali convergence theorem) gives strong pre-compactness of (u_ε) in $\mathrm{L}^1(0,1)$ as $\varepsilon \longrightarrow 0$. Such compactness result is the prerequisite for the proof of Γ-convergence of $(\varepsilon^{-1} I_0^\varepsilon)$ on $\mathrm{L}^1(0,1)$ as $\varepsilon \longrightarrow 0$. One possibility of relaxing the assumptions on W is to consider the case of W which satisfies a suitable non-integrability condition

$$\int_0^{+\infty} \sqrt{V(\xi)}d\xi = +\infty, \tag{1.2}$$

where $V : [0, +\infty) \longrightarrow [0, +\infty)$ is defined by $V(\xi) := \min\{W(\zeta) : |\zeta| = \xi\}$. We refer to such a case as the non-integrable case. In the relatively recent paper [20], G. Leoni obtained strong pre-compactness of FE sequences (u_ε) for $(\varepsilon^{-1}I_0^\varepsilon)$ (equipped with the mass constraint) in $\mathrm{L}^1(0,1)$ under the non-integrability assumption (1.2). Boundedness of an FE sequence (u_ε) in $\mathrm{L}^p(0,1)$, where $p > 0$, is already sufficient to bring about its underlying geometry as $\varepsilon \longrightarrow 0$ (cf. Theorem 3.2, (iii)). As a typical example of W which satisfies (1.2) ($\int_{\mathbf{R}} \sqrt{W(\zeta)}d\zeta < +\infty$, resp.), we consider W such that for $0 \leq q \leq 2$ ($q > 2$, resp.) and $R_0 > 1$ it holds that

$$\frac{c_0}{|\zeta|^q} \leq W(\zeta) \leq \frac{C_0}{|\zeta|^q}, \tag{1.3}$$

for every $|\zeta| \geq R_0$, where $0 < c_0 \leq C_0 < +\infty$. In this paper, a particular emphasis is placed on the optimality of the assumptions on W. We always assume that W satisfies $W(\zeta) = 0$ iff $\zeta \in \{\pm 1\}$, $W \geq 0$. If no additional properties of W are assumed, we refer to such W as an arbitrary two-well potential. Furthermore, we introduce the following integrability condition (non-integrability condition, resp.), which is essential to our setting:

$$\int_0^{+\infty} \sqrt{V(\xi)}d\xi < +\infty \tag{1.4}$$

$$\Big(\int_0^{+\infty} \sqrt{V^p(\xi)}d\xi = +\infty, \text{ where } 1 < p < +\infty, \text{ resp.}\Big). \tag{1.5}$$

In particular, our assumptions on W allow the consideration of the case $\liminf_{\xi \longrightarrow +\infty} V(\xi) = 0$ and/or $\limsup_{\xi \longrightarrow +\infty} V(\xi) = +\infty$. These assumptions constitute a non-standard behavior of W as infinity, and are not well covered in the literature. The main technical tool in the proofs of our results is the area formula (cf. [22], Theorem 3.65, p. 100). Herein, we present certain underlying asymptotic properties of FE sequences of the one-dimensional Cahn-Hilliard functional $\varepsilon^{-1}I_0^\varepsilon$, where W is a two-well potential with symmetrically placed wells endowed with aforementioned integrability condition, whereby FE sequences are not a priori bounded in $\mathrm{L}^1(0,1)$. To this end, in Section 2, we introduce a number of new classes of sequences, we describe the relation between them, and we provide some examples. In Section 3, we present a priori estimates for FE sequences in the case of an arbitrary two-well potential W. In Section 4 (Section 5, resp.), we recover further results, subject to appropriate assumptions on the two-well potential W, which can be interpreted as regularity results parallel to Theorem 1.3 in [20], including a variant of classical compactness result (Theorem 5.2). As such, our results should be primarily viewed as a further development of the considerations in [20]. We were not able to find these results in widely available sources. To the best of our knowledge, observations stated in Theorem 4.1, Proposition 4.2 and Proposition 5.1 are new, as well as several notions introduced in Definition 2.1 and Definition 2.2. While Theorem 5.2 is well-known (cf. [21]), herein we present its technical improvement. Results of Proposition 3.3 and Corollary 3.4 are most-likely also well-known, but not easily found in the literature. In Theorem 4.3 and Theorem 5.3 we summed up the main results of this paper.

A very similar functional to I_0^ε is $J_0^\varepsilon : \mathrm{H}^2(0,1) \longrightarrow \mathbf{R}$, where

$$J_0^\varepsilon(v) := \int_0^1 \Big(\varepsilon^2 v''^2(s) + W(v'(s))\Big) ds. \tag{1.6}$$

The functional (1.1) is usually equipped with the mass constraint $\int_0^1 u(s)ds = 0$, which, in the notation (1.6) (see also [3] or [24]), is equivalent to the constraint $v \in \mathrm{H}^2_\#(0,1)$, where $\mathrm{H}^2_\#(0,1) := \{v \in \mathrm{H}^2(0,1) : v(0) = v(1)\}$ (cf. Corollary 3.4). In one-dimensional case considered herein, the replacement of u by v' is no loss of generality, but it has the advantage of enabling a simple introduction of the lower-order perturbation of I_0^ε. One example of such lower-order perturbation is $J_\alpha^\varepsilon : \mathrm{H}^2(0,1) \longrightarrow \mathbf{R}$,

$$J_\alpha^\varepsilon(v) := J_0^\varepsilon(v) + \alpha_0 \int_0^1 v^2(s)ds, \tag{1.7}$$

where $\alpha > 0$ (cf. [3]). Another way of interpreting the functional (1.1) is to view it as a very special case of singularly perturbed functional of the type $u \mapsto \int_0^1 f(s, u(s), u'(s))ds$ (cf. [6]). If $q = -1$ is chosen in (1.3), the standard approach to the asymptotic problem associated with (1.6) uses the so-called Zhang's lemma (cf. [26]) which provides direct approximation of an arbitrary FE sequence for the rescaled functional (1.6) by an equi-Lipschitz FE sequence (compare [33]).

Following [3], we consider a compact metric space (K, d) (the space of patterns), which is the set of all measurable mappings $x : \mathbf{R} \longrightarrow [-\infty, +\infty]$ (modulo equivalence λ-almost everywhere, where λ is one-dimensional Lebesgue measure), endowed with the metric d defined by

$$d(x_1, x_2) := \sum_{k=1}^{\infty} \frac{1}{2^k \alpha_k} \left| \int_{\mathbf{R}} y_k \Big(\frac{2}{\pi} \arctan x_1 - \frac{2}{\pi} \arctan x_2\Big) d\lambda \right|,$$

where (y_k) is a sequence of bounded functions which are dense in $\mathrm{L}^1(\mathbf{R})$, such that the support of y_k is a subset of $(-k, k)$, with $\alpha_k := \|y_k\|_{\mathrm{L}^1} + \|y_k\|_{\mathrm{L}^\infty}$. As shown in [3], p. 806, $\mathrm{L}^p_{loc}(\mathbf{R})$ continuously embeds in K for every $p \in [1, +\infty]$. The notation $\mathrm{C}(K)$ ($\mathrm{C}_0(\mathbf{R})$, resp.) stands for the space of all continuous real functions on K (the space of all continuous real functions on $\mathbf{R}$ which vanish at infinity, resp.), whose dual is identified with the space of all real

Radon measures on K (all real bounded Radon measures on $\mathbf{R}$, resp.), denoted by $\mathcal{M}(K)$ ($\mathcal{M}_b(\mathbf{R})$, resp.), endowed with the corresponding weak-star topology. By $\mathcal{P}(K)$ ($\mathcal{P}(\mathbf{R})$, resp.) we denote the set of all probability measures in $\mathcal{M}(K)$ ($\mathcal{M}_b(\mathbf{R})$, resp.). By $\mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}(K))$ ($\mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}_b(\mathbf{R}))$, resp.) we denote the dual of $\mathrm{L}^1(\Omega;\mathrm{C}(K))$ ($\mathrm{L}^1(\Omega;\mathrm{C}_0(\mathbf{R}))$, resp.), where $\Omega\subseteq\mathbf{R}$ is measurable set such that $0<\lambda(\Omega)<+\infty$. The set of all K-valued ($\mathbf{R}$-valued, resp.) Young measures on Ω, denoted by $YM(\Omega;K)$ ($\mathrm{L}^\infty_{w*}(\Omega;\mathcal{P}(\mathbf{R}))$, resp.), is the set of all $\boldsymbol{\nu}\in\mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}(K))$ ($\boldsymbol{\nu}\in\mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}_b(\mathbf{R}))$, resp.) such that $\nu_s\in\mathcal{P}(K)$ ($\nu_s\in\mathcal{P}(\mathbf{R})$, resp.) for almost every $s\in\Omega$, where $\boldsymbol{\nu}(s):=\nu_s$, $s\in\Omega$, and it is always endowed with the weak-star topology of $\mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}(K))$ ($\mathrm{L}^\infty_{w*}(\Omega;\mathcal{M}_b(\mathbf{R}))$, resp.). The elementary Young measure associated to a measurable map $u:\Omega\longrightarrow K$ ($u:\Omega\longrightarrow\mathbf{R}$, resp.) is the map $\underline{\delta}_u:\Omega\longrightarrow\mathcal{M}(K)$ ($\underline{\delta}_u:\Omega\longrightarrow\mathcal{M}_b(\mathbf{R})$, resp.) given by $\underline{\delta}_u(s):=\delta_{u(s)}$, $s\in\Omega$. Besides the fundamental theorem of Young measures which involves $\mathbf{R}$-valued Young measures (cf. [5] or [27]), we use the version of the theorem which involves K-valued Young measures (cf. [3]). The main advantage of the introduction of the notion of K-valued Young measures comes from the fact that compactness of $YM((0,1);K)$ is guaranteed by compactness of K (such a compactness fails in the case of $\mathbf{R}$-valued Young measures). We say that a sequence of measurable sets (K_η) increases to a non-empty bounded interval $J\subseteq\mathbf{R}$ as $\eta\searrow 0$, and we write $K_\eta\nearrow J$ as $\eta\searrow 0$, if it holds that $\chi_{K_\eta}\nearrow\chi_J$ (a.e. $s\in\mathbf{R}$) as $\eta\searrow 0$, and if there exists $0<\overline{\eta}<\lambda(J)$ such that for every $0<\eta\le\overline{\eta}$ we have $\lambda(J\backslash K_\eta)\le\eta$, and such that $0<\eta_2\le\eta_1\le\overline{\eta}$ implies $K_{\eta_1}\subseteq K_{\eta_2}\subseteq J$. We also introduce abbreviations $m_\varepsilon:=\min_{[0,1]}|u_\varepsilon|$, $M_\varepsilon:=\max_{[0,1]}|u_\varepsilon|$, and $m_\varepsilon(A):=\inf_A|u_\varepsilon|$, $M_\varepsilon(A):=\sup_A|u_\varepsilon|$, where $A\subseteq[0,1]$ is an arbitrary measurable set of strictly positive measure λ, and where (u_ε) is an FE sequence for $(\varepsilon^{-1}I_0^\varepsilon)$. By $u\llcorner\omega$ we denote the restriction of the function $u:\Omega\longrightarrow\mathbf{R}$ on the set $\omega\subset\Omega$, while by $u^{\leftarrow}(\xi)$ we denote the pre-image of ξ with respect to u, i.e., $u^{\leftarrow}(\xi):=\{s\in\Omega:u(s)=\xi\}$. We recall that we say that a sequence (u_ε) is a.e. point-wise bounded on Ω as $\varepsilon\longrightarrow 0$ if it holds that $\limsup_{\varepsilon\longrightarrow 0}|u_\varepsilon(s)|<+\infty$ (a.e. $s\in\Omega$). By $\mathrm{card}S$ we denote the cardinality of a set S. Throughout the paper, we assume that every Sobolev function $u\in\mathrm{W}^{1,p}(0,1)$, where $1\le p\le+\infty$, is already replaced by its absolutely continuous representantive (cf. [12], Theorem 1, p. 163). By $\mathrm{BV}(0,1)$ we denote the set of all real functions of bounded variation on $(0,1)$ (cf. [12], p. 166), and we say that a set $E\subseteq(0,1)$ is a set of finite perimeter in $(0,1)$ if it holds that $\chi_E\in\mathrm{BV}(0,1)$. If a measurable function $\psi:[0,+\infty)\longrightarrow[0,+\infty)$ belongs to $\mathrm{L}^1_{loc}((\rho,+\infty);d\mu)$, where $\rho\ge 0$, $d\mu=gd\lambda$, $g\in\mathrm{L}^1_{loc}(\rho,+\infty)$, $g\ge 0$, we define $\liminf_{\xi\longrightarrow+\infty}\psi(\xi):=\liminf_{\xi\longrightarrow+\infty}\psi_\star(\xi)$, where $\psi_\star$ denotes the precise representative of ψ (cf. [12], p. 46), which is well-defined for every $\xi\in[0,+\infty)$. By $\mathrm{supp}\mu$ we denote the support of $\mu\in\mathcal{M}_b(\mathbf{R})$ ($\mu\in\mathcal{M}(K)$, resp.). Finally, we write "WLG" as an abbreviation instead of the expression "without loss of generality".

2. Terminology

To simplify the statements of our main results, we introduce the following terminology. Note that in the definitions below we do not require that (u_ε) is an FE sequence, but we primarily do have FE sequences in mind. We are focused on two different sets of properties of FE sequences, which are stated in the definitions below on the level of arbitrary sequences: boundedness and oscillation. We begin with the definition pertaining to boundedness of a given sequence.

Definition 2.1. *Consider a sequence (u_ε) in $\mathrm{C}(\overline{J})$, where $J \subseteq \mathbf{R}$ is a non-empty bounded open interval. We say that the sequence (u_ε) is:*

(i) normal (or N) on J if there exists a sequence (c_ε) in $\overline{J}$ such that $(u_\varepsilon(c_\varepsilon))$ is bounded,
(ii) uniformly normal (or UN) on J if there exists $\varepsilon_0 > 0$ and a measurable set $G \subseteq J$ such that $\lambda(G) > 0$ with the property: for every measurable set $A \subseteq G$ such that $\lambda(A) > 0$ it holds that $\sup_{0<\varepsilon\leq\varepsilon_0} m_\varepsilon(A) < +\infty$.

If $v_\varepsilon \in \mathrm{C}^1[0,1]$ is T_ε-periodic for some $0 < T_\varepsilon \leq 1$ for every $0 < \varepsilon \leq \varepsilon_0$, then the Rolle theorem implies the normality of the sequence (v'_ε) on $(0,1)$. The Rolle theorem also implies that for a sequence (v_ε) of periodic smooth functions it holds that $(\varepsilon^{\frac{1}{2}} v'_\varepsilon)$ is normal iff (v'_ε) is normal. In particular, for every FE sequence (v_ε) for $(\varepsilon^{-1} J_0^\varepsilon)$ in $\mathrm{H}^2_\#(0,1)$ it follows that (v'_ε) is a normal FE sequence. Typical condition which is imposed on FE sequences (u_ε) for $(\varepsilon^{-1} I_0^\varepsilon)$ in $\mathrm{H}^1(0,1)$ is the mass constraint condition $\tilde{m}_\varepsilon = m$ for some $m \in \mathbf{R}$, where $\tilde{m}_\varepsilon := \int_0^1 u_\varepsilon(s)ds$ (cf. [20]), which (by the integral mean value theorem) also implies the normality of (u_ε). Roughly speaking, if there exist at least two disjoint non-empty open intervals $J \subseteq (0,1)$ such that $\mathrm{osc}(|u|;J) \geq b - a > 0$, where $\mathrm{osc}(|u|;J) := \sup_J |u| - \inf_J |u|$ and $0 \leq a < b < +\infty$, we have some kind of oscillatory sequence. In particular, if a normal sequence (u_ε) is unbounded in $\mathrm{L}^\infty(0,1)$, then u_ε fulfills the later condition for at least one non-empty open interval J_ε for every $0 < \varepsilon \leq \varepsilon_0$ (up to a subsequence). In the next definition, we introduce several notions pertaining to oscillatory properties of a given sequence.

Definition 2.2. *Consider a sequence (u_ε) in $\mathrm{C}(J)$, where $J \subseteq \mathbf{R}$ is a non-empty open interval. We say that the sequence (u_ε) is:*

(i) lower pre-oscillatory (or LPO) on J if there exists a sequence of measurable sets (K_η) which increases to J as $\eta \searrow 0$, such that there exists $0 < \overline{\eta} < \lambda(J)$ with the following property: for every $0 < \eta \leq \overline{\eta}$ we have

$$\liminf_{\xi \longrightarrow +\infty} \liminf_{\varepsilon \longrightarrow 0} \mathrm{card}\{|u_{\varepsilon,\eta}|^{\leftarrow}(\xi)\} = 0, \text{ where } u_{\varepsilon,\eta} := u_\varepsilon \llcorner K_\eta, \tag{2.1}$$

and where the mapping $\xi \mapsto \liminf_{\varepsilon\longrightarrow 0} \mathrm{card}\{|u_{\varepsilon,\eta}|^{\leftarrow}(\xi)\}$ *belongs to* $\mathrm{L}^1_{loc}((\rho,+\infty);d\mu)$*, with* $d\mu = g d\lambda$*,* $\rho \geq 0$*,* $g \in \mathrm{L}^1_{loc}(\rho,+\infty)$*, and* $g(\xi) > 0$ *(a.e.* $\xi \in (\rho,+\infty)$*),*

(ii) (a,b)*-steady oscillatory (or* (a,b)*-SO) on* J *if there exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon \leq \varepsilon_0$ *there exists at most* $N_\varepsilon = N_\varepsilon(a,b)$ *pairwise disjoint non-empty open intervals* $(J_i^\varepsilon)_{i=1}^{N_\varepsilon}$ *in* J *such that* $\inf_{J_i^\varepsilon}|u_\varepsilon| \leq a$*,* $\sup_{J_i^\varepsilon}|u_\varepsilon| \geq b$*,* $\sup_{0<\varepsilon\leq\varepsilon_0} N_\varepsilon \leq N_0$ *for some* $N_0 = N_0(a,b) \in \mathbf{N}$*,*

(iii) steady oscillatory (or SO) on J *if it is* (a,b)*-steady oscillatory on* J *for every* $0 \leq a < b < +\infty$*,*

(iv) rapidly oscillatory (or RO) on J *if there exists* $0 \leq a < b < +\infty$ *such that* $\limsup_{\varepsilon\longrightarrow 0} N_\varepsilon(a,b) = +\infty$*.*

To proceed, we remark that the choice of the domain (being $(0,1)$ or some other choice) and periodicity of the sequence (u_ε) may significantly affect oscillatory properties in the definition above. Consider open non-empty intervals $J_1, J_2 \subseteq \mathbf{R}$ such that $J_1 \subseteq J_2$. Then it holds that: if (u_ε) is an SO sequence (an RO sequence on J_1, resp.) on J_2 , then it is also an SO sequence on J_1 (an RO sequence on J_2, resp.). For instance, every sequence of non-constant functions in $\mathrm{H}^1_{per}(0,1)$ is an RO sequence on $\mathbf{R}$, but not necessarily on $(0,1)$. By contrast, an extreme example of a sequence with no oscillations at all is a constant sequence, for which we have $N_\varepsilon(a,b) = 0$ for arbitrary $0 \leq a < b < \infty$ and for sufficiently small ε, and which is an LPO sequence. To illustrate occurrence of those oscillatory properties, we consider $\alpha, \beta \in \mathbf{R}$ and we set $u_\varepsilon(s) := \varepsilon^\beta \sin\big(\frac{2\pi}{\varepsilon^\alpha} s\big)$, $s \in \mathbf{R}$. If $\beta \geq 0$ ($\beta < 0$, resp.), then (u_ε) is an LPO sequence on $\mathbf{R}$ ((u_ε) is not an LPO sequence on $\mathbf{R}$, resp.) for every $\alpha \in \mathbf{R}$. If it holds that $\beta = 0$ and $\alpha > 0$, then $\lim_{\varepsilon\longrightarrow 0} N_\varepsilon(a,b) = +\infty$ provided $0 < a < b < 1$, and $\lim_{\varepsilon\longrightarrow 0} N_\varepsilon(a,b) = 0$ provided $0 \leq a < 1 < b < +\infty$ or $1 \leq a < b < +\infty$. Thus, (u_ε) is an RO sequence on $(0,1)$, and an (a,b)-SO sequence on $(0,1)$ as well (and also on $\mathbf{R}$) provided $0 \leq a < 1 < b < +\infty$ or $1 \leq a < b < +\infty$. Furthermore, if it holds that $\beta = 0$ and $\alpha \leq 0$, then $0 < \lim_{\varepsilon\longrightarrow 0} N_\varepsilon(a,b) < +\infty$ provided $0 < a < b < 1$, and $\lim_{\varepsilon\longrightarrow 0} N_\varepsilon(a,b) = 0$ provided $0 \leq a < 1 < b < +\infty$ or $1 \leq a < b < +\infty$, and therefore, (u_ε) is an SO sequence on $\mathbf{R}$. Finally, if $\beta \neq 0$, for every $\alpha \in \mathbf{R}$ we have $\lim_{\varepsilon\longrightarrow 0} N_\varepsilon(a,b) = 0$ for every $0 \leq a < b < +\infty$, and it results again that (u_ε) is an SO sequence on $\mathbf{R}$. On the level of FE sequences (u_ε) in $\mathrm{H}^1(0,1)$ ($\mathrm{H}^1_{per}(0,1)$, resp.), we always set $J := (0,1)$, and we use abbreviations like "(u_ε) is an SO sequence" instead of "(u_ε) is an SO sequence on $(0,1)$" whenever there is no confusion what the chosen domain J is. In the definitions above, we attempted to capture different properties of FE sequences, the absence of which can result in the loss of strong pre-compactness in $\mathrm{L}^1(0,1)$. Essential properties of such FE sequences are prescribed on sets K_η, whereas on "bits" $(0,1)\backslash K_\eta$ of arbitrarily small measure we allow any type of behavior of (u_ε). In particular, as we pass to the limit as $\eta \longrightarrow 0$, we can extract a subsequence (u_{ε_η}), where $\lim_{\eta\longrightarrow 0} \varepsilon_\eta = 0$, such that $u_{\varepsilon_\eta}\chi_{K_\eta}$ behaves in a certain way, as expressed in definitions above. The asymptotic analysis as $\varepsilon \longrightarrow 0$ of FE sequences (u_ε) which are strongly pre-compact in $\mathrm{L}^1(0,1)$ is well-covered in the literature (for instance cf. [1, 2, 4, 7, 8, 20, 21]). Typically, pre-compactness is obtained by proving (or assuming) that a chosen FE sequence is bounded in $\mathrm{L}^\infty(0,1)$ as $\varepsilon \longrightarrow 0$. In the language introduced in Definition 2.1, we restate Theorem 1.3 in [20].

Proposition 2.3. *Consider a two-well-potential W which satisfies (1.2). Then every normal FE sequence is bounded in* $\mathrm{L}^\infty(0,1)$ *as* $\varepsilon \longrightarrow 0$. *As a consequence, every normal FE sequence is strongly pre-compact in* $\mathrm{L}^1(0,1)$ *as* $\varepsilon \longrightarrow 0$.

Our results are a continuation of Leoni's remarks in [20] (cf. Example 1.4 and Remark 1.5 therein). We begin with some observations for an arbitrary two-well potential W.

3. The case of arbitrary W

As the following results show, in the case of an arbitrary two-well potential W, it is necessary to impose at least some (though very weak) point-wise bounds to an arbitrary FE sequence in order to obtain $\mathrm{L}^p(0,1)$ bounds (compare [20], Example 1.4). In the first lemma, we also provide L^∞-bound for a quite large class of FE sequences (u_ε).

Lemma 3.1. *Consider an arbitrary W and an arbitrary FE sequence (u_ε). If $(\varepsilon^{\frac{1}{2}}u_\varepsilon)$ is a normal sequence, then we have the following:*

(i) $\limsup_{\varepsilon\longrightarrow 0} \|\varepsilon^{\frac{1}{2}}u_\varepsilon\|_{\mathrm{L}^\infty(0,1)} < +\infty$,

(ii) If for some $p > 0$ there exists $\varepsilon_0 > 0$ and $R_0 >> 1$ such that we have $\sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{-\frac{p}{2}}\lambda\{|u_\varepsilon| > R_0\} < +\infty$, then $(\int_0^1 |u_\varepsilon|^p)$ is bounded,

(iii) If there exists $\varepsilon_0 > 0$ such that $\lim_{R\longrightarrow+\infty} \sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{-\frac{1}{2}}\lambda\{|u_\varepsilon| > R\} = 0$, then (u_ε) is weakly pre-compact in $\mathrm{L}^1(1,0)$.

Proof. First, we address the proof of (i). Jensen's inequality implies that $M \geq \varepsilon\int_0^1 |u_\varepsilon'|^2 \geq \varepsilon\|u_\varepsilon'\|^2_{\mathrm{L}^1(1,0)}$, and so $\varepsilon^{\frac{1}{2}}\|u_\varepsilon'\|_{\mathrm{L}^1(1,0)} \leq M^{\frac{1}{2}}$. We consider $a_\varepsilon, b_\varepsilon \in [0,1]$ such that $|u_\varepsilon(a_\varepsilon)| = m_\varepsilon$, $|u_\varepsilon(b_\varepsilon)| = M_\varepsilon$, and we recall that we have $|u_\varepsilon'(s)| = ||u_\varepsilon|'(s)|$ (a.e. $s \in (0,1)$) (cf. [35], Theorem 2.1.11, p. 48). Next, we apply the fundamental theorem of calculus for absolutely continuous functions (cf. [17], Theorem 6.52, p. 364), which gives

$$M^{\frac{1}{2}} \geq \varepsilon^{\frac{1}{2}}\int_0^1 |u_\varepsilon'| \geq \varepsilon^{\frac{1}{2}}\left|\int_{a_\varepsilon}^{b_\varepsilon} |u_\varepsilon|'\right| = \varepsilon^{\frac{1}{2}}(M_\varepsilon - m_\varepsilon),$$

getting $\varepsilon^{\frac{1}{2}}M_\varepsilon \leq M^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}m_\varepsilon$. Because of the normality of $(\varepsilon^{\frac{1}{2}}u_\varepsilon)$, it results that the sequence $(\varepsilon^{\frac{1}{2}}m_\varepsilon)$ is bounded, and so is $(\varepsilon^{\frac{1}{2}}M_\varepsilon)$.

The assertion (ii) follows by the estimate

$$\int_{\{|u_\varepsilon|>R_0\}} |u_\varepsilon|^p \leq C\varepsilon^{-\frac{p}{2}}\lambda\{|u_\varepsilon| > R_0\}.$$

The assertion (iii) is a consequence of the Dunford-Pettis theorem (cf. [14], Theorem 2.54, or [9]), since the assumption in (iii) provides uniform integrability of the sequence (u_ε). □

The following theorem captures (the integrable case being of particular interest here) asymptotic properties shared by all FE sequences which are L^p-bounded on a sequence of measurable subsets of arbitrarily large measure in $(0,1)$.

Theorem 3.2. *Consider an arbitrary W and an arbitrary FE sequence (u_ε) for $(\varepsilon^{-1}I_0^\varepsilon)$ with the property: there exist $0 < p \leq +\infty$ and $0 < \overline{\eta} < 1$ such that for every $0 < \eta \leq \overline{\eta}$ (u_ε) is bounded in $L^p(K_\eta)$, where the sequence of sets (K_η) satisfies $K_\eta \nearrow (0,1)$ as $\eta \searrow 0$. Then the following conclusions hold:*

(i) $|u_\varepsilon| \xrightarrow{\lambda} 1$ on $(0,1)$ as $\varepsilon \longrightarrow 0$. Moreover, every subsequence of (u_ε) allows a further subsequence (not relabeled) such that

$$|u_\varepsilon(s)| \longrightarrow 1 \ \ (a.e\ s \in (0,1)) \text{ as } \varepsilon \longrightarrow 0, \tag{3.1}$$

(ii) (u_ε) is a sequence which is a.e. point-wise bounded on $(0,1)$ as $\varepsilon \longrightarrow 0$,

(iii) an arbitrary subsequence of (u_ε) allows a further subsequence (not relabeled) which satisfies

$$\underline{\delta}_{u_\varepsilon} \xrightarrow{\ *\ } \theta_0\delta_{-1} + (1-\theta_0)\delta_1 \text{ in } YM((0,1);K) \text{ as } \varepsilon \longrightarrow 0, \tag{3.2}$$

for some measurable function (which depends on the chosen subsequence) $\theta_0 = \theta_0(s)$ which satisfies $0 \leq \theta_0(s) \leq 1$ (a.e. $s \in (0,1)$),

(iv) if $1 < p < +\infty$ ($p = +\infty$, resp.), then there exists a further subsequence which satisfies

$$u_\varepsilon \rightharpoonup 1 - 2\theta_0 \text{ in } L^p(K_\eta) \text{ as } \varepsilon \longrightarrow 0 \tag{3.3}$$

$$\Big(u_\varepsilon \xrightarrow{\ *\ } 1 - 2\theta_0 \text{ in } L^\infty(K_\eta) \text{ as } \varepsilon \longrightarrow 0, resp.\Big), \tag{3.4}$$

(v) if for every $0 < \eta \leq \overline{\eta}$ there exists a subsequence of (u_ε) (not relabeled) such that $u_\varepsilon \rightharpoonup 0$ in $L^1(K_\eta)$ as $\varepsilon \longrightarrow 0$ (where (K_η) increases to $(0,1)$ as $\eta \searrow 0$), then we get

$$\underline{\delta}_{u_\varepsilon} \xrightarrow{\ *\ } \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1 \text{ in } YM((0,1);K) \text{ as } \varepsilon \longrightarrow 0. \tag{3.5}$$

If $u_\varepsilon \rightharpoonup \varphi$ in $L^1(0,1)$ as $\varepsilon \longrightarrow 0$, then it holds that $\|\varphi\|_{L^\infty(0,1)} \leq 1$, and in (3.2) it holds that $\int_0^1 \varphi(s)ds = 0$ iff $\int_0^1 \theta_0(s)ds = \frac{1}{2}$. In particular, we have $\lim_{\varepsilon\longrightarrow 0}\int_0^1 u_\varepsilon(s)ds = 0$ iff $\int_0^1 \theta_0(s)ds = \frac{1}{2}$.

Proof. Firstly, we deal with the proof of the assertion (i). To this end, we note that, by the assertion (i) we have

$$\sup_{0<\varepsilon\leq\varepsilon_0}\int_{\{s\in K_\eta:|u_\varepsilon(s)|>R\}} R^p \leq \sup_{0<\varepsilon\leq\varepsilon_0}\int_{K_\eta}|u_\varepsilon|^p \leq C<+\infty. \tag{3.6}$$

As we pass to the limit as $R\longrightarrow+\infty$, we conclude that Ball's condition

$$\lim_{R\longrightarrow+\infty}\sup_{0<\varepsilon\leq\varepsilon_0}\lambda\{s\in K_\eta:|u_\varepsilon(s)|>R\}=0, \tag{3.7}$$

is fulfilled, and, according to the fundamental theorem of Young measures (cf. [25], Theorem 3.1, p. 31), for every $0<\eta\leq\overline{\eta}$ arbitrary subsequence of (u_ε) allows a further subsequence (which depends on η and which is not relabeled) which satisfies

$$\underline{\delta}_{u_\varepsilon}\xrightarrow{*}\theta_\eta\delta_{-1}+(1-\theta_\eta)\delta_1 \text{ in } \mathrm{L}^\infty_{w*}(K_\eta;\mathcal{P}(\mathbf{R})) \text{ as } \varepsilon\longrightarrow 0, \tag{3.8}$$

$$|u_\varepsilon|\xrightarrow{\lambda}1 \text{ on } K_\eta, \tag{3.9}$$

for some measurable functions $\theta_\eta:K_\eta\longrightarrow\mathbf{R}$ (which depend on the chosen subsequence), $\theta_\eta=\theta_\eta(s)$, with the property $0\leq\theta_\eta(s)\leq 1$ (a.e. $s\in K_\eta$). We recall that convergence in measure λ of measurable functions on $(0,1)$ is equivalent to convergence of measurable functions with respect to the metric

$$\rho(f,g):=\int_0^1\frac{|f(s)-g(s)|}{1+|f(s)-g(s)|}ds$$

(cf. [13], p. 60). Thus, by the uniqueness of the cluster point, it is enough to show that every subsequence of (u_ε) allows a further subsequence (u_{ε_η}) such that $\lim_{\eta\longrightarrow 0}\varepsilon_\eta=0$ and $|u_{\varepsilon_\eta}|\xrightarrow{\lambda}1$ on $(0,1)$ as $\eta\longrightarrow 0$. Indeed, given arbitrary subsequence of (u_ε), by (3.9) (since $\lim_{\eta\longrightarrow 0}\lambda((0,1)\backslash K_\eta)=0$) and by the usual diagonal argument, we conclude that there exists a subsequence (u_{ε_η}) such that $|u_{\varepsilon_\eta}|-1\xrightarrow{\lambda}0$ on $(0,1)$ as $\eta\longrightarrow 0$. In particular, for a given subsequence of (u_ε), we can select a further subsequence (not relabeled) such that (3.1) holds, which yields the assertion (ii).

Next, we address the proof of the assertion (iii). Since $YM((0,1);K)$ is compact, every subsequence of (u_ε) allows a further subsequence which converges to some limit $\boldsymbol{\nu}$ in $YM((0,1);K)$ as $\varepsilon\longrightarrow 0$. In the following, we show that all such cluster points $\boldsymbol{\nu}$ satisfy $\mathrm{supp}\nu_s\subseteq\{-1,1\}$ (a.e. $s\in(0,1)$). We consider $\theta_\eta=\theta_\eta(s)$ as in the assertion (i), and we define $\tilde{\theta}_\eta:(0,1)\longrightarrow[0,1]$ by $\tilde{\theta}_\eta(s):=\theta_\eta(s)$ if $s\in K_\eta$; $\tilde{\theta}_\eta(s):=0$ if $s\in(0,1)\backslash K_\eta$. Since the sequence $(\tilde{\theta}_\eta)$ is bounded on $(0,1)$, there exists a subsequence of $\tilde{\theta}_\eta$ (not relabeled) and a function $\theta_0\in\mathrm{L}^\infty(0,1)$ such that $\tilde{\theta}_\eta\xrightarrow{*}\theta_0$ in $\mathrm{L}^\infty(0,1)$ as $\eta\longrightarrow 0$, whereby $0\leq\theta_0(s)\leq 1$ (a.e. $s\in(0,1)$). If we define $\boldsymbol{\nu}^\eta,\boldsymbol{\nu}^0\in YM((0,1);K)$ by $\nu^\eta_s:=\theta_\eta(s)\delta_{-1}+(1-\theta_\eta(s))\delta_1$ (a.e. $s\in(0,1)$), $\nu^0_s:=\theta_0(s)\delta_{-1}+(1-\theta_0(s))\delta_1$

(a.e. $s \in (0,1)$), we get $\boldsymbol{\nu}^\eta \xrightarrow{\;*\;} \boldsymbol{\nu}^0$ in $\mathrm{L}^\infty_{w*}((0,1);\mathcal{P}(\mathbf{R}))$ and in $YM((0,1);K)$ as $\eta \longrightarrow 0$. Now it holds that

$$\text{(3.10)}\quad \Phi(\underline{\delta}_{u_\varepsilon}, \boldsymbol{\nu}^0) \le \Phi(\underline{\delta}_{u_\varepsilon}, \boldsymbol{\nu}^\eta) + \Phi(\boldsymbol{\nu}^\eta, \boldsymbol{\nu}^0) \le \Phi_{K_\eta}(\underline{\delta}_{u_\varepsilon}, \boldsymbol{\nu}^\eta) + O(1)\eta + \Phi(\boldsymbol{\nu}^\eta, \boldsymbol{\nu}^0),$$

where Φ (Φ_{K_η}, resp.) is the metric on $YM((0,1);K)$ ($YM(K_\eta;K)$, resp.). Finally, by the embedding $\mathrm{L}^\infty_{w*}(K_\eta;\mathcal{P}(\mathbf{R})) \hookrightarrow YM(K_\eta;K)$ and by an application of (3.8), we pass to the limit in (3.10), first as $\varepsilon \longrightarrow 0$, then as $\eta \longrightarrow 0$.

To prove the assertion (iv), we note that, by the assumption $1 < p < +\infty$ ($p = +\infty$, resp.), for every $0 < \eta \le \overline{\eta}$ there exists a subsequence of (u_ε) (which depends on η and which is not relabeled) such that $u_\varepsilon \rightharpoonup \tilde{u}_\eta$ in $\mathrm{L}^1(K_\eta)$ as $\varepsilon \longrightarrow 0$, where the measurable function $\tilde{u}_\eta : K_\eta \longrightarrow \mathbf{R}$ depends on the chosen subsequence, and where the sets K_η satisfy $K_\eta \nearrow (0,1)$ as $\eta \searrow 0$. Thus, it follows that for arbitrary measurable set $A \subseteq (0,1)$ we have

$$\int_{A\cap K_\eta} u_\varepsilon(s)ds \longrightarrow \int_{A\cap K_\eta} \tilde{u}_\eta(s)ds \text{ as } \varepsilon \longrightarrow 0.$$

On the other hand, by the fundamental theorem of Young measures, for every $h \in \mathrm{L}^\infty(K_\eta)$ and $g \in \mathrm{C}(\mathbf{R})$ such that $(g \circ u_\varepsilon)$ is weakly pre-compact in $\mathrm{L}^1(K_\eta)$ as $\varepsilon \longrightarrow 0$, we get

$$\lim_{\varepsilon \longrightarrow 0} \int_{K_\eta} h(s) g(u_\varepsilon(s))ds = \int_{K_\eta} h(s)\Big(\theta_\eta(s)g(-1) + (1-\theta_\eta(s))g(1)\Big)ds.$$

If we choose $h := \chi_{K_\eta \cap A}$ and $g(\zeta) := \zeta$, it results that

$$\lim_{\varepsilon \longrightarrow 0} \int_{A\cap K_\eta} u_\varepsilon(s)ds = \int_{A\cap K_\eta} (1 - 2\theta_\eta(s))ds.$$

Hence, $⨍_{A\cap K_\eta}(1 - 2\theta_\eta(s))ds = ⨍_{A\cap K_\eta} \tilde{u}_\eta(s)ds$ for every $0 < \eta \le \overline{\eta}$. By arbitrariness of the set A we obtain $\theta_\eta(s) = \frac{1-\tilde{u}_\eta(s)}{2}$ (a.e. $s \in K_\eta$), which yields

$$\underline{\delta}_{u_\varepsilon} \xrightarrow{\;*\;} \frac{1-\tilde{u}_\eta}{2}\delta_{-1} + \frac{1+\tilde{u}_\eta}{2}\delta_1 \text{ in } \mathrm{L}^\infty_{w*}(K_\eta;\mathcal{P}(\mathbf{R})) \text{ as } \varepsilon \longrightarrow 0.$$

Since the sequence (K_η) increases to $(0,1)$ as $\eta \searrow 0$, it follows that $\theta_\eta(s) = \theta_0(s)$ (a.e. $s \in K_\eta$). In effect, by the diagonal argument there exists a further subsequence (which is independent of η and which is not relabeled) which satisfies (3.3) ((3.4), resp.) for every $0 < \eta \le \overline{\eta}$. In particular, in the case of $\tilde{u}_\eta(s) = 0$ (a.e. $s \in K_\eta$), by passing to the limit as $\eta \longrightarrow 0$, we obtain $⨍_A \theta_0(s)ds = \frac{1}{2}$ for arbitrary measurable set $A \subseteq (0,1)$, getting $\theta_0(s) = \frac{1}{2}$ (a.e. $s \in (0,1)$), which gives (3.5). The remaining assertions follow similarly as above. □

Proposition 3.3. *Consider an arbitrary two-well potential W.*

(i) *If* (u_ε) *is an N FE sequence such that* $\lim_{\varepsilon\longrightarrow 0} M_\varepsilon = +\infty$*, then there exists* $\delta > 0$ *such that the mapping* $\xi \mapsto \liminf_{\varepsilon\longrightarrow 0} \mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\}$ *belongs to* $\mathrm{L}^1((1+\delta,+\infty); d\mu)$*, where* $d\mu := \sqrt{V} d\lambda$*.*

(ii) *If* $u \in \mathrm{H}^1(0,1)$ *satisfies* $M \geq \int_0^1 \sqrt{V(|u(s)|)}|u'(s)|ds$*, then for arbitrary* $0 \leq a < b < +\infty$ *there exist at most finitely many pairwise disjoint non-empty open intervals* (J_i) *in* $(0,1)$ *such that* $\inf_{J_i}|u| \leq a$ *and* $\sup_{J_i}|u| \geq b$*.*

(iii) *If* (u_ε) *is an FE sequence, then* (u_ε) *is an SO FE sequence.*

Proof. Since $(|u_\varepsilon|)$ is a sequence of absolutely continuous functions, by the area formula (cf. [22], Theorem 3.65, p. 100, and Corollary 3.41, p. 92) we have

$$M \geq \varepsilon^{-1} I_0^\varepsilon(u_\varepsilon) \geq \int_0^1 \sqrt{V(|u_\varepsilon|)}||u_\varepsilon|'| = \int_{m_\varepsilon}^{M_\varepsilon} \sqrt{V(\xi)}\mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\}d\xi\,.$$

By the normality of (u_ε), there exist $\delta > 0$ and $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$ it holds that $m_\varepsilon \leq 1+\delta$. As we pass to the limit as $\varepsilon \longrightarrow 0$, Fatou's Lemma yields

$$M \geq \int_{1+\delta}^{+\infty} \sqrt{V(\xi)} \liminf_{\varepsilon\longrightarrow 0} \mathrm{card}\{|u_\varepsilon|^{\leftarrow}(\xi)\}d\xi,$$

which gives the assertion (i).

To prove (ii), we assume the opposite. Then, there exist infinitely many pairwise disjoint non-empty open intervals $(J_i)_{i=1}^{+\infty}$ such that $\inf_{J_i}|u| \leq a$ and $\sup_{J_i}|u| \geq b$. By the area formula, it results that

$$M \geq \sum_{i=1}^{+\infty} \int_{J_i} \sqrt{V(|u(s)|)}|u'(s)|ds \geq \sum_{i=1}^{+\infty} \int_a^b \sqrt{V(\xi)}d\xi,$$

which yields a contradiction.

Similarly, to prove (iii), we estimate

$$\begin{aligned} M &\geq \sum_{i=1}^{N_\varepsilon} \int_{J_i^\varepsilon} \Big(\varepsilon u_\varepsilon'^2 + \varepsilon^{-1} V(|u_\varepsilon|)\Big) \geq \sum_{i=1}^{N_\varepsilon} \int_{J_i^\varepsilon} \sqrt{V(|u_\varepsilon|)}|u_\varepsilon'| \\ &\geq \sum_{i=1}^{N_\varepsilon} \int_a^b \sqrt{V(\xi)}d\xi = N_\varepsilon(a,b) \int_a^b \sqrt{V(\xi)}d\xi. \end{aligned}$$

As we apply $\sup_{0<\varepsilon\leq\varepsilon_0}$ in the last inequality, the assertion (iii) follows by arbitrariness of a and b. □

Corollary 3.4 (Non-existence of periodic FE sequences with vanishing period as $\varepsilon \longrightarrow 0$). *If* W *is an arbitrary two-well potential, then there exists no* T_ε*-periodic FE sequence for* $(\varepsilon^{-1} J_0^\varepsilon)$ *such that* $\lim_{\varepsilon\longrightarrow 0} T_\varepsilon = 0$*.*

PROOF. We consider an arbitrary T_ε-periodic FE sequence (v_ε) such that $v_\varepsilon \in \mathrm{H}^2_{loc}(\mathbf{R})$. Since (v_ε) is T_ε-periodic, by Rolle's theorem there exists $\theta_\varepsilon \in (0, T_\varepsilon)$ such that $v'_\varepsilon(\theta_\varepsilon) = 0$. By n_ε we denote the largest integer below $\frac{1}{T_\varepsilon}$, whereby $\lim_{\varepsilon \longrightarrow 0} n_\varepsilon = +\infty$. As in the proof of Proposition 3.3, by the periodicity and by the area formula, it follows that

$$M \geq \varepsilon^{-1} J_0^\varepsilon(v_\varepsilon) \geq \int_0^1 \sqrt{V(|v'_\varepsilon|)}||v'_\varepsilon|'| \geq \int_0^{n_\varepsilon T_\varepsilon} \sqrt{V(|v'_\varepsilon|)}||v'_\varepsilon|'|$$
$$= n_\varepsilon \int_0^{T_\varepsilon} \sqrt{V(|v'_\varepsilon|)}||v'_\varepsilon|'| \geq n_\varepsilon \int_{\min_{[0,1]}|v'_\varepsilon|}^{\max_{[0,1]}|v'_\varepsilon|} \sqrt{V(\xi)}d\xi.$$

If the sequence (v'_ε) is bounded in $\mathrm{L}^\infty(0,1)$, by Theorem 3.2, (i), we have $|v'_\varepsilon| \xrightarrow{\lambda} 1$ on $(0,1)$ as $\varepsilon \longrightarrow 0$. Consequently, there exist $0 < \delta < \frac{1}{4}$ and $\varepsilon_0(\delta) > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0(\delta)$ we have $\min_{[0,1]} |v'_\varepsilon| \leq \delta$ and $\max_{[0,1]} |v'_\varepsilon| \geq 1 - \delta$. If, on the other hand, the sequence (v'_ε) is not bounded in $\mathrm{L}^\infty(0,1)$, we obtain the same estimates. Hence, we get $M \geq n_\varepsilon \int_\delta^{1-\delta} \sqrt{V(\xi)}d\xi$, which, as we pass to the limit as $\varepsilon \longrightarrow 0$, yields the contradiction. □

4. The non-integrable case

In the next theorem we show that, for a fairly large class of non-integrable two-well potentials, it is not necessary to assume the normality of FE sequences in order to obtain underlying geometric properties of FE sequences as $\varepsilon \longrightarrow 0$. On the other hand, the following result can be viewed, in part, as L^p-variant of Theorem 1.3 in [20].

THEOREM 4.1. *Assume that W is a two-well potential such that for some $C_0 > 0$, $R_0 > 1$, $-2 < r < 0$ and for every $\xi \geq R_0$ it holds that $V(\xi) \geq C_0 \xi^r$, and that (u_ε) is an arbitrary FE sequence. If $(\varepsilon^{\frac{1}{2}} u_\varepsilon)$ is a normal sequence, then the following conclusions hold.*

- *(i) $(\int_0^1 |u_\varepsilon|^{p_r})$ is bounded, where $p_r := r + 2 > 0$.*
- *(ii) If $-1 < r < 0$, then every subsequence of (u_ε) allows a subsequence (not relabeled) which satisfies $u_\varepsilon \rightharpoonup u$ in $\mathrm{L}^{p_r}(0,1)$ as $\varepsilon \longrightarrow 0$, where $u \in \mathrm{L}^{p_r}(0,1)$ depends on the chosen subsequence.*
- *(iii) every subsequence of (u_ε) allows a subsequence (not relabeled) such that*

$$\underline{\delta}_{u_\varepsilon} \xrightarrow{*} \theta_0 \delta_{-1} + (1-\theta_0)\delta_1 \text{ in } \mathrm{L}^\infty_{w*}((0,1); \mathcal{P}(\mathbf{R})) \text{ as } \varepsilon \longrightarrow 0, \tag{4.1}$$

for some measurable function $\theta_0 = \theta_0(s)$, $s \in (0,1)$, which depends on the chosen subsequence and which satisfies $0 \leq \theta_0(s) \leq 1$ (a.e. $s \in (0,1)$).

PROOF. We note that WLG we can assume that (u_ε) is a FE sequence which satisfies $\lim_{\varepsilon \longrightarrow 0} M_\varepsilon = +\infty$ (otherwise the assertions (i)-(iii) are obvious). Hence, there exists $R \geq R_0 > 1$ such that there exists $0 < \varepsilon_0 < 1$ such that for every $0 < \varepsilon \leq \varepsilon_0$ it holds that $\lambda\{|u_\varepsilon| > R\} > 0$. Since it holds that $V(|u_\varepsilon(s)|) > 0$ (a.e. $s \in \{|u_\varepsilon| > R\}$), we estimate

$$M \geq \varepsilon^{-1} I_0^\varepsilon(u_\varepsilon) \geq \varepsilon^{-1} \int_0^1 V(|u_\varepsilon|) \geq \varepsilon^{-1} C_0 \int_{\{|u_\varepsilon|>R\}} |u_\varepsilon|^r.$$

Hence, by Lemma 3.1, (i), it results $\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-\frac{r}{2}-1} \lambda\{|u_\varepsilon| > R\} < +\infty$. By Lemma 3.1, (ii), with p replaced by $p_r := r + 2$, we get the assertion (i). In particular, the assumption $-1 < r < 0$ gives $p_r > 1$, getting the assertion (ii). The convergence in (4.1) follows quite similarly as in the proof of the assertion (i) in Theorem 3.2, with K_η replaced by $(0, 1)$ in (3.6) and (3.7).

□

From Lemma 3.1 we infer that, in the case of arbitrary two-well potential W, not all FE sequences are necessarily LPO FE sequences. However, the next proposition shows that, if we add a non-integrability assumption on W, every UN FE sequence is an LPO FE sequence.

Proposition 4.2 (Sufficient condition for existence of LPO FE sequences). *If W is a two-well potential such that for some $1 \leq p < +\infty$ W satisfies (1.5), then every UN FE sequence (u_ε) is an LPO FE sequence.*

Proof. In the case of $p = 1$, the assertion follows from Proposition 2.3. In the case of $1 < p < +\infty$, we argue as follows. We consider an arbitrary UN FE sequence (u_ε) and an arbitrary subsequence of (u_ε) (not relabeled). If (u_ε) is an FE sequence which is bounded in $\mathrm{L}^\infty(K_\eta)$, for some choice of measurable sets (K_η) such that $K_\eta \nearrow (0, 1)$ as $\eta \searrow 0$, the proof is finished. Therefore, we assume the opposite, i.e., that for every sequence of measurable sets (K_η) which increases to $(0, 1)$ as $\eta \searrow 0$ there exists an increasing subsequence of $\Big(M_\varepsilon(K_\eta)\Big)$ (not relabeled) such that $\lim_{\varepsilon \longrightarrow 0} M_\varepsilon(K_\eta) = +\infty$. Since it holds that $V \circ |u_\varepsilon| \longrightarrow 0$ strongly in $\mathrm{L}^1(0, 1)$ as $\varepsilon \longrightarrow 0$, there exists a subsequence (not relabeled) such that $V(|u_\varepsilon(s)|) \longrightarrow 0$ (a.e. $s \in (0, 1)$) as $\varepsilon \longrightarrow 0$. By Egoroff's Theorem, for every $0 < \eta < 1$ there exists a measurable set $\tilde{K}_\eta \subseteq (0, 1)$ such that $V \circ |u_\varepsilon| \longrightarrow 0$ uniformly on $\tilde{K}_\eta$ and such that $\lambda((0, 1) \backslash \tilde{K}_\eta) \leq \eta$. By the Borel regularity of the Lebesgue measure WLG we can assume that each $\tilde{K}_\eta$ is a compact set. Furthermore, WLG we assume that $(\tilde{K}_\eta)$ increases to $(0, 1)$ as $\eta \searrow 0$. Hence, there exists $0 < \tilde{\eta}_0 < 1$ such that for every $0 < \eta \leq \tilde{\eta}_0 < 1$ there exists an increasing subsequence of $\Big(M_\varepsilon(\tilde{K}_\eta)\Big)$ (not relabeled) such that $\lim_{\varepsilon \longrightarrow 0} M_\varepsilon(\tilde{K}_\eta) = +\infty$. On the other hand, if $0 < \delta < 1$ is given, then for every $0 < \eta \leq \tilde{\eta}_0$ there exists $\varepsilon_1 = \varepsilon_1(\delta, \eta) > 0$ such that for every $0 < \varepsilon \leq \varepsilon_1(\delta, \eta)$ it holds that $V(|u_\varepsilon(s)|) \leq \delta < 1$ (a.e. $s \in \tilde{K}_\eta$). In effect, $p > 1$ implies $0 \leq V^p(|u_\varepsilon(s)|) \leq V(|u_\varepsilon(s)|) \leq \delta < 1$ (a.e.

$s \in \tilde{K}_\eta$). For $0 < \varepsilon < \varepsilon_1(\delta, \eta)$ we obtain

$$M \geq \int_{\tilde{K}_\eta} \Big(\varepsilon |u_\varepsilon'(s)|^2 + \varepsilon^{-1} W(u_\varepsilon(s))\Big) ds \geq \int_{\tilde{K}_\eta} \sqrt{V^p(|u_\varepsilon(s)|)} ||u_\varepsilon|'(s)| ds,$$

and so

$$M \geq \int_{m_\varepsilon(\tilde{K}_\eta)}^{M_\varepsilon(\tilde{K}_\eta)} \sqrt{V^p(\xi)} \mathrm{card}\{|u_{\varepsilon,\eta}|^{\leftarrow}(\xi)\} d\xi, \tag{4.2}$$

where $m_\varepsilon(\tilde{K}_\eta) = \min\{|u_\varepsilon(s)| : s \in \tilde{K}_\eta\}$, $M_\varepsilon(\tilde{K}_\eta) = \max\{|u_\varepsilon(s)| : s \in \tilde{K}_\eta\}$. Since the sequence $(\tilde{K}_\eta)$ increases to $(0,1)$ as $\eta \searrow 0$, there exists $0 < \tilde{\eta}_1 < 1$ such that for every $0 < \eta \leq \tilde{\eta}_1$ we have $\lambda(\tilde{K}_\eta \cap G) > 0$. Then, by the uniform normality of (u_ε), for every $0 < \eta \leq \tilde{\eta}_1$ there exists $0 < \varepsilon_0(\eta) \leq \min\{\varepsilon_0, \varepsilon_1(\delta, \eta))\}$ (where $\varepsilon_0 > 0$ is chosen as in Definition 2.1, (ii)) such that

$$\rho_{\varepsilon_0}(\eta) := \mathrm{sup}_{0<\varepsilon\leq\varepsilon_0(\eta)} m_\varepsilon(\tilde{K}_\eta) \leq \mathrm{sup}_{0<\varepsilon\leq\varepsilon_0(\eta)} m_\varepsilon(\tilde{K}_\eta \cap G) < +\infty.$$

By passing to the limit in (4.2) as $\varepsilon \longrightarrow 0$, for every $0 < \eta < \min\{\tilde{\eta}_0, \tilde{\eta}_1\}$ we get

$$M \geq \int_{\rho_{\varepsilon_0}(\eta)}^{+\infty} \sqrt{V^p(\xi)} \liminf_{\varepsilon \longrightarrow 0} \mathrm{card}\{|u_{\varepsilon,\eta}|^{\leftarrow}(\xi)\} d\xi, \tag{4.3}$$

where we used Fatou's Lemma. Finally, from (1.5) and (4.3), arguing by contradiction, we conclude that (u_ε) is an LPO FE sequence. □

In the next theorem, we sum up the main results of this section.

THEOREM 4.3. *Assume that* W *is a two-well potential such that for some* $C_0 > 0$, $R_0 > 1$, $-2 < r < 0$ *and for every* $\xi \geq R_0$ *it holds that* $V(\xi) \geq C_0 \xi^r$, *and that an FE sequence* (u_ε) *satisfies that* $(\varepsilon^{\frac{1}{2}} u_\varepsilon)$ *is a normal sequence. Then* (u_ε) *is a normal FE sequence. Therefore, by Theorem 1.3 in [20],* (u_ε) *is bounded in* $\mathrm{L}^\infty(0,1)$ *and strongly pre-compact in* $\mathrm{L}^1(0,1)$ *as* $\varepsilon \longrightarrow 0$.

5. The integrable case

We recall that the compactness result in the non-integrable case relies on boundedness of (u_ε) in $\mathrm{L}^\infty(0,1)$ (cf. [20]). In the next proposition, we adapt this argument to the case of an integrable function $\sqrt{V}$. We describe underlying geometric properties of uniformly normal FE sequences under the integrability assumption (1.4), coupled with the assumption $0 < \limsup_{\xi \longrightarrow +\infty} V(\xi) \leq +\infty$. The latter assumption is crucial. The expression "locality" in the next proposition refers to the fact that local property of uniform normality of an FE sequence yields global property of being a lower pre-oscillatory FE sequence. Also, the result below shows that oscillations of V trigger oscillations of (u_ε), where (u_ε) is a normal sequence which is not bounded in $\mathrm{L}^\infty(0,1)$ as $\varepsilon \longrightarrow 0$. The proof is carried out by reduction to the non-integrable case.

Proposition 5.1 (Locality: the integrable case). *If W satisfies (1.4) and $0 < L \leq +\infty$, where $L := \limsup_{\xi \longrightarrow +\infty} V(\xi)$, then every UN FE sequence (u_ε) for $(\varepsilon^{-1} I_0^\varepsilon)$ is an LPO FE sequence.*

Proof. The proof is very similar to the proof of Proposition 4.2. We consider an arbitrary UN FE sequence (u_ε) and an arbitrary subsequence of (u_ε) (not relabeled). If (u_ε) is not bounded in $\mathrm{L}^\infty(K_\eta)$ for an arbitrary choice of measurable sets (K_η) such that $K_\eta \nearrow (0,1)$ as $\eta \searrow 0$, by Egoroff's Theorem, we get $V \circ |u_\varepsilon| \longrightarrow 0$ uniformly on the compact set $\tilde{K}_\eta$, where WLG $(\tilde{K}_\eta)$ increases to $(0,1)$ as $\eta \searrow 0$, and there exists $0 < \tilde{\eta}_0 < 1$ such that for every $0 < \eta \leq \tilde{\eta}_0 < 1$ it holds that $\lim_{\varepsilon \longrightarrow 0} M_\varepsilon(\tilde{K}_\eta) = +\infty$. On the other hand, if $\delta > 0$ is given, then for every $0 < \eta \leq \tilde{\eta}_0$ there exists $\varepsilon_1 = \varepsilon_1(\delta, \eta) > 0$ such that for every $0 < \varepsilon \leq \varepsilon_1(\delta, \eta)$ it holds that $\max\{V(|u_\varepsilon(s)|) : s \in \tilde{K}_\eta\} \leq \delta$. Consider $\varphi_\delta \in \mathrm{C}(0, +\infty)$ such that $\int_0^{+\infty} \sqrt{\varphi_\delta(V(\xi))} d\xi = +\infty$ and $\varphi_\delta(z) = z$ for every $0 \leq z \leq \delta$. Indeed, to prove that such a function φ_δ exists, we argue as follows. If it holds $0 < L < +\infty$ ($L = +\infty$, resp.), there exists a strictly increasing sequence (c_k) of strictly positive numbers such that $\lim_{k \longrightarrow +\infty} c_k = +\infty$ and $\lim_{k \longrightarrow +\infty} V(c_k) = L$ ($\lim_{k \longrightarrow +\infty} V(c_k) = +\infty$, resp.). This means that for chosen $\delta \in (0, \frac{1}{5}L)$ ($\delta > 0$, resp.) there exists $k_0(\delta) \in \mathbf{N}$ such that for every $k \geq k_0(\delta)$ we have $V(c_k) > L - \delta$ ($V(c_k) > 3\delta$, resp.). On the other hand, by continuity of V, there exists a sequence $(J_k)_{k \geq k_0(\delta)}$ of pairwise disjoint compact intervals $J_k := [a_k, b_k]$ such that for every $k \geq k_0(\delta)$ we have $a_k < c_k < b_k$ and $\min\{V(\xi) : \xi \in [a_k, b_k]\} \geq \frac{L-\delta}{2} \geq 2\delta$ ($\min\{V(\xi) : \xi \in [a_k, b_k]\} \geq 2\delta$, resp.). We define $\varphi_\delta : [0, +\infty) \longrightarrow [0, +\infty)$ by $\varphi_\delta(z) := z$, if $0 \leq z \leq \delta$; $\varphi(z) := \max\{V(\xi) : \xi \in [a_k, b_k]\} |b_k - a_k|^{-1}$, if $z \in [a_k, b_k]$ (for every $k \geq k_0(\delta)$); by continuity, otherwise. Since we have

$$\sum_{k=k_0(\delta)}^{+\infty} \int_{a_k}^{b_k} \sqrt{\varphi_\delta(V(\xi))} d\xi \geq \sum_{k=k_0(\delta)}^{+\infty} \sqrt{V(c_k)} \geq \sum_{k=k_0(\delta)}^{+\infty} \sqrt{\delta} = +\infty,$$

it results that

$$\int_0^{+\infty} \sqrt{\varphi_\delta(V(\xi))} d\xi = +\infty. \tag{5.1}$$

By the same argument as in the proof of Proposition 4.2, for sufficiently small $\eta > 0$ and $\varepsilon_0 > 0$ we get

$$M \geq \int_{\rho_{\varepsilon_0}(\eta)}^{+\infty} \sqrt{\varphi_\delta(V(\xi))} \liminf_{\varepsilon \longrightarrow 0} \mathrm{card}\{|u_{\varepsilon,\eta}|^{\leftarrow}(\xi)\} d\xi, \tag{5.2}$$

where $0 \leq \rho_{\varepsilon_0}(\eta) < +\infty$. Thus, the assertion follows from (5.1) and (5.2). □

A version of the next theorem can be found in [21]. Note that for general sequences (u_ε) which are a.e. point-wise bounded on $(0,1)$ there is no compactness result which ensures almost everywhere convergence or convergence in measure λ on $(0,1)$. The following result shows, however, that under constraint that an a.e. point-wise bounded sequence is in fact an FE sequence, we get such a convergence (up to a subsequence).

THEOREM 5.2 (Compactness: the integrable case). *Suppose that W satisfies (1.4). Then arbitrary subsequence of an a.e. point-wise bounded FE sequence (u_ε) allows a further subsequence (not relabeled) which satisfies: there exists a function $u_0 \in \mathrm{BV}(0,1)$ (which depends on the chosen subsequence) such that*

$$u_\varepsilon(s) \longrightarrow u_0(s) \text{ (a.e. } s \in (0,1)) \text{ as } \varepsilon \longrightarrow 0, \tag{5.3}$$

where $|u_0(s)| = 1$ (a.e. $s \in (0,1)$). Moreover, such a subsequence (u_ε) is a sequence which is bounded in $\mathrm{L}^\infty(K_\eta)$, for some sequence of measurable sets (K_η) which increases to $(0,1)$ as $\eta \searrow 0$. Consequently, for every $1 \le p < +\infty$ and every $0 < \eta \le \overline{\eta}$ it holds that

$$u_\varepsilon \longrightarrow u_0 \text{ in } \mathrm{L}^p(K_\eta) \text{ as } \varepsilon \longrightarrow 0. \tag{5.4}$$

In particular, (u_ε) is an a.e. point-wise bounded FE sequence iff (u_ε) is an FE sequence which is is bounded in $\mathrm{L}^\infty(K_\eta)$, where a sequence of measurable sets (K_η) increases to $(0,1)$ as $\eta \searrow 0$.

PROOF. We repeat the classical argument in [21]. Indeed, we define $F : [0,+\infty) \longrightarrow (0,+\infty)$ by $F(t) := \int_t^{+\infty} \sqrt{V(\xi)} d\xi$, where $t \in [0,+\infty)$. Thus $F \in \mathrm{C}^1(0,+\infty)$, $F \in \mathrm{W}^{1,1}_{loc}(0,+\infty)$, $F(t) \le \|\sqrt{V}\|_{\mathrm{L}^1(0,+\infty)}$, $t \in (0,+\infty)$, $F(0) = \|F\|_{\mathrm{L}^\infty(0,+\infty)}$, $F'(t) = -\sqrt{V(t)}$ for every $t \in [0,+\infty)$. Hence, F is strictly decreasing and continuous bijection from $[0,+\infty)$ onto $(0,F(0)]$. We also define $w_\varepsilon : (0,1) \longrightarrow \mathbf{R}$ by $w_\varepsilon(s) := F(|u_\varepsilon(s)|)$. Then, we obtain

$$\begin{aligned}
&w'_\varepsilon(s) = \sqrt{V(|u_\varepsilon(s)|)}|u_\varepsilon|'(s) \text{ (a.e. } s \in (0,1)),\\
&\sup_{0<\varepsilon\le\varepsilon_0} \|w_\varepsilon\|_{\mathrm{L}^\infty(0,1)} \le \|F\|_{\mathrm{L}^\infty(0,+\infty)} < +\infty,\\
&\sup_{0<\varepsilon\le\varepsilon_0} \|w'_\varepsilon\|_{\mathrm{L}^1(0,1)} \le M.
\end{aligned}$$

By the Rellich-Kondrachov compactness theorem (cf. [12], section 5.2.3, Theorem 4, p. 176), there exists a subsequence of (w_ε) (not relabeled) and a function $w_0 \in \mathrm{BV}(0,1)$ such that $w_\varepsilon \longrightarrow w_0$ in $\mathrm{L}^1(0,1)$ as $\varepsilon \longrightarrow 0$. We extract a further subsequence such that $w_\varepsilon(s) \longrightarrow w_0(s)$ and $V(|u_\varepsilon(s)|) \longrightarrow 0$ (a.e. $s \in (0,1)$) as $\varepsilon \longrightarrow 0$. Notice that the assumptions on (u_ε) imply that $w_0(s) \ne 0$ (a.e. $s \in (0,1)$). Indeed, by passing to the limit as $\varepsilon \longrightarrow 0$ in $w_\varepsilon(s) = \int_{|u_\varepsilon(s)|}^{+\infty} \sqrt{V(\xi)} d\xi$, we get $w_0(s) = \lim_{\varepsilon\longrightarrow 0} \int_{|u_\varepsilon(s)|}^{+\infty} \sqrt{V(\xi)} d\xi$ (a.e. $s \in (0,1)$). In turn, the assumption $w_0(s) = 0$ (a.e. $s \in (0,1)$) gives $\limsup_{\varepsilon\longrightarrow 0} |u_\varepsilon(s)| = +\infty$ (a.e. $s \in (0,1)$), which is not possible since (u_ε) is an a.e. point-wise bounded sequence. By the construction, it follows that $F^{-1} : (0, \|F\|_{\mathrm{L}^\infty(0,+\infty)}] \longrightarrow [0,+\infty)$ is strictly decreasing and continuous bijection (cf. [16], Proposition 6.4.5, p. 163). It results that $|u_\varepsilon(s)| = F^{-1}(w_\varepsilon(s))$, $u_\varepsilon(s) \longrightarrow u_0(s)$ and $V(|u_\varepsilon(s)|) \longrightarrow V(|u_0(s)|)$ (a.e. $s \in (0,1)$) as $\varepsilon \longrightarrow 0$, where $|u_0(s)| := F^{-1}(w_0(s))$ (a.e. $s \in (0,1)$). Since $w_0(s) \neq 0$ (a.e. $s \in (0,1)$), u_0 is a well-defined almost everywhere on $(0,1)$. Hence, $u_0(s) \in \{-1,1\}$ and $w_0(s) \in \{F(-1), F(1)\}$ (a.e. $s \in (0,1)$). Therefore for some measurable subset $E \subseteq (0,1)$ we can write $w_0(s) = F(-1)\chi_E(s) + F(1)\chi_{(0,1)\setminus E}(s)$, where, by $w_0 \in \mathrm{BV}(0,1)$, it follows that E is a set of finite perimeter in $(0,1)$. We conclude that we have $u_0(s) = (-1)\cdot\chi_E(s) + 1\cdot\chi_{(0,1)\setminus E}(s)$, whereby $u_0 \in \mathrm{BV}(0,1)$. Next, by Egoroff's Theorem it follows that such a sequence (u_ε) satisfies: for every $0 < \eta < 1$ there exists a measurable set $K_\eta \subseteq (0,1)$ such that $\lambda((0,1)\setminus K_\eta) \leq \eta$ and such that $u_\varepsilon \longrightarrow u_0$ uniformly on K_η. In effect, we get

$$\|u_\varepsilon\|_{\mathrm{L}^\infty(K_\eta)} \leq \|u_\varepsilon - u_0\|_{\mathrm{L}^\infty(K_\eta)} + \|u_0\|_{\mathrm{L}^\infty(K_\eta)}. \tag{5.5}$$

Estimate (5.5) shows that (u_ε) is bounded in $\mathrm{L}^\infty(K_\eta)$, while by (5.3) and by the dominated convergence theorem we get (5.4). Finally, we note that we just proved that every subsequence of an a.e. point-wise bounded FE sequence allows a further subsequence which is bounded in $\mathrm{L}^\infty(K_\eta)$. Arguing by contradiction, it is easy to see that this suffices to conclude that the initial a.e. point-wise bounded FE sequence shares the same property. The converse of the last assertion of the theorem is obvious. ◻

In the following theorem, we sum up the main results of this section.

THEOREM 5.3. *If a two-well potential W satisfies (1.4) and $0 < L \leq +\infty$, where $L := \limsup_{\xi \longrightarrow +\infty} V(\xi)$, then every UN FE sequence (u_ε) satisfies:*

(i) (u_ε) is an SO FE sequence,
(ii) (u_ε) is an LPO FE sequence,
(iii) if (u_ε) is an a.e. point-wise bounded FE sequence, then (u_ε) is precompact in measure λ on $(0,1)$ as $\varepsilon \longrightarrow 0$,
(iv) if $(\int_0^1 |u_\varepsilon|^p)$ is bounded for some $0 < p < +\infty$, then every subsequence of (u_ε) allows a further subsequence (not relabeled) such that we have $\lim_{\varepsilon \longrightarrow 0} \int_0^1 |u_\varepsilon - u_0|^q = 0$ for every $0 < q < p$, where $u_0 \in \mathrm{BV}(0,1)$ depends on the chosen subsequence.

PROOF. According to Proposition 3.3, Proposition 5.1 and Theorem 5.2, it remains to prove the assertion (iv). To this end, we assume that (u_ε) satisfies $\int_0^1 |u_\varepsilon|^p \leq C_0$, where $C_0 > 0$. By Theorem 3.2, (ii), it follows that (u_ε) is an a.e. point-wise bounded sequence. By Theorem 5.2, for every subsequence of (u_ε) there exists a subsequence of (u_ε) (not relabeled) and $u_0 \in \mathrm{BV}(0,1)$, $|u_0(s)| = 1$ (a.e. $s \in (0,1)$), such that $u_\varepsilon \longrightarrow u_0$ (a.e. $s \in (0,1)$) as $\varepsilon \longrightarrow 0$. Furthermore, by the Hölder inequality for every $0 < q < p$ we have (cf. [13], Proposition 6.12, p. 178)

$$\int_{B_\eta} |u_\varepsilon - u_0|^q \le \Big(\int_{B_\eta} |u_\varepsilon - u_0|^p \Big)^{\frac{q}{p}} \lambda(B_\eta)^{1-\frac{q}{p}} \le C_1 \lambda(B_\eta)^{1-\frac{q}{p}},$$

where $B_\eta := (0,1)\backslash K_\eta$ and $C_1 > 0$. Since $\int_0^1 |u_\varepsilon - u_0|^q$ can be written as $\int_{K_\eta} |u_\varepsilon - u_0|^q + \int_{B_\eta} |u_\varepsilon - u_0|^q$, we pass to the limit, first as $\varepsilon \longrightarrow 0$, then as $\eta \longrightarrow 0$, and, by (5.4) and (5.6), we recover the assertion (iv). □

Acknowledgements.
The author is grateful to the anonymous referee for his useful suggestions.

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Received: 1.11.2021.
Revised: 7.11.2022.

NEKI REZULTATI U ASIMPTOTIČKOJ ANALIZI NIZOVA KONAČNE ENERGIJE CAHN–HILLIARDOVOG FUNKCIONALA S NESTANDARDNIM POTENCIJALNOM S DVIJE JAME

Andrija Raguž

Sažetak. U ovom radu proširujemo razmatranja G. Leonija koja se odnose na nizove konačne energije jednodimenzionalnog Cahn-Hilliardovog funkcionala

$$I_0^\varepsilon(u) = \int_0^1 \Big(\varepsilon^2 u'^2(s) + W(u(s))\Big) ds,$$

gdje je $u \in \mathrm{H}^1(0,1)$ i gdje je W potencijal s dvije simetrično postavljene jame s nestandardnim uvjetom integrabilnosti. U radu uvodimo nekoliko novih klasa nizova s konačnom energijom, donosimo njihova temeljna geometrijska svojstva kad $\varepsilon \longrightarrow 0$ i dokazujemo rezultat kompaktnosti.