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CONCEPTIONS ON TOPOLOGICAL TRANSITIVITY IN PRODUCTS AND SYMMETRIC PRODUCTS II

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Abstract. We continue the work initiated by A. Rojas, F. Barragán and S. Macías in Conceptions on topological transitivity in products and symmetric products. Thurk. J. Math. 44 (2020), 491–523. We consider classes of functions not included in the mentioned paper, namely: exact in the sense of Akin-Auslander-Nagar, fully exact, strongly transitive in the sense of Akin-Auslander-Nagar, very strongly transitive, exact transitive, strongly exact transitive and strongly product transitive.

1. Introduction

A *dynamical system* is a pair (X, f), where X is a nondegenerate topological space and $f : X \to X$ is a function. Given a topological space X and a positive integer n, we consider the *n-fold symmetric product* of X, $\mathcal{F}_n(X)$, consisting of all nonempty subsets of X with at most n points [8]. A function $f : X \to X$ induces a function on $\mathcal{F}_n(X)$ denoted by $\mathcal{F}_n(f) : \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ and defined by $\mathcal{F}_n(f)(A) = f(A)$, for each $A \in \mathcal{F}_n(X)$. In this way, the discrete dynamical system (X, f) induces the discrete dynamical system $(\mathcal{F}_n(X), \mathcal{F}_n(f))$.

Let $X_1, \ldots, X_m$ be topological spaces, with $m \geq 2$, and for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. We define the function $\prod_{i=1}^m f_i : \prod_{i=1}^m X_i \to \prod_{i=1}^m X_i$ by $\prod_{i=1}^m f_i((x_1, \ldots, x_m)) = (f_1(x_1), \ldots, f_m(x_m))$ for each $(x_1, \ldots, x_m) \in \prod_{i=1}^m X_i$. This function is called *product function*. In this way, we can analyze the relationships between the dynamical properties of the systems (1) $(\mathcal{F}_n(\prod_{i=1}^m X_i), \mathcal{F}_n(\prod_{i=1}^m f_i))$; (2) $(\mathcal{F}_n(X_i), \mathcal{F}_n(f_i))$,

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for each $i \in \{1, \ldots, m\}$; (3) $(\prod_{i=1}^{m} X_i, \prod_{i=1}^{m} f_i)$ and (4) (X_i, f_i), for each $i \in \{1, \ldots, m\}$. In [18], Rojas et al. analyzed the relationships between the dynamical systems (1), (2), (3) and (4) when any of them is: exact, mixing, transitive, weakly mixing, totally transitive, strongly transitive, chaotic, minimal, orbit-transitive, strictly orbit-transitive, ω-transitive, TT_{++}, mild mixing, exactly Devaney chaotic, backward minimal, totally minimal, scattering, Touhey or an F-system. However, there exist more classes of functions that we can study in the same way, namely: exact in the sense of Akin-Auslander-Nagar, fully exact, strongly transitive in the sense of Akin-Auslander-Nagar, very strongly transitive, exact transitive, strongly exact transitive and strongly product transitive [2].

Let $\mathcal{M}$ be one of the following classes of functions: exact in the sense of Akin-Auslander-Nagar, fully exact, strongly transitive in the sense of Akin-Auslander-Nagar, very strongly transitive, exact transitive, strongly exact transitive and strongly product transitive. Continuing with the work in [18] we will study the relationships between the following four statements.

1. For each $i \in \{1, \ldots, m\}$, $f_i \in \mathcal{M}$.
2. $\prod_{i=1}^{m} f_i \in \mathcal{M}$.
3. $\mathcal{F}_n(\prod_{i=1}^{m} f_i) \in \mathcal{M}$.
4. For each $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(f_i) \in \mathcal{M}$.

This paper is organized as follows: In Section 2, we recall basic definitions and introduce some notation. In Section 3, we present some preliminary results needed for the rest of the paper. In particular, we prove results concerning the product function. Section 4 is devoted to study the relationships between the functions $\prod_{i=1}^{m} f_i$ and f_i, for each $i \in \{1, \ldots, m\}$. Finally, in Section 5, we study the relationships between the functions $\mathcal{F}_n(\prod_{i=1}^{m} f_i)$, $\mathcal{F}_n(f_i)$ and f_i, for each $i \in \{1, \ldots, m\}$.

2. Definitions and notations

Throughout this paper, m is an integer greater than one. A set is said to be *nondegenerate* if it has more than one point. By a (discrete) *dynamical system* we mean a pair (X, f), where X is a nondegenerate topological space and $f : X \to X$ is a not necessarily continuous function, X is called *the phase space*. Let X be a topological space and let A be a subset of X, $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the closure and interior of the set A in X, respectively. A function $f : X \to Y$ between topological spaces is said to be *open* provided $f(U)$ is open in Y for each open subset U of X. The symbols $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ denote the set of integers, the set of nonnegative integers and the set of positive integers, respectively. The *cartesian product* of a finite collection of topological spaces $X_1, \ldots, X_m$ endowed with the product topology [16] is denoted by $\prod_{i=1}^{m} X_i$. On the other hand, given a finite collection of functions, $f_1 : X_1 \to X_1, \ldots, f_m : X_m \to X_m$, we define the *product function* $\prod_{i=1}^{m} f_i$:

$\prod_{i=1}^{m} X_i \to \prod_{i=1}^{m} X_i$ by $\prod_{i=1}^{m} f_i((x_1, \ldots, x_m)) = (f_1(x_1), \ldots, f_m(x_m))$, for each $(x_1, \ldots, x_m) \in \prod_{i=1}^{m} X_i$. In particular, if X is a topological space and $f : X \to X$ is a function, the cartesian product of X with itself m times is denoted by $X^{\times m}$ and the product of f with itself m times is denoted by $f^{\times m}$.

Given a dynamical system (X, f), for each $k \in \mathbb{N}$, the kth iteration of f is defined as the repeated composition of f with itself k times and it is denoted by f^k, that is $f^k = f \circ f^{k-1}$, where $f^1 = f$ and $f^0 = id_X$, the identity function on X. For a subset A of X and $k \in \mathbb{Z}$, we denote by $f^k(A)$ the image of A under f^k, when $k \geq 0$, and the preimage under $f^{|k|}$ when $k < 0$.

REMARK 2.1. For each $i \in \{1, \ldots, m\}$, let (X_i, f_i) be a dynamical system, let U_i, V_i be nonempty subsets of X_i, and let $k \in \mathbb{N}$. Then the following hold.

1. $(\prod_{i=1}^{m} f_i)^k = \prod_{i=1}^{m} f_i^k$.
2. If $(\prod_{i=1}^{m} f_i)^k(\prod_{i=1}^{m} U_i) = \prod_{i=1}^{m} V_i$, then for all $i \in \{1, \ldots, m\}$, $f_i^k(U_i) = V_i$.

Let (X, f) be a dynamical system, and let $x, y \in X$. Then x is a:

- *fixed point of* f if $f(x) = x$,
- *periodic point of* f if there exists $k \in \mathbb{N}$ such that $f^k(x) = x$. The set of periodic points of f is denoted by $Per(f)$,
- *recurrent point of* f provided that for each open subset U of X such that $x \in U$, there exists $l \in \mathbb{N}$ such that $f^l(x) \in U$,
- *quasi-periodic point of* f provided that for each open subset U of X such that $x \in U$, there exists $l \in \mathbb{N}$ such that $f^{kl}(x) \in U$ for every $k \geq 0$,
- *nonwandering point of* f if for each open subset U in X such that $x \in U$ there exists $l \in \mathbb{N}$ such that $f^l(U) \cap U \neq \emptyset$,
- *ω-limit point of y under* f if for any $k \in \mathbb{N}$ and for any open subset U of X such that $x \in U$, there exists a positive integer $l \geq k$ such that $f^l(y) \in U$. The set of ω-limit points of y under f, is denoted by $\omega(y, f)$ and is called *ω-limit set* of y.

The *orbit* of x under f is the set $\mathcal{O}(x, f) = \{f^k(x) : k \in \mathbb{Z}_+\}$. Given a subset A of X, we say that A is *+invariant* under f if $f(A) \subseteq A$. A topological space X is *+invariant over open subsets under* f, if for each open subset U of X, U is +invariant under f. For subsets A and B of X, the following subset of $\mathbb{N}$ is defined as $n_f(A, B) = \{k \in \mathbb{N} : A \cap f^{-k}(B) \neq \emptyset\}$.

The following definitions can be found in [1, 3, 15, 19].

Let (X, f) be a dynamical system. Then f is:

- *exact*, if for each nonempty open subset U of X, there exists $k \in \mathbb{N}$ such that $f^k(U) = X$,
- *mixing*, if for each pair of nonempty open subsets U and V of X, there exists $N \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$, for all $k \geq N$,

- *transitive*, if for every pair of nonempty open subsets U and V of X, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$,
- *weakly mixing*, if $f^{\times 2}$ is transitive,
- *totally transitive*, if f^s is transitive, for all $s \in \mathbb{N}$,
- *strongly transitive*, if for each nonempty open subset U of X, there exists $s \in \mathbb{N}$ such that $\bigcup_{k=0}^{s} f^k(U) = X$,
- *chaotic*, if it is transitive and $Per(f)$ is dense in X (this definition corresponds to chaos in the sense of Devaney [9, Remark 1]),
- *orbit-transitive*, if there exists $x \in X$ such that $\mathrm{cl}(\mathcal{O}(x, f)) = X$,
- *strictly orbit-transitive*, if there exists $x \in X$ such that $\mathrm{cl}(\mathcal{O}(f(x), f)) = X$,
- *ω-transitive*, if there exists $x \in X$ such that $\omega(x, f) = X$,
- *TT_{++}*, if for any pair of nonempty open subsets U and V of X, the set $n_f(U, V)$ is infinite,
- *mild mixing*, if for any transitive function $f_1 : X_1 \to X_1$, the function $f \times f_1$ is transitive,
- *exactly Devaney chaotic*, if f is exact and $Per(f)$ is dense in X,
- *scattering*, if for any minimal function (see [18]), $f_1 : X_1 \to X_1$, the function $f \times f_1$ is transitive,
- *Touhey*, if for every pair of nonempty open subsets U and V of X, there exists a periodic point $x \in U$ and $k \in \mathbb{Z}_+$ such that $f^k(x) \in V$,
- *an F-system*, if f is totally transitive and $Per(f)$ is dense in X,
- *exact in the sense of Akin-Auslander-Nagar*, if for any pair of nonempty open subsets U and V of X, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap f^k(V) \neq \emptyset$,
- *fully exact*, if for each pair of nonempty open subsets U and V of X, there exists $k \in \mathbb{N}$ such that $\mathrm{int}(f^k(U) \cap f^k(V)) \neq \emptyset$,
- *strongly transitive in the sense of Akin-Auslander-Nagar*, if for each nonempty open subset U of X, $\bigcup_{k=1}^{\infty} f^k(U) = X$,
- *very strongly transitive*, if for each nonempty open subset U of X, there exists $N \in \mathbb{N}$ such that $\bigcup_{k=1}^{N} f^k(U) = X$,
- *exact transitive*, if for each pair of nonempty open subsets U and V of X, $\bigcup_{k=1}^{\infty}(f^k(U) \cap f^k(V))$ is dense in X,
- *strongly exact transitive*, if for each pair of nonempty open subsets U and V of X, we have that $\bigcup_{k=1}^{\infty}(f^k(U) \cap f^k(V)) = X$,
- *strongly product transitive*, if for each $k \in \mathbb{Z}_+$, $f^{\times k}$ is strongly transitive in the sense of Akin-Auslander-Nagar.

The concepts of *exact in the sense of Akin-Auslander-Nagar* and *strongly transitive in the sense of Akin-Auslander-Nagar* were studied by Akin et al. [2] just as *exact* and *strongly transitive* respectively. For convenience, from now on we will refer to them as AAN exact and AAN strongly transitive.

Also, since every exact function is surjective, we have that $f : X \to X$ is exact if and only if, for all nonempty open subset U of X, there exists $N \in \mathbb{N}$ such that $f^k(U) = X$, for each $k \geq N$. Throughout this paper we use this equivalence.

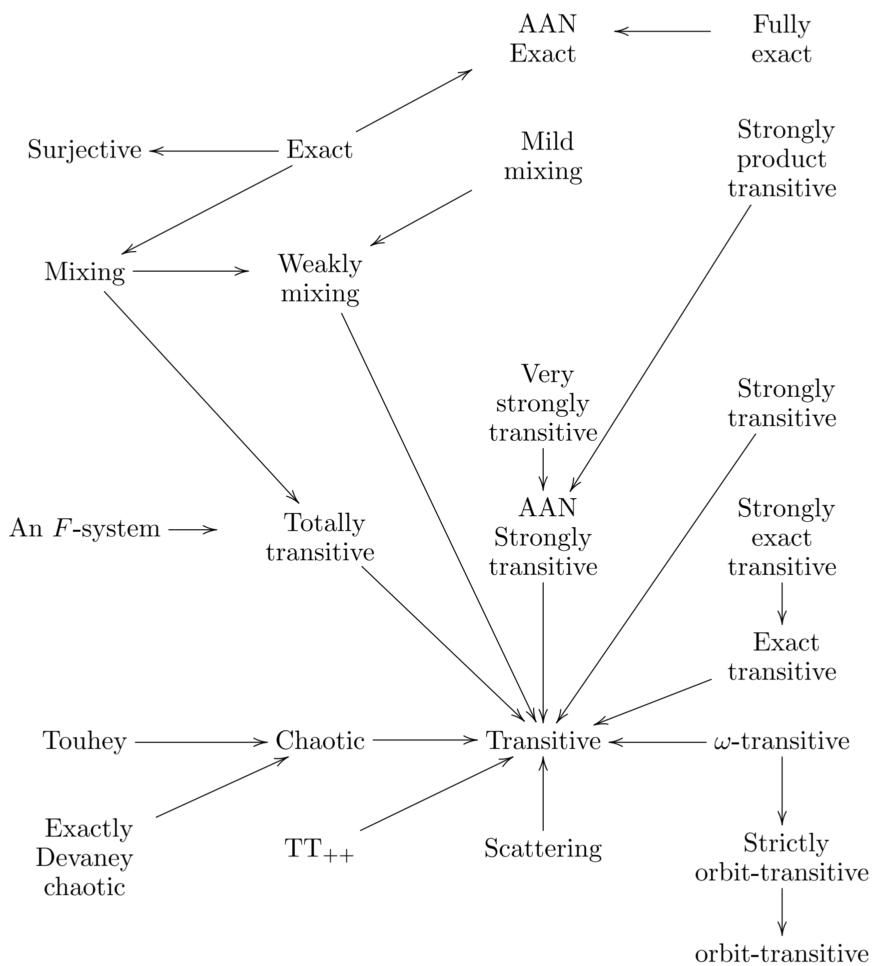

FIGURE 1. Inclusions between some classes of functions.

Diagram of Figure 1 shows some inclusions between the classes of functions we are working with. Readers interested in the proofs of these inclusions can see, for instance, [1, 3, 6, 15].

Note that by adding certain properties to a phase space or to a function, we may obtain different (stronger) dynamical properties, namely: Let X be a compact metric space and let $f : X \to X$ be a function, then every exact transitive function is weakly mixing [1, Theorem 2.18]; in addition if f is an open function, then the concepts of AAN strongly transitive and very strongly transitive are equivalent [1, Theorem 2.13]. If X is compact and f is open and continuous, we have that AAN strongly transitive and very strongly transitive are equivalent [1, Theorem 2.13]. If X is compact and f is continuous, it follows that if f is exact transitive then f is weakly mixing.

3. Preliminary results

In this section we establish some preliminary results about the function $\prod_{i=1}^m f_i$.

In [18, Theorem 3.3], Rojas et al. studied relationships between transitive, ω-limit, isolated and periodic points of the functions $\prod_{i=1}^m f_i$ and f_i, for all $i \in \{1, \dots, m\}$. Now, we analyzed the same problem considering recurrent, quasi-periodic and nonwandering points.

Clearly, the following implications hold: fixed $\Longrightarrow$ periodic $\Longrightarrow$ quasi-periodic $\Longrightarrow$ recurrent $\Longrightarrow$ nonwandering. However, the reverse implications do not hold.

Theorem 3.1. *For each $i \in \{1, \dots, m\}$, let (X_i, f_i) be a dynamical system, and let $(x_1, \dots, x_m) \in \prod_{i=1}^m X_i$. If $(x_1, \dots, x_m)$ is a recurrent, quasi-periodic or nonwandering point of $\prod_{i=1}^m f_i$, then, for each $i \in \{1, \dots, m\}$, x_i is a recurrent, quasi-periodic or nonwandering point of f_i, respectively.*

Proof. Suppose that $(x_1, \dots, x_m)$ is a recurrent point of $\prod_{i=1}^m f_i$. Let $i_0 \in \{1, \dots, m\}$ and let U_{i_0} be an open subset of X_{i_0} such that $x_{i_0} \in U_{i_0}$. For each $i \in \{1, \dots, m\} \backslash \{i_0\}$, let $U_i = X_i$. Then $\mathcal{U} = \prod_{i=1}^m U_i$ is an open subset of $\prod_{i=1}^m X_i$ such that $(x_1, \dots, x_m) \in \mathcal{U}$. By hypothesis, there exists $k \in \mathbb{N}$ such that $(\prod_{i=1}^m f_i)^k((x_1, \dots, x_m)) \in \mathcal{U}$, that is $(\prod_{i=1}^m f_i^k)((x_1, \dots, x_m)) \in \mathcal{U}$. Therefore, $f_{i_0}^k(x_{i_0}) \in U_{i_0}$ and thus, x_{i_0} is a recurrent point of f_{i_0}.

The proof for quasi-periodic and nonwandering points is similar to that given for recurrent points. ◻

Theorem 3.2. *For each $i \in \{1, \dots, m\}$, let (X_i, f_i) be a dynamical system, let $(x_1, \dots, x_m) \in \prod_{i=1}^m X_i$, let $i_0 \in \{1, \dots, m\}$, and for each $i \in \{1, \dots, m\} \backslash \{i_0\}$, let $x_i \in X_i$ be a fixed point of f_i. If x_{i_0} is a recurrent, quasi-periodic or nonwandering point of f_{i_0}, then $(x_1, \dots, x_m)$ is a recurrent, quasi-periodic or nonwandering point of $\prod_{i=1}^m f_i$, respectively.*

Proof. Suppose that x_{i_0} is a recurrent point of f_{i_0}. Let Ω be an open subset of $\prod_{i=1}^m X_i$ such that $(x_1, \dots, x_m) \in \Omega$. Then, for every $i \in \{1, \dots, m\}$, there exists an open subset U_i of X_i such that $(x_1, \dots, x_m) \in \mathcal{U} = \prod_{i=1}^m U_i \subseteq$

Ω. By hypothesis, there exists $k_{i_0} \in \mathbb{N}$ such that $f_{i_0}^{k_{i_0}}(x_{i_0}) \in U_{i_0}$. On the other hand, since, for all $i \in \{1, \ldots, m\} \setminus \{i_0\}$, x_i is a fixed point, we have that $f_i^{k_{i_0}}(x_i) \in U_i$. Consequently, $(\prod_{i=1}^m f_i^{k_{i_0}})((x_1, \ldots, x_m)) \in \mathcal{U}$, that is to say, $(\prod_{i=1}^m f_i)^{k_{i_0}}((x_1, \ldots, x_m)) \in \mathcal{U} \subseteq \Omega$. Therefore, $(x_1, \ldots, x_m)$ is a recurrent point of $\prod_{i=1}^m f_i$.

The proof for quasi-periodic and nonwandering points is similar to that given for recurrent points. □

Definition 3.3. *Let (X, f) and (Y, g) be dynamical systems. Then f and g are said to be topologically conjugate if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. The homeomorphism h is called a topological conjugacy between f and g.*

Remark 3.4. Let (X, f) and (Y, g) be dynamical systems, let $h : X \to Y$ be a homeomorphism, and let $k \in \mathbb{N}$. If f and g are topologically conjugate via h then:

(1) g and f are topologically conjugate via h^{-1},
(2) $f^k = h^{-1} \circ g^k \circ h$ and $g^k = h \circ f^k \circ h^{-1}$.

The following result follows from [10, Proposition 2.3.29] and [7, Lemma 7].

Theorem 3.5. *Let $X_1, \ldots, X_m$, $Y_1, \ldots, Y_m$ be topological spaces, and for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to Y_i$ be a function. For each $i \in \{1, \ldots, m\}$, f_i is a homeomorphism if and only if $\prod_{i=1}^m f_i$ is a homeomorphism.*

Our next result follows from Theorem 3.5.

Theorem 3.6. *Let (X, f) and (Y, g) be dynamical systems, let $h : X \to Y$ be a homeomorphism, and let $k \in \mathbb{N}$. Then f and g are topologically conjugate via h if and only if $f^{\times k}$ and $g^{\times k}$ are topologically conjugate via $h^{\times k}$.*

There are many dynamical properties that are preserved under topological conjugacy.

Theorem 3.7. *Let (X, f) and (Y, g) be dynamical systems, let $h : X \to Y$ be a topological conjugacy between f and g, and let $\mathcal{M}$ be one of the following classes of functions: AAN exact, fully exact, AAN strongly transitive, very strongly transitive, exact transitive, strongly exact transitive or strongly product transitive. Then $f \in \mathcal{M}$ if and only if $g \in \mathcal{M}$.*

Proof. Suppose that f is AAN exact. Let U and V be nonempty open subsets of Y. Then $h^{-1}(U)$ and $h^{-1}(V)$ are nonempty open subsets of X. By hypothesis, there exists $k \in \mathbb{N}$ such that $f^k(h^{-1}(U)) \cap f^k(h^{-1}(V)) \neq \emptyset$, that is, $h^{-1}(g^k(U)) \cap h^{-1}(g^k(V)) = h^{-1}(g^k(U) \cap g^k(V)) \neq \emptyset$. Therefore, $g^k(U) \cap g^k(V) \neq \emptyset$ and thus, g is AAN exact.

Suppose that f is fully exact. Let U and V be nonempty open subsets of Y. Then $h^{-1}(U)$ and $h^{-1}(V)$ are nonempty open subsets of X. Since f is fully exact, there exists $k \in \mathbb{N}$ such that $\text{int}(f^k(h^{-1}(U)) \cap f^k(h^{-1}(V))) \neq \emptyset$. Thus, $\text{int}(h^{-1}(g^k(U)) \cap h^{-1}(g^k(V))) = \text{int}(h^{-1}(g^k(U) \cap g^k(V))) \neq \emptyset$. Let $x \in \text{int}(h^{-1}(g^k(U) \cap g^k(V)))$. Hence, there exists an open subset W of X such that $x \in W \subseteq h^{-1}(g^k(U) \cap g^k(V))$. Finally, $h(x) \in h(W) \subseteq g^k(U) \cap g^k(V)$. Therefore, $h(x) \in \text{int}(g^k(U) \cap g^k(V))$ and thus, g is fully exact.

Suppose that f is AAN strongly transitive. Let U be a nonempty open subset of Y. It follows that:

$$\begin{aligned}\bigcup_{k=1}^{\infty} g^k(U) &= \bigcup_{k=1}^{\infty} (h \circ f^k \circ h^{-1})(U) \\ &= \bigcup_{k=1}^{\infty} h(f^k(h^{-1}(U))) \\ &= h\left(\bigcup_{k=1}^{\infty} f^k(h^{-1}(U))\right).\end{aligned}$$

Since f is AAN strongly transitive, $\bigcup_{k=1}^{\infty} g^k(U) = h(X) = Y$. Therefore, g is AAN strongly transitive.

The proof for very strongly transitivity is similar to that given for AAN strongly transitivity.

Suppose that f is exact transitive. Let U and V be nonempty open subsets of Y. Then, $h^{-1}(U)$ and $h^{-1}(V)$ are nonempty open subsets of X. By hypothesis, $\bigcup_{k=1}^{\infty}(f^k(h^{-1}(U)) \cap f^k(h^{-1}(V)))$ is dense in X. Since h is a homeomorphism, we have that $h(\bigcup_{k=1}^{\infty}(f^k(h^{-1}(U)) \cap f^k(h^{-1}(V))))$ is dense in Y. Also:

$$\begin{aligned}h\left(\bigcup_{k=1}^{\infty}(f^k(h^{-1}(U)) \cap f^k(h^{-1}(V)))\right) &= \bigcup_{k=1}^{\infty} h(f^k(h^{-1}(U)) \cap f^k(h^{-1}(V))) \\ &= \bigcup_{k=1}^{\infty}(g^k(U) \cap g^k(V)).\end{aligned}$$

Therefore, g is exact transitive.

Suppose that f is strongly exact transitive. Let U and V be nonempty open subsets of Y. Hence, $h^{-1}(U)$ and $h^{-1}(V)$ are nonempty open subsets of X. By hypothesis, $\bigcup_{k=1}^{\infty}(f^k(h^{-1}(U)) \cap f^k(h^{-1}(V))) = X$. It follows that $h(\bigcup_{k=1}^{\infty}(f^k(h^{-1}(U)) \cap f^k(h^{-1}(V)))) = Y$. Finally, $\bigcup_{k=1}^{\infty}(g^k(U) \cap g^k(V)) = Y$. Therefore, g is strongly exact transitive.

Suppose that f is strongly product transitive. Then, $f^{\times k}$ is AAN strongly transitive for all $k \in \mathbb{Z}_+$. On the other hand, by Theorem 3.6, $f^{\times k}$ and $g^{\times k}$ are topologically conjugate via $h^{\times k}$. Finally, by the third paragraph of the proof of this theorem, $g^{\times k}$ is AAN strongly transitive for each $k \in \mathbb{Z}_+$. Therefore, g is strongly product transitive.

By Remark 3.4, part (1), we have the converse. $\square$

We conclude this section establishing that the functions $(\prod_{i=1}^m f_i)^{\times k}$ and $\prod_{i=1}^m f_i^{\times k}$ are topologically conjugate.

THEOREM 3.8. *For each $i \in \{1,\dots,m\}$, let (X_i, f_i) be a dynamical system, let $k \in \mathbb{N}$, and let $h : (\prod_{i=1}^m X_i)^{\times k} \to \prod_{i=1}^m X_i^{\times k}$ be a function given by:*

$$h(((x_1^1,\dots,x_m^1),\dots,(x_1^k,\dots,x_m^k))) = ((x_1^1,\dots,x_1^k),\dots,(x_m^1,\dots,x_m^k)).$$

Then h is a homeomorphism.

PROOF. It is not difficult to prove that h is a bijective function. We shall show that h is a continuous and open function. Let $\mathcal{U}$ be an open subset of $\prod_{i=1}^m X_i^{\times k}$ and let $\mathcal{X} \in h^{-1}(\mathcal{U})$. It follows that, for each $i \in \{1,\dots,m\}$ and for all $j \in \{1,\dots,k\}$, there exists a nonempty open subset U_i^j of X_i such that $h(\mathcal{X}) \in \prod_{i=1}^m(\prod_{j=1}^k U_i^j) \subseteq \mathcal{U}$, that is, $\mathcal{X} \in h^{-1}\left(\prod_{i=1}^m\left(\prod_{j=1}^k U_i^j\right)\right) \subseteq h^{-1}(\mathcal{U})$. Even more, since $h^{-1}\left(\prod_{i=1}^m\left(\prod_{j=1}^k U_i^j\right)\right) = \prod_{j=1}^k\left(\prod_{i=1}^m U_i^j\right)$, we have that $h^{-1}(\mathcal{U})$ is an open subset of $(\prod_{i=1}^m X_i)^{\times k}$. Therefore, h is continuous.

Now, let $\mathcal{U}$ be an open subset of $(\prod_{i=1}^m X_i)^{\times k}$ and let $\mathcal{X} \in h(\mathcal{U})$. Then, for each $j \in \{1,\dots,k\}$ and for all $i \in \{1,\dots,m\}$, there exists a nonempty open subset U_i^j of X_i such that $h^{-1}(\mathcal{X}) \in \prod_{j=1}^k\left(\prod_{i=1}^m U_i^j\right) \subseteq \mathcal{U}$. Even more, since $h(\prod_{j=1}^k(\prod_{i=1}^m U_i^j)) = \prod_{i=1}^m(\prod_{j=1}^k U_i^j)$, we conclude that $\mathcal{X} \in \prod_{i=1}^m(\prod_{j=1}^k U_i^j) \subseteq h(\mathcal{U})$. Therefore, h is open and thus, h is a homeomorphism. $\square$

As a consequence of Theorem 3.8, we have the following result.

THEOREM 3.9. *For each $i \in \{1,\dots,m\}$, let (X_i, f_i) be a dynamical system, let $k \in \mathbb{N}$, and let $h : (\prod_{i=1}^m X_i)^{\times k} \to \prod_{i=1}^m X_i^{\times k}$ be the same as in Theorem 3.8. Then $(\prod_{i=1}^m f_i)^{\times k}$ is topologically conjugate to $\prod_{i=1}^m f_i^{\times k}$ via the homeomorphism h.*

By Remark 3.4, part (1), and Theorems 3.9 and 3.7, we immediately obtain the following theorem.

THEOREM 3.10. *For each $i \in \{1,\dots,m\}$, let (X_i, f_i) be a dynamical system, let $k \in \mathbb{N}$, and let $\mathcal{M}$ be one of the following classes of functions: AAN exact, fully exact, AAN strongly transitive, very strongly transitive, exact transitive, strongly exact transitive or strongly product transitive. Then $(\prod_{i=1}^m f_i)^{\times k} \in \mathcal{M}$ if and only if $\prod_{i=1}^m f_i^{\times k} \in \mathcal{M}$.*

4. Dynamic properties of product functions

In this section, we study the relationships that exist between the functions $\prod_{i=1}^m f_i$ and f_i, for each $i \in \{1,\dots,m\}$, when any of them is AAN exact, fully exact, AAN strongly transitive, very strongly transitive, exact transitive, strongly exact transitive or strongly product transitive.

Theorem 4.1. *For each $i \in \{1,\dots,m\}$, let (X_i, f_i) be a dynamical system. Then, for each $i \in \{1,\dots,m\}$, f_i is AAN exact if and only if $\prod_{i=1}^m f_i$ is AAN exact.*

Proof. Suppose that $\prod_{i=1}^m f_i$ is AAN exact. Let $i_0 \in \{1,\dots,m\}$ and let U_{i_0} and V_{i_0} be nonempty open subsets of X_{i_0}. For each $i \in \{1,\dots,m\}\backslash\{i_0\}$, let $U_i = X_i$ and $V_i = X_i$. Then $\mathcal{U} = \prod_{i=1}^m U_i$ and $\mathcal{V} = \prod_{i=1}^m V_i$ are nonempty open subsets of $\prod_{i=1}^m X_i$. By hypothesis, there exists $k \in \mathbb{N}$ such that $(\prod_{i=1}^m f_i)^k(\mathcal{U}) \cap (\prod_{i=1}^m f_i)^k(\mathcal{V}) \neq \emptyset$, that is to say, $(\prod_{i=1}^m f_i^k(U_i)) \cap (\prod_{i=1}^m f_i^k(V_i)) \neq \emptyset$. Therefore, $f_{i_0}^k(U_{i_0}) \cap f_{i_0}^k(V_{i_0}) \neq \emptyset$ and thus, f_{i_0} is AAN exact.

Now, suppose that for all $i \in \{1,\dots,m\}$, f_i is AAN exact. Let Ω_1 and Ω_2 be nonempty open subsets of $\prod_{i=1}^m X_i$. Then, for each $i \in \{1,\dots,m\}$, there exist nonempty open subsets U_i and V_i of X_i such that $\mathcal{U} = \prod_{i=1}^m U_i \subseteq \Omega_1$ and $\mathcal{V} = \prod_{i=1}^m V_i \subseteq \Omega_2$. By hypothesis, for each $i \in \{1,\dots,m\}$, there exists $k_i \in \mathbb{N}$ such that $f_i^{k_i}(U_i) \cap f_i^{k_i}(V_i) \neq \emptyset$. Let $k = \max\{k_1,\dots,k_m\}$. It follows that, for each $i \in \{1,\dots,m\}$, there exists $l_i \in \mathbb{N} \cup \{0\}$ such that $k = k_i + l_i$ and thus, $f_i^k(U_i) \cap f_i^k(V_i) = f_i^{l_i}(f_i^{k_i}(U_i) \cap f_i^{k_i}(V_i)) \neq \emptyset$. In consequence, $(\prod_{i=1}^m f_i^k(U_i)) \cap (\prod_{i=1}^m f_i^k(V_i)) \neq \emptyset$. Finally, $(\prod_{i=1}^m f_i)^k(\mathcal{U}) \cap (\prod_{i=1}^m f_i)^k(\mathcal{V}) \neq \emptyset$. Therefore, $(\prod_{i=1}^m f_i)^k(\Omega_1) \cap (\prod_{i=1}^m f_i)^k(\Omega_2) \neq \emptyset$ and thus, $\prod_{i=1}^m f_i$ is AAN exact. □

Theorem 4.2. *For each $i \in \{1,\dots,m\}$, let (X_i, f_i) be a dynamical system, and let $\mathcal{M}$ be one of the following classes of functions: fully exact, AAN strongly transitive, very strongly transitive, exact transitive, strongly exact transitive or strongly product transitive. If $\prod_{i=1}^m f_i \in \mathcal{M}$, then, for each $i \in \{1,\dots,m\}$, $f_i \in \mathcal{M}$.*

Proof. Let $i_0 \in \{1,\dots,m\}$, let U_{i_0} and V_{i_0} be nonempty open subsets of X_{i_0}, and for each $i \in \{1,\dots,m\}\backslash\{i_0\}$, let $U_i = X_i$ and $V_i = X_i$. It follows that, $\mathcal{U} = \prod_{i=1}^m U_i$ and $\mathcal{V} = \prod_{i=1}^m V_i$ are nonempty open subsets of $\prod_{i=1}^m X_i$.

Suppose that $\prod_{i=1}^m f_i$ is fully exact. This implies that, there exists $n \in \mathbb{N}$ such that $\operatorname{int}((\prod_{i=1}^m f_i)^k(\mathcal{U}) \cap (\prod_{i=1}^m f_i)^k(\mathcal{V})) \neq \emptyset$. Then

$$\operatorname{int}\left(\left(\prod_{i=1}^m f_i^k(U_i)\right) \cap \left(\prod_{i=1}^m f_i^k(V_i)\right)\right) \neq \emptyset.$$

In consequence, $\left(\prod_{i=1}^m \operatorname{int}\left(f_i^k(U_i)\right)\right) \cap \left(\prod_{i=1}^m \operatorname{int}\left(f_i^k(V_i)\right)\right) \neq \emptyset$ and thus, we obtain that $\operatorname{int}(f_{i_0}^k(U_{i_0}) \cap f_{i_0}^k(V_{i_0})) \neq \emptyset$. Therefore, f_{i_0} is fully exact.

Suppose that $\prod_{i=1}^m f_i$ is AAN strongly transitive. Then, we have that $\bigcup_{k=1}^\infty(\prod_{i=1}^m f_i)^k(\mathcal{U}) = \prod_{i=1}^m X_i$. Let $x_{i_0} \in X_{i_0}$, and for all $i \in \{1,\dots,m\}\setminus\{i_0\}$, let $x_i \in X_i$. If follows that, $\varphi = (x_1,\dots,x_m) \in \prod_{i=1}^m X_i$. Consequently, there exists $k_{i_0} \in \mathbb{N}$ such that $\varphi \in (\prod_{i=1}^m f_i)^{k_{i_0}}(\mathcal{U})$, that is, $\varphi \in \prod_{i=1}^m f_i^{k_{i_0}}(U_i)$. Therefore, $x_{i_0} \in f_{i_0}^{k_{i_0}}(U_{i_0}) \subseteq \bigcup_{k=1}^\infty f_{i_0}^k(U_{i_0})$ and thus, f_{i_0} is AAN strongly transitive.

The proof for very strongly transitivity is similar to that given for AAN strongly transitivity.

Suppose that $\prod_{i=1}^m f_i$ is exact transitive. Then, $\bigcup_{k=1}^\infty((\prod_{i=1}^m f_i)^k(\mathcal{U}) \cap (\prod_{i=1}^m f_i)^k(\mathcal{V}))$ is dense in $\prod_{i=1}^m X_i$. Let W_{i_0} be a nonempty open subset of X_{i_0} and for each $i \in \{1,\dots,m\}\setminus\{i_0\}$, let $W_i = X_i$. It follows that, $\mathcal{W} = \prod_{i=1}^m W_i$ is a nonempty open subset of $\prod_{i=1}^m X_i$. Consequently, there exists $k_{i_0} \in \mathbb{N}$ such that $((\prod_{i=1}^m f_i)^{k_{i_0}}(\mathcal{U}) \cap (\prod_{i=1}^m f_i)^{k_{i_0}}(\mathcal{V})) \cap \mathcal{W} \neq \emptyset$, that is, $(\prod_{i=1}^m f_i^{k_{i_0}}(U_i) \cap \prod_{i=1}^m f_i^{k_{i_0}}(V_i)) \cap \mathcal{W} \neq \emptyset$. In consequence, $(f_{i_0}^{k_{i_0}}(U_{i_0}) \cap f_{i_0}^{k_{i_0}}(V_{i_0})) \cap W_{i_0} \neq \emptyset$. Therefore, $\bigcup_{k=1}^\infty(f_{i_0}^k(U_{i_0}) \cap f_{i_0}^k(V_{i_0}))$ is dense in X_{i_0} and thus, f_{i_0} is exact transitive.

Suppose that $\prod_{i=1}^m f_i$ is strongly exact transitive. Then,

$$\bigcup_{k=1}^\infty\left(\left(\prod_{i=1}^m f_i\right)^k(\mathcal{U}) \cap \left(\prod_{i=1}^m f_i\right)^k(\mathcal{V})\right) = \prod_{i=1}^m X_i.$$

Let $x_{i_0} \in X_{i_0}$, and for each $i \in \{1,\dots,m\}\setminus\{i_0\}$, let $x_i \in X_i$. Thus, there exists $k_{i_0} \in \mathbb{N}$ such that $\varphi = (x_1,\dots,x_m) \in (\prod_{i=1}^m f_i)^{k_{i_0}}(\mathcal{U}) \cap (\prod_{i=1}^m f_i)^{k_{i_0}}(\mathcal{V})$, that is, $\varphi \in (\prod_{i=1}^m f_i^{k_{i_0}}(U_i)) \cap (\prod_{i=1}^m f_i^{k_{i_0}}(V_i))$. Consequently, $x_{i_0} \in f_{i_0}^{k_{i_0}}(U_{i_0}) \cap f_{i_0}^{k_{i_0}}(V_{i_0})$. Therefore, $\bigcup_{k=1}^\infty(f_{i_0}^k(U_{i_0}) \cap f_{i_0}^k(V_{i_0})) = X_{i_0}$ and thus, f_{i_0} is strongly exact transitive.

Suppose that $\prod_{i=1}^m f_i$ is strongly product transitive. Then, $(\prod_{i=1}^m f_i)^{\times k}$ is AAN strongly transitive, for all $k \in \mathbb{Z}_+$. By Theorem 3.10, we have that $\prod_{i=1}^m f_i^{\times k}$ is AAN strongly transitive. Finally, by the second paragraph of the proof of this theorem, we have that $f_{i_0}^{\times k}$ is AAN strongly transitive for all $k \in \mathbb{Z}_+$. Therefore, f_{i_0} is strongly product transitive. □

For each $i \in \{1,\dots,m\}$, let (X_i, f_i) be a dynamical system where X_i is +invariant over open subsets under f_i. Rojas et al. [18, Theorem 4.10] prove that: if for all $i \in \{1,\dots,m\}$, f_i is transitive, weakly mixing, totally transitive, chaotic, orbit-transitive, strictly orbit-transitive, ω-transitive, TT_{++}, Touhey, scattering, an F-system or mild mixing, then $\prod_{i=1}^m f_i$ has the same property. In Theorems 4.6, 4.7 and 4.8, we present an alternative form of [18, Theorem 4.10] when $\mathcal{M}$ is one of the following classes of functions: transitive, weakly mixing, totally transitive, chaotic, TT_{++}, Touhey or an F-system. Specifically, we replace condition "for all $i \in \{1,\dots,m\}$, X_i is +invariant over open subsets under f_i and $f_i \in \mathcal{M}$" by "for all $i \in \{1, \ldots, m\} \backslash \{i_0\}$, f_i is mixing, and f_{i_0} is surjective, continuous and transitive (respectively, weakly mixing, totally transitive or TT_{++})" (Theorem 4.6), "for all $i \in \{1, \ldots, m\} \backslash \{i_0\}$, f_i is mixing and $Per(f_i)$ is dense in X_i, and f_{i_0} is surjective, continuous and chaotic (respectively, an F-system)" (Theorem 4.7) and "for all $i \in \{1, \ldots, m\} \backslash \{i_0\}$, f_i is continuous and mixing, and $Per(f_i)$ is dense in X_i, and f_{i_0} surjective, continuous and Touhey" (Theorem 4.8). Furthermore, we explore the following classes of functions: AAN exact, fully exact, AAN strongly transitive, very strongly transitive, exact transitive, strongly exact transitive and strongly product transitive and we obtain results that are analogous to those presented in [18, Theorem 4.10].

Remark 4.3 (Compare with [18, Remark 4.9]). Let (X, f) be a dynamical system. Observe that, if X is +invariant over open subsets under f, then f cannot be AAN strongly transitive, very strongly transitive, strongly exact transitive or strongly product transitive unless X has the trivial topology.

Theorem 4.4. *For each $i \in \{1, \ldots, m\}$, let (X_i, f_i) be a dynamical system, let $i_0 \in \{1, \ldots, m\}$, for all $i \in \{1, \ldots, m\} \backslash \{i_0\}$, let f_i be exact, and let f_{i_0} be surjective. If f_{i_0} is AAN strongly transitive, strongly transitive, very strongly transitive, strongly exact transitive or strongly product transitive, then $\prod_{i=1}^m f_i$ is AAN strongly transitive, strongly transitive, very strongly transitive, strongly exact transitive or strongly product transitive, respectively.*

Proof. Suppose that f_{i_0} is AAN strongly transitive. Let Ω be a nonempty open subset of $\prod_{i=1}^m X_i$. Then, for each $i \in \{1, \ldots, m\}$, there exists a nonempty open subset U_i of X_i such that $\mathcal{U} = \prod_{i=1}^m U_i \subseteq \Omega$. Since for each $i \in \{1, \ldots, m\} \backslash \{i_0\}$, f_i is exact, there exists $N_i \in \mathbb{N}$ such that $f_i^l(U_i) = X_i$, for all $l \geq N_i$. By hypothesis, $\bigcup_{k=1}^\infty f_{i_0}^k(U_{i_0}) = X_{i_0}$. Let $N = \max\{N_i : i \in \{1, \ldots, m\} \backslash \{i_0\}\}$. Then, $\bigcup_{k=1}^\infty f_{i_0}^{N+k}(U_{i_0}) = X_{i_0}$. Finally, if $(x_1, \ldots, x_m) \in \prod_{i=1}^m X_i$, there exists $k_{i_0} \in \mathbb{N}$ such that $x_{i_0} \in f_{i_0}^{N+k_{i_0}}(U_{i_0})$, and for all $i \in \{1, \ldots, m\} \backslash \{i_0\}$, $x_i \in f_i^{N+k_{i_0}}(U_i)$. Then, $(x_1, \ldots, x_m) \in \prod_{i=1}^m f_i^{N+k_{i_0}}(U_i)$, that is, $(x_1, \ldots, x_m) \in (\prod_{i=1}^m f_i)^{N+k_{i_0}}(\mathcal{U}) \subseteq \bigcup_{k=1}^\infty (\prod_{i=1}^m f_i)^k(\Omega)$. Therefore, $\prod_{i=1}^m f_i$ is AAN strongly transitive.

The proof for very strongly transitivity and strongly transitivity is similar to that given for AAN strongly transitivity.

Suppose that f_{i_0} is strongly exact transitive. Let Ω_1 and Ω_2 be nonempty open subsets of $\prod_{i=1}^m X_i$. For all $i \in \{1, \ldots, m\}$, there exist nonempty open subsets U_i and V_i of X_i such that $\mathcal{U} = \prod_{i=1}^m U_i \subseteq \Omega_1$ and $\mathcal{V} = \prod_{i=1}^m V_i \subseteq \Omega_2$. Since for each $i \in \{1, \ldots, m\} \backslash \{i_0\}$, f_i is exact, we have that there exist $N_i^1, N_i^2 \in \mathbb{N}$ such that $f_i^{k_1}(U_i) = X_i$ and $f_i^{k_2}(V_i) = X_i$ for all $k_1 \geq N_i^1$ and $k_2 \geq N_i^2$. By hypothesis, $\bigcup_{k=1}^\infty (f_{i_0}^k(U_{i_0}) \cap f_{i_0}^k(V_{i_0})) = X_{i_0}$. Let $N = \max\{N_i^1, N_i^2 : i \in \{1, \ldots, m\} \backslash \{i_0\}\}$. Note that $\bigcup_{k=1}^\infty (f_{i_0}^{N+k}(U_{i_0}) \cap f_{i_0}^{N+k}(V_{i_0})) = X_{i_0}$. Let $(x_1, \ldots, x_m) \in \prod_{i=1}^m X_i$. Then, there exists

$k_{i_0} \in \mathbb{N}$ such that $x_{i_0} \in f_{i_0}^{N+k_{i_0}}(U_{i_0}) \cap f_{i_0}^{N+k_{i_0}}(V_{i_0})$. On the other hand, for each $i \in \{1,\ldots,m\}\setminus\{i_0\}$, $x_i \in f_i^{N+k_{i_0}}(U_i) \cap f_i^{N+k_{i_0}}(V_i)$. In consequence, $(x_1,\ldots,x_m) \in \prod_{i=1}^m f_i^{N+k_{i_0}}(U_i) \cap \prod_{i=1}^m f_i^{N+k_{i_0}}(V_i)$, that is to say,

$$(x_1,\ldots,x_m) \in \left(\prod_{i=1}^m f_i\right)^{N+k_{i_0}}(\mathcal{U}) \cap \left(\prod_{i=1}^m f_i\right)^{N+k_{i_0}}(\mathcal{V}).$$

This implies that

$$(x_1,\ldots,x_m) \in \bigcup_{k=1}^{\infty}\left(\left(\prod_{i=1}^m f_i\right)^k(\Omega_1) \cap \left(\prod_{i=1}^m f_i\right)^k(\Omega_2)\right).$$

Therefore, $\prod_{i=1}^m f_i$ is strongly exact transitive.

Suppose that f_{i_0} is strongly product transitive. Then, $f_{i_0}^{\times k}$ is AAN strongly transitive for all $k \in \mathbb{Z}_+$. Also, by [18, Theorem 4.3], for each $i \in \{1,\ldots,m\}\setminus\{i_0\}$, $f_i^{\times k}$ is exact. By the first paragraph of the proof of this theorem, we have that $\prod_{i=1}^m f_i^{\times k}$ is AAN strongly transitive for every $k \in \mathbb{Z}_+$. Finally, by Theorem 3.10, $(\prod_{i=1}^m f_i)^{\times k}$ is AAN strongly transitive for all $k \in \mathbb{Z}_+$. Therefore, $\prod_{i=1}^m f_i$ is strongly product transitive. ☐

THEOREM 4.5. *For each $i \in \{1,\ldots,m\}$, let (X_i, f_i) be a dynamical system, let $i_0 \in \{1,\ldots,m\}$, let f_{i_0} be continuous and exact transitive, and for each $i \in \{1,\ldots,m\}\setminus\{i_0\}$ let f_i be exact. Then $\prod_{i=1}^m f_i$ is exact transitive.*

PROOF. Let Ω_1 and Ω_2 be nonempty open subsets of $\prod_{i=1}^m X_i$. Then, for each $i \in \{1,\ldots,m\}$, there exist nonempty open subsets U_i and V_i of X_i such that $\mathcal{U} = \prod_{i=1}^m U_i \subseteq \Omega_1$ and $\mathcal{V} = \prod_{i=1}^m V_i \subseteq \Omega_2$. Since for every $i \in \{1,\ldots,m\}\setminus\{i_0\}$, f_i is exact, there exist $N_i^1, N_i^2 \in \mathbb{N}$ such that $f_i^{k_1}(U_i) = X_i$, for each $k_1 \geq N_i^1$ and $f_i^{k_2}(V_i) = X_i$, for every $k_2 \geq N_i^2$. On the other hand, since f_{i_0} is exact transitive, $\bigcup_{k=1}^{\infty}(f_{i_0}^k(U_{i_0}) \cap f_{i_0}^k(V_{i_0}))$ is dense in X_{i_0}. Let $N = \max\{N_i^1, N_i^2 : i \in \{1,\ldots,m\}\setminus\{i_0\}\}$. Since $f_{i_0}^N$ is continuous, we have that $\bigcup_{k=1}^{\infty}(f_{i_0}^{N+k}(U_{i_0}) \cap f_{i_0}^{N+k}(V_{i_0}))$ is dense in X_{i_0}. Let Ω_3 be a nonempty open subset of $\prod_{i=1}^m X_i$. Then, for each $i \in \{1,\ldots,m\}$, there exists a nonempty open subset W_i of X_i such that $\mathcal{W} = \prod_{i=1}^m W_i \subseteq \Omega_3$. In consequence, there exist $k_{i_0} \in \mathbb{N}$ and $x_{i_0} \in X_{i_0}$ such that $x_{i_0} \in f_{i_0}^{N+k_{i_0}}(U_{i_0}) \cap f_{i_0}^{N+k_{i_0}}(V_{i_0}) \cap W_{i_0}$. On the other hand, for all $i \in \{1,\ldots,m\}\setminus\{i_0\}$, $f_i^{N+k_{i_0}}(U_i) = X_i$ and $f_i^{N+k_{i_0}}(V_i) = X_i$. Finally, for all $i \in \{1,\ldots,m\}\setminus\{i_0\}$, let $x_i \in W_i$. Notice that $(x_1,\ldots,x_m)\in(\prod_{i=1}^m f_i^{N+k_{i_0}}(U_i))\cap(\prod_{i=1}^m f_i^{N+k_{i_0}}(V_i))$, that is to say,

$$\begin{aligned}(x_1,\ldots,x_m)&\in\left(\left(\prod_{i=1}^m f_i\right)^{N+k_{i_0}}(\mathcal{U})\cap\left(\prod_{i=1}^m f_i\right)^{N+k_{i_0}}(\mathcal{V})\right)\cap\mathcal{W}\\&\subseteq\left(\left(\prod_{i=1}^m f_i\right)^{N+k_{i_0}}(\Omega_1)\cap\left(\prod_{i=1}^m f_i\right)^{N+k_{i_0}}(\Omega_2)\right)\cap\Omega_3.\end{aligned}$$

Therefore, $\bigcup_{k=1}^{\infty}((\prod_{i=1}^m f_i)^k(\Omega_1)\cap(\prod_{i=1}^m f_i)^k(\Omega_2))$ is dense in $\prod_{i=1}^m X_i$ and thus, $\prod_{i=1}^m f_i$ is exact transitive. □

Theorem 4.6. *For each $i\in\{1,\ldots,m\}$, let (X_i,f_i) be a dynamical system, let $i_0\in\{1,\ldots,m\}$, let f_{i_0} be surjective and continuous, and for each $i\in\{1,\ldots,m\}\backslash\{i_0\}$, let f_i be mixing. If f_{i_0} is transitive, weakly mixing, totally transitive or TT_{++}, then $\prod_{i=1}^m f_i$ is transitive, weakly mixing, totally transitive or TT_{++}, respectively.*

Proof. Suppose that f_{i_0} is transitive. Let Ω_1 and Ω_2 be nonempty open subsets of $\prod_{i=1}^m X_i$. For each $i\in\{1,\ldots,m\}$, there exist nonempty open subsets U_i and V_i of X_i such that $\mathcal{U}=\prod_{i=1}^m U_i\subseteq\Omega_1$ and $\mathcal{V}=\prod_{i=1}^m V_i\subseteq\Omega_2$. Since for all $i\in\{1,\ldots,m\}\backslash\{i_0\}$, f_i is mixing, there exists $N_i\in\mathbb{N}$ such that $f_i^k(U_i)\cap V_i\neq\emptyset$, for all $k\geq N_i$. Let $N=\max\{N_i : i\in\{1,\ldots,m\}\backslash\{i_0\}\}$. It is clear that $f_{i_0}^N$ is surjective and continuous and thus, we have that $f_{i_0}^{-N}(V_{i_0})$ is a nonempty open subset of X_{i_0}. By hypothesis, there exists $l\in\mathbb{N}$ such that $f_{i_0}^l(U_{i_0})\cap f_{i_0}^{-N}(V_{i_0})\neq\emptyset$. Hence, $f_{i_0}^N(f_{i_0}^l(U_{i_0})\cap f_{i_0}^{-N}(V_{i_0}))=f_{i_0}^{N+l}(U_{i_0})\cap V_{i_0}\neq\emptyset$. On the other hand, for each $i\in\{1,\ldots,m\}\backslash\{i_0\}$, $f_i^{N+l}(U_i)\cap V_i\neq\emptyset$. Consequently, there exists $(x_1,\ldots,x_m)\in(\prod_{i=1}^m f_i^{N+l}(U_i))\cap\mathcal{V}=((\prod_{i=1}^m f_i)^{N+l}(\mathcal{U}))\cap\mathcal{V}$ and thus, $((\prod_{i=1}^m f_i)^{N+l}(\Omega_1))\cap\Omega_2\neq\emptyset$. Therefore, $\prod_{i=1}^m f_i$ is transitive.

Suppose that f_{i_0} is weakly mixing. Let $\Omega_1,\Omega_2,\Sigma_1$ and Σ_2 be nonempty open subsets of $\prod_{i=1}^m X_i$. Then, for all $i\in\{1,\ldots,m\}$, there exists nonempty open subsets U_i^1,U_i^2,V_i^1 and V_i^2 of X_i such that $\mathcal{U}_1=\prod_{i=1}^m U_i^1\subseteq\Omega_1$, $\mathcal{U}_2=\prod_{i=1}^m U_i^2\subseteq\Omega_2$, $\mathcal{V}_1=\prod_{i=1}^m V_i^1\subseteq\Sigma_1$ and $\mathcal{V}_2=\prod_{i=1}^m V_i^2\subseteq\Sigma_2$. Since for each $i\in\{1,\ldots,m\}\backslash\{i_0\}$, f_i is mixing, there exists $N_i^1,N_i^2\in\mathbb{N}$ such that $f_i^{k_1}(U_i^1)\cap V_i^1\neq\emptyset$ and $f_i^{k_2}(U_i^2)\cap V_i^2\neq\emptyset$, for all $k_1\geq N_i^1$ and $k_2\geq N_i^2$. Let $N=\max\{N_i^1,N_i^2 : i\in\{1,\ldots,m\}\backslash\{i_0\}\}$. Since $f_{i_0}^N$ is surjective and continuous, we have that $f_{i_0}^{-N}(V_{i_0}^1)$ and $f_{i_0}^{-N}(V_{i_0}^2)$ are nonempty open subsets of X_{i_0}. By hypothesis, there exists $l\in\mathbb{N}$ such that $f_{i_0}^l(U_{i_0}^1)\cap f_{i_0}^{-N}(V_{i_0}^1)\neq\emptyset$ and $f_{i_0}^l(U_{i_0}^2)\cap f_{i_0}^{-N}(V_{i_0}^2)\neq\emptyset$. This implies that $f_{i_0}^{N+l}(U_{i_0}^1)\cap V_{i_0}^1\neq\emptyset$ and $f_{i_0}^{N+l}(U_{i_0}^2)\cap V_{i_0}^2\neq\emptyset$. On the other hand, for each $i\in\{1,\ldots,m\}\backslash\{i_0\}$, $f_i^{N+l}(U_i^1)\cap V_i^1\neq\emptyset$ and $f_i^{N+l}(U_i^2)\cap V_i^2\neq\emptyset$. Consequently, there exist $(x_1,\ldots,x_m)\in(\prod_{i=1}^m f_i)^{N+l}(\mathcal{U}_1)\cap\mathcal{V}_1$ and $(y_1,\ldots,y_m)\in(\prod_{i=1}^m f_i)^{N+l}(\mathcal{U}_2)\cap\mathcal{V}_2$ and thus,

$(\prod_{i=1}^{m} f_i)^{N+l}(\Omega_1)\cap\Sigma_1 \neq \emptyset$ and $(\prod_{i=1}^{m} f_i)^{N+l}(\Omega_2)\cap\Sigma_2 \neq \emptyset$. Therefore, $\prod_{i=1}^{m} f_i$ is weakly mixing.

Suppose that f_{i_0} is totally transitive. Let $s \in \mathbb{N}$, and let Ω_1 and Ω_2 be nonempty open subsets of $\prod_{i=1}^{m} X_i$. Then, for each $i \in \{1,\dots,m\}$, there exist nonempty open subsets U_i and V_i of X_i such that $\mathcal{U} = \prod_{i=1}^{m} U_i \subseteq \Omega_1$ and $\mathcal{V} = \prod_{i=1}^{m} V_i \subseteq \Omega_2$. Since for all $i \in \{1,\dots,m\}\backslash\{i_0\}$, f_i is mixing, there exists $N_i \in \mathbb{N}$ such that $f_i^k(U_i) \cap V_i \neq \emptyset$, for each $k \geq N_i$. Let $N = \max\{N_i : i \in \{1,\dots,m\}\backslash\{i_0\}\}$. It is clear that $f_{i_0}^{sN}$ is surjective and continuous and thus, $f_{i_0}^{-sN}(V_{i_0})$ is a nonempty open subset of X_{i_0}. By hypothesis, since $f_{i_0}^{s}$ is transitive, there exists $l \in \mathbb{N}$ such that $(f_{i_0}^{s})^l(U_{i_0})\cap f_{i_0}^{-sN}(V_{i_0}) \neq \emptyset$. Consequently, $f_{i_0}^{sN+sl}(U_{i_0})\cap V_{i_0} \neq \emptyset$. On the other hand, for all $i \in \{1,\dots,m\}\backslash\{i_0\}$, $f_i^{sN+sl}(U_i)\cap V_i \neq \emptyset$. This implies that, there exists $(x_1,\dots,x_m) \in (\prod_{i=1}^{m} f_i^{sN+sl}(U_i)) \cap (\prod_{i=1}^{m} V_i) = (\prod_{i=1}^{m} f_i)^{sN+sl}(\mathcal{U})\cap\mathcal{V}$ and thus, $((\prod_{i=1}^{m} f_i)^s)^{N+l}(\mathcal{U})\cap\mathcal{V} \neq \emptyset$. Finally, $((\prod_{i=1}^{m} f_i)^s)^{N+l}(\Omega_1)\cap\Omega_2 \neq \emptyset$. Therefore, $(\prod_{i=1}^{m} f_i)^s$ is transitive and thus, $\prod_{i=1}^{m} f_i$ is totally transitive.

Suppose that f_{i_0} is TT_{++}. Let Ω_1 and Ω_2 be nonempty open subsets of $\prod_{i=1}^{m} X_i$. Then, for all $i \in \{1,\dots,m\}$, there exist nonempty open subsets U_i and V_i of X_i such that $\mathcal{U} = \prod_{i=1}^{m} U_i \subseteq \Omega_1$ and $\mathcal{V} = \prod_{i=1}^{m} V_i \subseteq \Omega_2$. Since for all $i \in \{1,\dots,m\}\backslash\{i_0\}$, f_i is mixing, there exists $N_i \in \mathbb{N}$ such that $f_i^k(U_i)\cap V_i \neq \emptyset$, for all $k \geq N_i$. Let $N = \max\{N_i : i \in \{1,\dots,m\}\backslash\{i_0\}\}$. Notice that f_{i_0} is surjective and continuous and thus, $f_{i_0}^{-N}(V_{i_0})$ is a nonempty open subset of X_{i_0}. Furthermore, since f_{i_0} is TT_{++}, we have that $n_{f_{i_0}}(U_{i_0}, f_{i_0}^{-N}(V_{i_0}))$ is infinite. Let $l \in n_{f_{i_0}}(U_{i_0}, f_{i_0}^{-N}(V_{i_0}))$. Then, $f_{i_0}^{l}(U_{i_0}) \cap f_{i_0}^{-N}(V_{i_0}) \neq \emptyset$. Consequently, $f_{i_0}^{N+l}(U_{i_0}) \cap V_{i_0} \neq \emptyset$. On the other hand, for all $i \in \{1,\dots,m\}\backslash\{i_0\}$, $f_i^{N+l}(U_i) \cap V_i \neq \emptyset$. By the above, $(\prod_{i=1}^{m} f_i)^{N+l}(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$ and thus, $N + l \in n_{\prod_{i=1}^{m} f_i}(\Omega_1,\Omega_2)$. Finally, since $n_{f_{i_0}}(U_{i_0}, f_{i_0}^{-N}(V_{i_0}))$ is infinite, we have that $n_{\prod_{i=1}^{m} f_i}(\Omega_1,\Omega_2)$ is infinite. Therefore, $\prod_{i=1}^{m} f_i$ is TT_{++}. □

As a consequence of [18, Theorem 3.15] and Theorem 4.6, we have the following statements.

THEOREM 4.7. *For each $i \in \{1,\dots,m\}$, let (X_i, f_i) be a dynamical system, let $i_0 \in \{1,\dots,m\}$, let f_{i_0} be surjective and continuous, and for each $i \in \{1,\dots,m\}\backslash\{i_0\}$, let f_i be mixing and $Per(f_i)$ is dense in X_i. If f_{i_0} is chaotic or an F-system, then $\prod_{i=1}^{m} f_i$ is chaotic or an F-system, respectively.*

THEOREM 4.8. *For each $i \in \{1,\dots,m\}$, let (X_i, f_i) be a dynamical system, let $i_0 \in \{1,\dots,m\}$, let f_{i_0} be surjective, continuous and Touhey, and for each $i \in \{1,\dots,m\}\backslash\{i_0\}$, let f_i be continuous and mixing and $Per(f_i)$ is dense in X_i. Then $\prod_{i=1}^{m} f_i$ is Touhey.*

PROOF. Let Ω_1 and Ω_2 be nonempty open subsets of $\prod_{i=1}^{m} X_i$. Then, for all $i \in \{1,\dots,m\}$, there exist nonempty open subsets U_i and V_i of

X_i such that $\mathcal{U} = \prod_{i=1}^m U_i \subseteq \Omega_1$ and $\mathcal{V} = \prod_{i=1}^m V_i \subseteq \Omega_2$. Since for every $i \in \{1,\dots,m\}\backslash\{i_0\}$, f_i is mixing, there exists $N_i \in \mathbb{N}$ such that $f_i^k(U_i) \cap V_i \neq \emptyset$, for all $k \geq N_i$. Let $N = \max\{N_i : i \in \{1,\dots,m\}\backslash\{i_0\}\}$. Notice that $f_{i_0}^N$ is surjective and continuous and thus, $f_{i_0}^{-N}(V_{i_0})$ is a nonempty open subset of X_{i_0}. By hypothesis, there exist a periodic point $x_{i_0} \in U_{i_0}$ and $k_{i_0} \in \mathbb{Z}_+$ such that $f_{i_0}^{k_{i_0}}(x_{i_0}) \in f_{i_0}^{-N}(V_{i_0})$. It follows that, $f_{i_0}^{N+k_{i_0}}(x_{i_0}) \in V_{i_0}$. On the other hand, for each $i \in \{1,\dots,m\}\backslash\{i_0\}$, $f_i^{N+k_{i_0}}(U_i) \cap V_i \neq \emptyset$. Consequently, for each $i \in \{1,\dots,m\}\backslash\{i_0\}$, $U_i \cap f_i^{-(N+k_{i_0})}(V_i)$ is a nonempty open subset of X_i. Even more, since for all $i \in \{1,\dots,m\}\backslash\{i_0\}$, $Per(f_i)$ is dense in X_i, we have that there exists $x_i \in U_i \cap f_i^{-(N+k_{i_0})}(V_i)$ such that $x_i \in Per(f_i)$. Finally, $(x_1,\dots,x_m) \in \Omega_1$, $(\prod_{i=1}^m f_i)^{N+k_{i_0}}((x_1,\dots,x_m)) \in \mathcal{V} \subseteq \Omega_2$ and by [18, Theorem 3.3], part (4), $(x_1,\dots,x_m)$ is a periodic point of $\prod_{i=1}^m f_i$. Therefore, $\prod_{i=1}^m f_i$ is Touhey. □

5. Induced functions to n-fold symmetric products of product spaces

Hyperspace theory started about 1900, with the work of F. Hausdorff [11] and L. Vietoris [20]. Nowadays hyperspaces are widely studied, mainly in continuum theory [13, 14, 17].

Given a topological space (X,τ) and a positive integer n, we define the *n-fold symmetric product of X* by:

$$\mathcal{F}_n(X) = \{A \subseteq X \ : \ A \neq \emptyset \quad \text{and} \quad A \quad \text{has at most } n \text{ elements}\}.$$

This hyperspace is considered with the Vietoris topology [17]. Next we describe this topology.

Let (X,τ) be a topological space. Given a finite collection of nonempty subsets $U_1,\dots,U_k$ of X, we denote by $\langle U_1,\dots,U_k\rangle$ the subset of $\mathcal{F}_n(X)$:

$$\left\{A \in \mathcal{F}_n(X) \ : \ A \subseteq \bigcup_{i=1}^k U_i \quad \text{and} \quad A \cap U_i \neq \emptyset, \quad \text{for each} \quad i \in \{1,\dots,k\}\right\}.$$

The family:

$$\mathcal{B} = \{\langle U_1,\dots,U_k\rangle \mid U_i \in \tau, \quad \text{for each} \quad i \in \{1,\dots,k\} \text{ and } k \in \mathbb{N}\}$$

forms a basis for a topology on $\mathcal{F}_n(X)$ which is denoted by τ_V and called the *Vietoris topology.*

Let n be a positive integer and let X be a topological space. If $f : X \to X$ is a function, we consider the function $\mathcal{F}_n(f) : \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ defined by $\mathcal{F}_n(f)(A) = f(A)$, for all $A \in \mathcal{F}_n(X)$; it is called the *induced map* of f on the n-fold symmetric product of X.

In addition, if n is an integer greater than or equal to two, we define the *n-fold symmetric product suspension of X*, $\mathcal{SF}_n(X)$, as the quotient space $\mathcal{F}_n(X)/\mathcal{F}_1(X)$, and the induced map $\mathcal{SF}_n(f) : \mathcal{SF}_n(X) \to \mathcal{SF}_n(X)$.

In this section we study relationships between the functions $\mathcal{F}_n(\prod_{i=1}^m f_i)$, $\mathcal{F}_n(f_i)$ and f_i, for all $i \in \{1, \ldots, m\}$, when any of them is AAN exact, fully exact, AAN strongly transitive, very strongly transitive, exact transitive, strongly exact transitive or strongly product transitive.

Let X be a continuum (nonempty compact connected metric space), let $f : X \to X$ be a continuous function, and let $\mathcal{M}$ be one of the following classes of functions: fully exact, AAN strongly transitive, very strongly transitive, exact transitive, strongly exact transitive or strongly product transitive. Barragán et al. [5] analyzed the relationships between the following statements: (1) $f \in \mathcal{M}$ and (2) $\mathcal{F}_n(f) \in \mathcal{M}$, via the function $\mathcal{SF}_n(f)$.

THEOREM 5.1. *Let (X, f) be a dynamical system, let $n \in \mathbb{N}$, and let $\mathcal{M}$ be one of the following classes of functions: fully exact, AAN strongly transitive, very strongly transitive, exact transitive, strongly exact transitive or strongly product transitive. If $\mathcal{F}_n(f) \in \mathcal{M}$, then $f \in \mathcal{M}$.*

PROOF. Suppose that $\mathcal{F}_n(f)$ is fully exact, and let U and V be nonempty open subsets of X. Then, $\langle U \rangle$ and $\langle V \rangle$ are nonempty open subsets of $\mathcal{F}_n(X)$. By hypothesis, there exists $k \in \mathbb{N}$ such that:

$$\text{int}((\mathcal{F}_n(f))^k(\langle U \rangle) \cap (\mathcal{F}_n(f))^k(\langle V \rangle)) \neq \emptyset.$$

Hence, there exists a nonempty open subset $\mathcal{U}$ of $\mathcal{F}_n(X)$ such that $\mathcal{U} \subseteq (\mathcal{F}_n(f))^k(\langle U \rangle) \cap (\mathcal{F}_n(f))^k(\langle V \rangle)$. By [4, Theorem 3.2], $\bigcup \mathcal{U}$ is a nonempty open subset of X. Let $x \in \bigcup \mathcal{U}$. Then, there exists $B \in \mathcal{U}$ such that $x \in B$. This implies that, there exist $C_1 \in \langle U \rangle$ and $C_2 \in \langle V \rangle$ such that $(\mathcal{F}_n(f))^k(C_1) = B$ and $(\mathcal{F}_n(f))^k(C_2) = B$. Consequently, $x \in f^k(U) \cap f^k(V)$, that is, $\bigcup \mathcal{U} \subseteq f^k(U) \cap f^k(V)$. Therefore, $\text{int}(f^k(U) \cap f^k(V)) \neq \emptyset$ and thus, f is fully exact.

Suppose that $\mathcal{F}_n(f)$ is AAN strongly transitive, and let U be a nonempty open subset of X. Then, $\langle U \rangle$ is a nonempty open subset of $\mathcal{F}_n(X)$. By hypothesis, $\bigcup_{k=1}^{\infty} (\mathcal{F}_n(f))^k(\langle U \rangle) = \mathcal{F}_n(X)$. Let $x \in X$. Hence, there exists $k_1 \in \mathbb{N}$ such that $\{x\} \in (\mathcal{F}_n(f))^{k_1}(\langle U \rangle)$. This implies that $x \in f^{k_1}(U)$. Therefore, $x \in \bigcup_{k=1}^{\infty} f^k(U)$ and thus, f is AAN strongly transitive.

The proof for very strongly transitivity is similar to that given for AAN strongly transitivity.

Suppose that $\mathcal{F}_n(f)$ is exact transitive, and let U and V be nonempty open subsets of X. Then $\langle U \rangle$ and $\langle V \rangle$ are nonempty open subsets of $\mathcal{F}_n(X)$. By hypothesis, $\bigcup_{k=1}^{\infty} ((\mathcal{F}_n(f))^k(\langle U \rangle) \cap (\mathcal{F}_n(f))^k(\langle V \rangle))$ is dense in $\mathcal{F}_n(X)$. Let W be a nonempty open subset of X. Since $\langle W \rangle$ is a nonempty open subset of $\mathcal{F}_n(X)$, we have that $\mathcal{A} = (\bigcup_{k=1}^{\infty} ((\mathcal{F}_n(f))^k(\langle U \rangle) \cap (\mathcal{F}_n(f))^k(\langle V \rangle))) \cap \langle W \rangle \neq \emptyset$. Let $A \in \mathcal{A}$. Then, there exist $k_1 \in \mathbb{N}$, $C_1 \in \langle U \rangle$ and $C_2 \in \langle V \rangle$ such that $(\mathcal{F}_n(f))^{k_1}(C_1) = A$ and $(\mathcal{F}_n(f))^{k_1}(C_2) = A$, that is to say, $f^{k_1}(C_1) = A$ and $f^{k_1}(C_2) = A$. Finally, for all $a \in A$, we have that $a \in (f^{k_1}(U) \cap f^{k_1}(V)) \cap W$.

This implies that $(\bigcup_{k=1}^{\infty}(f^k(U)\cap f^k(V)))\cap W\neq\emptyset$. Therefore, $\bigcup_{k=1}^{\infty}(f^k(U)\cap f^k(V))$ is dense in X and thus, f is exact transitive.

Suppose that $\mathcal{F}_n(f)$ is strongly exact transitive, and let U and V be nonempty open subsets of X. Then $\langle U\rangle$ and $\langle V\rangle$ are nonempty open subsets of $\mathcal{F}_n(X)$. By hypothesis, $\bigcup_{k=1}^{\infty}((\mathcal{F}_n(f))^k(\langle U\rangle)\cap(\mathcal{F}_n(f))^k(\langle V\rangle))=\mathcal{F}_n(X)$. Let $x\in X$. Since $\{x\}\in\mathcal{F}_n(X)$, there exists $k_1\in\mathbb{N}$ such that $\{x\}\in(\mathcal{F}_n(f))^{k_1}(\langle U\rangle)\cap(\mathcal{F}_n(f))^{k_1}(\langle V\rangle)$. This implies that there exist $C_1\in\langle U\rangle$ and $C_2\in\langle V\rangle$ such that $(\mathcal{F}_n(f))^{k_1}(C_1)=\{x\}$ and $(\mathcal{F}_n(f))^{k_1}(C_2)=\{x\}$. Hence, $f^{k_1}(C_1)=\{x\}$ and $f^{k_1}(C_2)=\{x\}$. In consequence, $x\in f^{k_1}(U)\cap f^{k_1}(V)$. Finally, $x\in\bigcup_{k=1}^{\infty}(f^k(U)\cap f^k(V))$. Therefore, $\bigcup_{k=1}^{\infty}(f^k(U)\cap f^k(V))=X$ and thus, f is strongly exact transitive.

Suppose that $\mathcal{F}_n(f)$ is strongly product transitive, and let $k\in\mathbb{Z}_+$. Let Ω be a nonempty open subset of $X^{\times k}$. Then, there exist nonempty open subsets $U_1,\dots,U_k$ of X such that $\mathcal{U}=\prod_{i=1}^{k}U_i\subseteq\Omega$. Note that $\langle U_1\rangle\times\cdots\times\langle U_k\rangle$ is a nonempty open subset of $(\mathcal{F}_n(X))^{\times k}$. Since $(\mathcal{F}_n(f))^{\times k}$ is AAN strongly transitive, we have that $\bigcup_{s=1}^{\infty}((\mathcal{F}_n(f))^{\times k})^s(\langle U_1\rangle\times\cdots\times\langle U_k\rangle)=(\mathcal{F}_n(X))^{\times k}$. Let $(x_1,\dots,x_k)\in X^{\times k}$. It follows that there exists $s_1\in\mathbb{N}$ such that $(\{x_1\},\dots,\{x_k\})\in((\mathcal{F}_n(f))^{\times k})^{s_1}(\langle U_1\rangle\times\cdots\times\langle U_k\rangle)$. Consequently, there exists $(A_1,\dots,A_k)\in\langle U_1\rangle\times\cdots\times\langle U_k\rangle$ such that $((\mathcal{F}_n(f))^{\times k})^{s_1}(A_1,\dots,A_k)=(\{x_1\},\dots,\{x_k\})$, that is, $((\mathcal{F}_n(f))^{s_1})^{\times k}(A_1,\dots,A_k)=(\{x_1\},\dots,\{x_k\})$. In consequence, for each $i\in\{1,\dots,k\}$, $f^{s_1}(A_i)=\{x_i\}$, and thus, there exists $a_i\in A_i$ such that $f^{s_1}(a_i)=x_i$. Even more, since $(a_1,\dots,a_k)\in\mathcal{U}$, we have that $(x_1,\dots,x_k)\in(f^{s_1})^{\times k}(\mathcal{U})=(f^{\times k})^{s_1}(\mathcal{U})$. Then, $(x_1,\dots,x_k)\in\bigcup_{s=1}^{\infty}(f^{\times k})^s(\mathcal{U})$ and thus, $\bigcup_{s=1}^{\infty}(f^{\times k})^s(\mathcal{U})=X^{\times k}$. Finally, $f^{\times k}$ is AAN strongly transitive. Therefore, f is strongly product transitive. ∎

Also, in [5, Theorem 4.6], Barragán et al. proved that: f is AAN exact if and only if $\mathcal{F}_n(f)$ is AAN exact, for continua and continuous functions. Actually, this result continue to hold true for topological spaces and not necessarily continuous functions. The proof of Theorem 5.2 is similar to that given in [5, Theorem 4.6].

Theorem 5.2. *Let (X,f) be a dynamical system, and let $n\in\mathbb{N}$. Then $\mathcal{F}_n(f)$ is AAN exact if and only if f is AAN exact.*

Now, let us consider the function $\mathcal{F}_n(\prod_{i=1}^{m}f_i)$.

Remark 5.3. For each $i\in\{1,\dots,m\}$, let (X_i,f_i) be a dynamical system, and let $k\in\mathbb{N}$. It is not difficult to prove that the kth iteration of the induced function $\mathcal{F}_n(\prod_{i=1}^{m}f_i)$ is equal to the induced function of the product function $\prod_{i=1}^{m}f_i^k$. That is, $[\mathcal{F}_n(\prod_{i=1}^{m}f_i)]^k=\mathcal{F}_n(\prod_{i=1}^{m}f_i^k)$.

Theorem 5.4. *For each $i\in\{1,\dots,m\}$, let (X_i,f_i) be a dynamical system, and let $n\in\mathbb{N}$. Then the following hold.*

1. *$\mathcal{F}_n(\prod_{i=1}^{m}f_i)$ is AAN exact if and only if, for each $i\in\{1,\dots,m\}$, $\mathcal{F}_n(f_i)$ is AAN exact.*

2. *$\mathcal{F}_n(\prod_{i=1}^m f_i)$ is AAN exact if and only if, for each $i \in \{1, \ldots, m\}$, f_i is AAN exact.*

PROOF. Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is AAN exact. Let $i_0 \in \{1, \ldots, m\}$, and let $\mathcal{U}$ and $\mathcal{V}$ be nonempty open subsets of $\mathcal{F}_n(X_{i_0})$. By [12, Lemma 4.2], there exist nonempty open subsets $U_{i_0}^1, \ldots, U_{i_0}^n$, $V_{i_0}^1, \ldots, V_{i_0}^n$ of X_{i_0} such that $\langle U_{i_0}^1, \ldots, U_{i_0}^n \rangle \subseteq \mathcal{U}$ and $\langle V_{i_0}^1, \ldots, V_{i_0}^n \rangle \subseteq \mathcal{V}$. For all $i \in \{1, \ldots, m\} \setminus \{i_0\}$ and every $j \in \{1, \ldots, n\}$, let $U_i^j = X_i$ and $V_i^j = X_i$. Notice that for each $j \in \{1, \ldots, n\}$, $U_j' = \prod_{i=1}^m U_i^j$ and $V_j' = \prod_{i=1}^m V_i^j$ are nonempty open subsets of $\prod_{i=1}^m X_i$. Then, $\langle U_1', \ldots, U_n' \rangle$ and $\langle V_1', \ldots, V_n' \rangle$ are nonempty open subsets of $\mathcal{F}_n(\prod_{i=1}^m X_i)$. By hypothesis, there exists $k \in \mathbb{N}$ such that $\mathcal{W} = (\mathcal{F}_n(\prod_{i=1}^m f_i))^k(\langle U_1', \ldots, U_n' \rangle) \cap (\mathcal{F}_n(\prod_{i=1}^m f_i))^k(\langle V_1', \ldots, V_n' \rangle) \neq \emptyset$. Let $\mathcal{X} = \{(x_1^l, \ldots, x_m^l) : l \in \{1, \ldots, p\} \text{ with } p \leq n\} \in \mathcal{W}$. Thus, there exist

$$\mathcal{A} = \{(a_1^{r_1}, \ldots, a_m^{r_1}) : r_1 \in \{1, \ldots, p_1\} \text{ with } p_1 \leq n\} \in \langle U_1', \ldots, U_n' \rangle$$

and

$$\mathcal{B} = \{(b_1^{r_2}, \ldots, b_m^{r_2}) : r_2 \in \{1, \ldots, p_2\} \text{ with } p_2 \leq n\} \in \langle V_1', \ldots, V_n' \rangle$$

such that $(\mathcal{F}_n(\prod_{i=1}^m f_i))^k(\mathcal{A}) = \mathcal{X}$ and $(\mathcal{F}_n(\prod_{i=1}^m f_i))^k(\mathcal{B}) = \mathcal{X}$, that is,

$$\{(f_1^k(a_1^{r_1}), \ldots, f_m^k(a_m^{r_1})) : r_1 \in \{1, \ldots, p_1\} \text{ with } p_1 \leq n\} = \mathcal{X}$$

and

$$\{(f_1^k(b_1^{r_2}), \ldots, f_m^k(b_m^{r_2})) : r_2 \in \{1, \ldots, p_2\} \text{ with } p_2 \leq n\} = \mathcal{X}.$$

Then, $\{f_{i_0}^k(a_{i_0}^1), \ldots, f_{i_0}^k(a_{i_0}^{p_1})\} = \{x_{i_0}^1, \ldots, x_{i_0}^p\}$ and $\{f_{i_0}^k(b_{i_0}^1), \ldots, f_{i_0}^k(b_{i_0}^{p_2})\} = \{x_{i_0}^1, \ldots, x_{i_0}^p\}$, that is to say, $(\mathcal{F}_n(f_{i_0}))^k(\{a_{i_0}^1, \ldots, a_{i_0}^{p_1}\}) = \{x_{i_0}^1, \ldots, x_{i_0}^p\}$ and $(\mathcal{F}_n(f_{i_0}))^k(\{b_{i_0}^1, \ldots, b_{i_0}^{p_2}\}) = \{x_{i_0}^1, \ldots, x_{i_0}^p\}$. On the other hand, by [18, Lemma 5.2], for each $i \in \{1, \ldots, m\}$, $\{a_i^1, \ldots, a_i^{p_1}\} \in \langle U_i^1, \ldots, U_i^n \rangle$ and $\{b_i^1, \ldots, b_i^{p_2}\} \in \langle V_i^1, \ldots, V_i^n \rangle$. Finally, we have that:

$$\{x_{i_0}^1, \ldots, x_{i_0}^p\} \in (\mathcal{F}_n(f_{i_0}))^k(\langle U_{i_0}^1, \ldots, U_{i_0}^n \rangle) \cap (\mathcal{F}_n(f_{i_0}))^k(\langle V_{i_0}^1, \ldots, V_{i_0}^n \rangle).$$

Therefore, $(\mathcal{F}_n(f_{i_0}))^k(\mathcal{U}) \cap (\mathcal{F}_n(f_{i_0}))^k(\mathcal{V}) \neq \emptyset$ and thus, $\mathcal{F}_n(f_{i_0})$ is AAN exact.

Suppose that for each $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(f_i)$ is AAN exact. Then, by Theorem 5.2, for all $i \in \{1, \ldots, m\}$, f_i is AAN exact and thus, by Theorem 4.1, $\prod_{i=1}^m f_i$ is AAN exact. Finally, by Theorem 5.2, we obtain that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is AAN exact.

The other implications follows from Theorems 4.1 and 5.2. ∎

LEMMA 5.5. *For each $i \in \{1, \ldots, m\}$, let (X_i, f_i) be a dynamical system, let $l, n, k \in \mathbb{N}$ such that $l \leq n$, for each $i \in \{1, \ldots, m\}$, let $U_i^1, \ldots, U_i^n$ be nonempty open subsets of X_i, and for each $j \in \{1, \ldots, l\}$, let $(x_1^j, \ldots, x_m^j) \in \prod_{i=1}^m X_i$. If*

$$\left\{(x_1^j, \ldots, x_m^j) : j \in \{1, \ldots, l\}\right\} \in \left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^k \left(\left\langle \prod_{i=1}^m U_i^1, \ldots, \prod_{i=1}^m U_i^n \right\rangle\right),$$

then, for each $i \in \{1,\ldots,m\}$, $\{x_i^1,\ldots,x_i^l\} \in (\mathcal{F}_n(f_i))^k(\langle U_i^1,\ldots,U_i^n\rangle)$.

PROOF. Let $\mathcal{F} = \left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^k\left(\left\langle \prod_{i=1}^m U_i^1,\ldots,\prod_{i=1}^m U_i^n\right\rangle\right)$. Suppose that $\mathcal{X} = \{(x_1^j,\ldots,x_m^j) : j \in \{1\ldots,l\}\} \in \mathcal{F}$. Then, there exists

$$\mathcal{A} = \{(a_1^s,\ldots,a_m^s) : s \in \{1,\ldots,p\} \text{ with } p \le n\} \in \left\langle \prod_{i=1}^m U_i^1,\ldots,\prod_{i=1}^m U_i^n \right\rangle$$

such that $\left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^k(\mathcal{A}) = \mathcal{X}$. It follows that $\{(f_1^k(a_1^s),\ldots,f_m^k(a_m^s)) : s \in \{1,\ldots,p\} \text{ with } p \le n\} = \mathcal{X}$. In consequence, for each $i \in \{1,\ldots,m\}$, $\{f_i^k(a_i^1),\ldots,f_i^k(a_i^p)\} = \{x_i^1,\ldots,x_i^l\}$ and thus,

$$(\mathcal{F}_n(f_i))^k(\{a_i^1,\ldots,a_i^p\}) = \{x_i^1,\ldots,x_i^l\}.$$

Even more, by [18, Lemma 5.2], for each $i \in \{1,\ldots,m\}$, $\{a_i^1,\ldots,a_i^p\} \in \langle U_i^1,\ldots,U_i^n\rangle$. Therefore, $\{x_i^1,\ldots,x_i^l\} \in (\mathcal{F}_n(f_i))^k(\langle U_i^1,\ldots,U_i^n\rangle)$, for every $i \in \{1,\ldots,m\}$. ☐

THEOREM 5.6. *For each* $i \in \{1,\ldots,m\}$, *let* (X_i, f_i) *be a dynamical system, let* $n \in \mathbb{N}$, *and let* $\mathcal{M}$ *be one of the following classes of functions: AAN strongly transitive, very strongly transitive, exact transitive or strongly exact transitive. If* $\mathcal{F}_n(\prod_{i=1}^m f_i) \in \mathcal{M}$, *then for each* $i \in \{1,\ldots,m\}$, $\mathcal{F}_n(f_i) \in \mathcal{M}$.

PROOF. Let $i_0 \in \{1,\ldots,m\}$, and let $\mathcal{U}$ and $\mathcal{V}$ be nonempty open subsets of $\mathcal{F}_n(X_{i_0})$. By [12, Lemma 4.2] there exist nonempty open subsets $U_{i_0}^1,\ldots,U_{i_0}^n, V_{i_0}^1,\ldots,V_{i_0}^n$ of X_{i_0} such that $\mathcal{T}_1 = \langle U_{i_0}^1,\ldots,U_{i_0}^n\rangle \subseteq \mathcal{U}$ and $\mathcal{T}_2 = \langle V_{i_0}^1,\ldots,V_{i_0}^n\rangle \subseteq \mathcal{V}$. Now, for each $i \in \{1,\ldots,m\}\backslash\{i_0\}$ and all $j \in \{1,\ldots,n\}$, let $U_i^j = X_i$ and $V_i^j = X_i$. Finally, for each $j \in \{1,\ldots,n\}$, let $U_j' = \prod_{i=1}^m U_i^j$ and $V_j' = \prod_{i=1}^m V_i^j$. Note that $\mathcal{F}_1 = \langle U_1',\ldots,U_n'\rangle$ and $\mathcal{F}_2 = \langle V_1',\ldots,V_n'\rangle$ are nonempty open subsets of $\mathcal{F}_n(\prod_{i=1}^m X_i)$.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is AAN strongly transitive. Then, we have that $\bigcup_{k=1}^\infty \left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^k(\mathcal{F}_1) = \mathcal{F}_n\left(\prod_{i=1}^m X_i\right)$. Let $\{z_{i_0}^1,\ldots,z_{i_0}^l\} \in \mathcal{F}_n(X_{i_0})$, for all $i \in \{1,\ldots,m\}\backslash\{i_0\}$ and every $j \in \{1,\ldots,l\}$, let $z_i^j \in X_i$, and for each $j \in \{1,\ldots,l\}$, let $x_j' = (z_1^j,\ldots,z_m^j)$. Note that $\{x_1',\ldots,x_l'\} \in \mathcal{F}_n(\prod_{i=1}^m X_i)$. Then, there exists $k_1 \in \mathbb{N}$ such that $\{x_1',\ldots,x_l'\} \in \left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^{k_1}(\mathcal{F}_1)$. By Lemma 5.5, $\{z_{i_0}^1,\ldots,z_{i_0}^l\} \in (\mathcal{F}_n(f_{i_0}))^{k_1}(\mathcal{T}_1)$. Hence:

$$\mathcal{F}_n(X_{i_0}) = (\mathcal{F}_n(f_{i_0}))^{k_1}(\mathcal{U}) \subseteq \bigcup_{k=1}^\infty (\mathcal{F}_n(f_{i_0}))^k(\mathcal{U}).$$

Therefore, $\mathcal{F}_n(X_{i_0}) = \bigcup_{k=1}^\infty (\mathcal{F}_n(f_{i_0}))^k(\mathcal{U})$ and thus, $\mathcal{F}_n(f_{i_0})$ is AAN strongly transitive.

The proof for very strongly transitivity is similar to that given for AAN strongly transitivity.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is exact transitive. Thus:

$$\bigcup_{k=1}^{\infty}\left(\left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^k(\mathcal{F}_1)\cap\left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^k(\mathcal{F}_2)\right)$$

is dense in $\mathcal{F}_n(\prod_{i=1}^m X_i)$. Let $\mathcal{W}$ be a nonempty open subset of $\mathcal{F}_n(X_{i_0})$. By [12, Lemma 4.2], there exist nonempty open subsets $W_{i_0}^1,\ldots,W_{i_0}^n$ of X_{i_0} such that $\langle W_{i_0}^1,\ldots,W_{i_0}^n\rangle\subseteq\mathcal{W}$. Now, for all $i\in\{1,\ldots,m\}\setminus\{i_0\}$ and each $j\in\{1,\ldots,n\}$, let $W_i^j=X_i$. Notice that for each $j\in\{1,\ldots,n\}$, $W'_j=\prod_{i=1}^m W_i^j$ is a nonempty open subset of $\prod_{i=1}^m X_i$ and thus, $\mathcal{F}_3=\langle W'_1,\ldots,W'_n\rangle$ is a nonempty open subset of $\mathcal{F}_n(\prod_{i=1}^m X_i)$. Then, there exists $k_1\in\mathbb{N}$ such that $\mathcal{A}=\Big((\mathcal{F}_n(\prod_{i=1}^m f_i))^{k_1}(\mathcal{F}_1)\cap(\mathcal{F}_n(\prod_{i=1}^m f_i))^{k_1}(\mathcal{F}_2)\Big)\cap\mathcal{F}_3\neq\emptyset$. Let $\{(z_1^l,\ldots,z_m^l):l\in\{1,\ldots,p\}\text{ with } p\leq n\}\in\mathcal{A}$. By Lemma 5.5 and [18, Lemma 5.2], we have that:

$$\{z_{i_0}^1,\ldots,z_{i_0}^p\}\in(\mathcal{F}_n(f_{i_0}))^{k_1}(\mathcal{T}_1)\cap(\mathcal{F}_n(f_{i_0}))^{k_1}(\mathcal{T}_2)\cap\langle W_{i_0}^1,\ldots,W_{i_0}^n\rangle.$$

Finally, $\big((\mathcal{F}_n(f_{i_0}))^{k_1}(\mathcal{U})\cap(\mathcal{F}_n(f_{i_0}))^{k_1}(\mathcal{V})\big)\cap\mathcal{W}\neq\emptyset$ and thus,

$$\left(\bigcup_{k=1}^{\infty}((\mathcal{F}_n(f_{i_0}))^k(\mathcal{U})\cap(\mathcal{F}_n(f_{i_0}))^k(\mathcal{V}))\right)\cap\mathcal{W}\neq\emptyset.$$

Therefore, $\bigcup_{k=1}^{\infty}((\mathcal{F}_n(f_{i_0}))^k(\mathcal{U})\cap(\mathcal{F}_n(f_{i_0}))^k(\mathcal{V}))$ is dense in $\mathcal{F}_n(X_{i_0})$ and thus, $\mathcal{F}_n(f_{i_0})$ is exact transitive.

Suppose that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is strongly exact transitive. Then,

$$\bigcup_{k=1}^{\infty}\left(\left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^k(\mathcal{F}_1)\cap\left(\mathcal{F}_n\left(\prod_{i=1}^m f_i\right)\right)^k(\mathcal{F}_2)\right)=\mathcal{F}_n\left(\prod_{i=1}^m X_i\right).$$

Let $\{x_{i_0}^1,\ldots,x_{i_0}^l\}\in\mathcal{F}_n(X_{i_0})$, for each $i\in\{1,\ldots,m\}\setminus\{i_0\}$ and all $j\in\{1,\ldots,n\}$, let $x_i^j\in X_i$. It follows that $\{(x_1^p,\ldots,x_m^p):p\in\{1,\ldots,l\}\}\in\mathcal{F}_n(\prod_{i=1}^m X_i)$ and thus, there exists $k_1\in\mathbb{N}$ such that $\{(x_1^p,\ldots,x_m^p):p\in\{1,\ldots,l\}\}\in(\mathcal{F}_n(\prod_{i=1}^m f_i))^{k_1}(\mathcal{F}_1)\cap(\mathcal{F}_n(\prod_{i=1}^m f_i))^{k_1}(\mathcal{F}_2)$. Finally, by Lemma 5.5,

$$\begin{aligned}\{x_{i_0}^1,\ldots,x_{i_0}^l\}&\in(\mathcal{F}_n(f_{i_0}))^{k_1}(\mathcal{T}_1)\cap(\mathcal{F}_n(f_{i_0}))^{k_1}(\mathcal{T}_2)\\&\subseteq\bigcup_{k=1}^{\infty}((\mathcal{F}_n(f_{i_0}))^k(\mathcal{U})\cap(\mathcal{F}_n(f_{i_0}))^k(\mathcal{V})).\end{aligned}$$

Therefore, $\bigcup_{k=1}^{\infty}((\mathcal{F}_n(f_{i_0}))^k(\mathcal{U})\cap(\mathcal{F}_n(f_{i_0}))^k(\mathcal{V}))=\mathcal{F}_n(X_{i_0})$ and thus, $\mathcal{F}_n(f_{i_0})$ is strongly exact transitive. □

By Theorems 4.2 and 5.1, we have the following result.

Theorem 5.7. *For each $i \in \{1, \ldots, m\}$, let (X_i, f_i) be a dynamical system, let $n \in \mathbb{N}$, and let $\mathcal{M}$ be one of the following classes of functions: fully exact, AAN strongly transitive, very strongly transitive, exact transitive, strongly exact transitive or strongly product transitive. If $\mathcal{F}_n(\prod_{i=1}^{m} f_i) \in \mathcal{M}$, then, for every $i \in \{1, \ldots, m\}$, $f_i \in \mathcal{M}$.*

We end this paper with the following questions.

Problem 5.1. *For each $i \in \{1, \ldots, m\}$, let (X_i, f_i) be a dynamical system, and let $n \in \mathbb{N}$.*

1. *Let $\mathcal{F}_n(\prod_{i=1}^{m} f_i)$ be fully exact. Is $\mathcal{F}_n(f_i)$ fully exact for every $i \in \{1, \ldots, m\}$?*
2. *Let $\mathcal{F}_n(\prod_{i=1}^{m} f_i)$ be strongly product transitive. Is $\mathcal{F}_n(f_i)$ strongly product transitive for every $i \in \{1, \ldots, m\}$?*

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References

[1] E. Akin, J. Auslander and A. Nagar, *Variations on the concept of topological transitivity*, Studia Math. **235** (2016), 225–249.
[2] E. Akin, J. Auslander and A. Nagar, *Dynamics of induced systems*, Ergodic Theory Dynam. Systems **37** (2017), 2034–2059.
[3] E. Akin and J. D. Carlson, *Conceptions of topological transitivity*, Topology Appl. **159** (2012), 2815–2830.
[4] F. Barragán, S. Macías and A. Rojas, *Conceptions of topological transitivity on symmetric products*, Math. Pannon. New Series **27** (2021), 61–80.
[5] F. Barragán, A. Santiago-Santos and J. F. Tenorio, *Dynamic properties for the induced maps on n-fold symmetric product suspensions II*, Topology Appl. **288** (2021), 107484.
[6] E. Bilokopytov and S. F. Kolyada, *Transitive maps on topological spaces*, Ukrainian Math. J. **65** (2014), 1293–1318.
[7] C. R. Borges, Elementary topology and applications, World Scientific, Singapore, 2000.
[8] K. Borsuk and S. Ulam, *On symmetric products of topological spaces*, Bull. Amer. Math. Soc. **37** (1931), 875–882.
[9] N. Değirmenci and Ş. Koçak, *Chaos in product maps*, Turkish J. Math. **34** (2010), 593–600.
[10] R. Engelking, General topology, Heldermann Verlag, Berlin, 1989.
[11] F. Hausdorff, Grundzüge der Mengenlehre, Leipzig, 1914.
[12] G. Higuera and A. Illanes, *Induced mappings on symmetric products*, Topology Proc. **37** (2011), 367–401.
[13] A. Illanes and S. B. Nadler Jr., Hyperspaces: fundamentals and recent advances, Monogr. Textbooks Pure Appl. Math., 216, Marcel Dekker, Inc., New York, 1999.
[14] S. Macías, Topics on continua, 2nd Ed. Springer, Cham, Switzerland, 2018.
[15] J. H. Mai and W. H. Sun, *Transitivities of maps of general topological spaces*, Topology Appl. **157** (2010), 946–953.
[16] J. R. Munkres, Topology, 2nd Ed., Prentice Hall, USA, 2000.

[17] S. B. Nadler Jr., Hyperspaces of sets, Monogr. Textbooks Pure Appl. Math., Vol. 49, Marcel Dekker, Inc., New York-Basel, 1978.
[18] A. Rojas, F. Barragán and S. Macías, *Conceptions on topological transitivity in products and symmetric products*, Turkish J. Math. **44** (2020), 491–523.
[19] P. Touhey, *Yet another definition of chaos*, Amer. Math. Monthly **104** (1997), 411–414.
[20] L. Vietoris, *Bereiche zweiter Ordnung*, Monatsh. Math. Phys. **32** (1922), 258–280.

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KONCEPCIJE O TOPOLOŠKOJ TRANZITIVNOSTI U PRODUKTIMA I SIMETRIČNIM PRODUKTIMA II

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