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ON THE VALIDITY OF THE CAUCHY–SCHWARZ INEQUALITY FOR THE BRACKET MAP

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Abstract. Due to its properties, the bracket map associated with a dual integrable unitary representation of a locally compact group can be viewed as a certain operator-valued inner product; however, in the non-commutative setting, the Cauchy–Schwarz property for bracket is no longer present in its full strength. In this paper, we show that fulfillment of the property, even in weaker forms, has strong consequences on the underlying group G and the corresponding von Neumann algebra $\mathrm{VN}(G)$. In particular, we show that for unimodular group G, positive elements of the L^1 space over $\mathrm{VN}(G)$ which are affiliated with the commutant of $\mathrm{VN}(G)$ are precisely those for which the weaker variant of the inequality is fulfilled, and that the validity of the Cauchy–Schwarz property for the appropriate set of elements indicates the existence of a closed abelian subgroup or abelian von Neumann subalgebra of $\mathrm{VN}(G)$.

1. Introduction

Dual integrable representations form a large and important class among the unitary representations of locally compact groups. The concept, whose main importance relies upon the existence of the associated *bracket map*, was first introduced in [11] for abelian groups, and was later studied in the non-commutative setting, including some particular groups, [1], as well the classes of discrete and compact groups ([2,13]; for the overview of the subject, consult [10]). Many reproducing function systems can be considered as orbits of such representations, with underlying group often being non-abelian (we refer the reader to [2] and [4] for various examples in the setting of non-commutative discrete groups).

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A unitary representation Π of an LCA group G on a separable Hilbert space $\mathbb{H}$ is called *dual integrable* if there exist a Haar measure $d\xi$ on the dual group $\widehat{G}$, and a function $[\cdot,\cdot] \equiv [\cdot,\cdot]_\Pi : \mathbb{H} \times \mathbb{H} \to L^1(\widehat{G}; d\xi)$, called the *bracket*, such that

$$\langle \varphi, \Pi(g)\psi \rangle = \int_{\widehat{G}} [\varphi, \psi](\xi) e_{-g}(\xi) d\xi, \tag{1.1}$$

for all $\varphi, \psi \in \mathbb{H}$, and all $g \in G$ (notation e_g is used for characters on $\widehat{G}$; consult [11]). Recently, in our collaboration [21] the concept was introduced for the entire class of locally compact groups; following [21], a unitary representation Π of a locally compact group G will be called *dual integrable* if all of its matrix coefficients

$$g \mapsto F_{\varphi,\psi}(g) := \langle \Pi(g)\varphi, \psi \rangle \tag{1.2}$$

belong to the Fourier algebra, $A(G)$. Using the theory of L^p-Fourier transform developed by Terp in [24] (and which is based on the theory of spatial derivatives developed by Connes, [7] and spatial L^p-spaces, $L^p(\psi_0)$, developed by Hilsum, [9]), the bracket is identified with an operator $[\cdot,\cdot] : \mathbb{H} \times \mathbb{H} \to L^1(\psi_0)$, satisfying

$$F_{\varphi,\psi} = \overline{\mathcal{F}_1}([\varphi, \psi]), \quad \forall \varphi, \psi \in \mathbb{H},$$

where ψ_0 is the canonical weight on $\mathrm{VN}(G)'$, the commutant of $\mathrm{VN}(G)$ (the von Neumann algebra generated by $\lambda(G)$), $\overline{\mathcal{F}_1}$ stands for the L^1*–Fourier contransform* (consult [24, Section 5]; precise definitions of all the notions mentioned in the Introduction which we use in the paper will be given in Section 2), and λ denotes the left regular representation. Using the fact that $\mathrm{VN}(G)$ can be identified with the Banach space dual of $A(G)$, the continuous positive definite function $F_{\psi,\psi}$ can be considered as a normal semifinite weight on $\mathrm{VN}(G)$, and $[\psi, \psi]$ takes the form of the corresponding spatial derivative. If $(F_{\varphi,\psi})\check{} \in L^2(G)$, then $[\varphi, \psi]$ is the closure of the operator

$$[\varphi, \psi]\xi = (F_{\varphi,\psi})\check{} * \Delta\xi, \quad \xi \in C_c(G), \tag{1.3}$$

where Δ denotes the multiplication operator by the modular function of G. Almost all of the properties of the bracket remain valid in this general setting; more precisely, it was proved in [21] that $[\cdot,\cdot]$ is a sesquilinear map, and the following properties hold for each $\varphi, \psi \in \mathbb{H}$:

(i) $[\psi, \psi] \geqslant 0$,
(ii) $[\varphi, \psi]^* = [\psi, \varphi]$,
(iii) $\|[\varphi, \psi]\|_{L^1(\psi_0)} \leqslant \|\varphi\|_{\mathbb{H}} \|\psi\|_{\mathbb{H}}$,
(iv) $[\Pi(g)\varphi, \psi] = \lambda(g)[\varphi, \psi]$, $[\varphi, \Pi(g)\psi] = [\varphi, \psi]\lambda(g)^*$, $\forall g \in G$.

Nevertheless, in the non-commutative setting, an important property of the bracket, which we refer to as the Cauchy–Schwarz inequality, is lost. Recall that, if G is an abelian group, then for every $\varphi, \psi \in \mathbb{H}$ we have

$$|[\varphi, \psi]|^2 \leqslant [\varphi, \varphi][\psi, \psi], \tag{1.4}$$

with

$$\varphi \in \langle\psi\rangle \quad \Leftrightarrow \quad |[\varphi,\psi]|^2 = [\varphi,\varphi][\psi,\psi] \text{ a.e. and } \Omega_\varphi \subseteq \Omega_\psi. \tag{1.5}$$

Here, Ω_ψ denotes the set $\{\xi \in \widehat{G} : [\psi,\psi](\xi) \neq 0\}$, defined up to a set of Haar measure zero.

The property played an important role in the analysis of Π-invariant subspaces in the abelian setting. Spaces invariant under unitary group representations were studied extensively during the last decades due to their importance in various areas, including the theory of wavelets, Gabor systems and approximation theory; we refer the reader to the recent papers [3,4,12,13,22] where the representation is dual integrable, and references therein. Let us point out that the recent novel and comprehensive approach to the theory of wavelets is largely based on the analysis of cyclic spaces of a particular dual integrable representation (consult [17]). The approach used in [22] was largely based on (1.5). Besides that, the inequality (1.4) played an important role in proving several key results in [11]. The fact that it does not hold in the non-commutative setting did not play a crucial part, though, for proving the analogue of the results of [11]; on the other hand, rather surprisingly, it suggested that the validity of the Cauchy-Schwarz property has strong repercussions on the underlying group (consult [21, Theorem 4.19 and Theorem 5.26]). The results of this paper additionally confirm such claim.

Given a linear operator $T \in L^1(\psi_0)$, the condition

$$\lambda(g)T = T\lambda(g), \quad \forall g \in G \tag{1.6}$$

implies that T commutes with every element of $\mathrm{VN}(G)$, i.e., T is affiliated with the commutant of $\mathrm{VN}(G)$; in particular, if T is bounded operator which belongs to $\mathrm{VN}(G)$, it belongs to the center of $\mathrm{VN}(G)$. As can be seen from [21], and additionally from this paper, this condition is closely tied with the Cauchy–Schwarz property for element $\psi \in \mathbb{H}$. Among other things, we prove that, for unimodular group G, the operator $[\psi,\psi]$ is affiliated with $\mathrm{VN}(G)'$ precisely when the weaker variant of the Cauchy–Schwarz inequality is fulfilled for ψ and all $\varphi \in \mathbb{H}$.

2. Preliminaries

In this section, we briefly recall the main results from [21] which will be used in this paper. We also give a short overview of the main concepts and notation. For the properties of operators and operator algebras which are not listed here, we refer to [14] and [26]; detailed exposition of the theory of L^p spaces associated with von Neumann algebras is given in [25].

Let G be a locally compact group. For $p \in [1,+\infty]$, we denote by $L^p(G)$ the standard Lebesgue spaces with respect to the left Haar measure on G, by $C(G)$ the space of all continuous functions on G, and by $C_c(G)$ the set of continuous functions on G with compact support; recall that $C_c(G)$ is dense in $L^p(G)$ for all $p \in [1, +\infty\rangle$. Let Δ denote the modular function on G. For a function $f : G \to \mathbb{C}$, we denote

$$\check{f}(x) = f(x^{-1}), \qquad \widetilde{f}(x) = \overline{f(x^{-1})},$$
$$Jf(x) = \Delta^{-1/2}(x)\widetilde{f}(x), \qquad f^*(x) = \Delta^{-1}(x)\widetilde{f}(x);$$

$f \mapsto f^*$ and $f \mapsto Jf$ are conjugate linear isometric involutions of $L^1(G)$ and $L^2(G)$, respectively, and $Jf = f^* = \widetilde{f}$ if G is unimodular. The convolution of two measurable functions $f, g : G \to \mathbb{C}$ is defined by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy,$$

whenever the integral makes sense.

Let $\mathbb{H}$ be a Hilbert space, $\mathcal{B}(\mathbb{H})$ the Banach algebra of bounded linear operators on $\mathbb{H}$, and $\mathcal{U}(\mathbb{H})$ the group of unitary operators on $\mathbb{H}$. By a *unitary representation* Π of G on $\mathbb{H}$, we mean a continuous homomorphism $\Pi : G \to \mathcal{U}(\mathbb{H})$. Continuity is considered either with respect to strong or weak operator topology on $\mathcal{U}(\mathbb{H})$ (recall that these two topologies coincide on $\mathcal{U}(\mathbb{H})$); hence, the continuity assumption is equivalent with $F_{\varphi,\psi} \in C(G)$ for all $\varphi, \psi \in \mathbb{H}$. Here, $F_{\varphi,\psi}$ denotes the *matrix coefficients*, $F_{\varphi,\psi}(g) = \langle \Pi(g)\varphi, \psi\rangle$, $g \in G$. We shall sometimes write $\mathbb{H}_\Pi$ instead of $\mathbb{H}$.

A closed subspace $\mathbb{V}$ of $\mathbb{H}$ is called Π-*invariant* if $\Pi(G)\mathbb{V} \subseteq \mathbb{V}$, i.e., if

$$\Pi(g)\psi \in \mathbb{V}, \quad \forall g \in G, \quad \forall \psi \in \mathbb{V}. \tag{2.1}$$

The smallest nontrivial Π-invariant subspaces $\langle\psi\rangle_\Pi$ are those which are generated by a single element $\psi \in \mathbb{H} \setminus \{0\}$, i.e.,

$$\langle\psi\rangle_\Pi := \mathrm{Cl}_{\mathbb{H}}(\mathrm{span}\{\Pi(g)\psi : g \in G\}); \tag{2.2}$$

we call these spaces *cyclic subspaces* generated by ψ. We shall mostly write $\langle\psi\rangle$ instead of $\langle\psi\rangle_\Pi$.

We recall the main concepts concerning von Neumann algebras (for more details, consult [14, Section 5]). A *von Neumann algebra* is a self-adjoint (i.e., it contains the adjoint of each of its elements) unital subalgebra $\mathcal{M}$ of the algebra of bounded linear operators on a Hilbert space which is closed in the weak operator topology. For a subset $S \subseteq \mathcal{B}(\mathbb{H})$,

$$S' = \{T \in \mathcal{B}(\mathbb{H}) : TU = UT, \forall U \in S\}, \tag{2.3}$$

is the *commutant* of S, and the *bicommutant*, $(S')'$, is denoted by S''. A linear operator T on $\mathbb{H}$ is said to be *affiliated* with von Neumann algebra $\mathcal{M}$ if $AT \subseteq TA$ for all $A \in \mathcal{M}'$ (here, $\subseteq$ denotes that TA is the extension of AT; consult [14, 2.7]).

On any locally compact group G, we can consider the *left* and the *right regular representation*, $\lambda_G : G \to \mathcal{U}(L^2(G))$ and $\rho_G : G \to \mathcal{U}(L^2(G))$ (throughout the paper we shall mostly use notation λ and ρ), defined by

$$\lambda_G(g)f(x) = f(g^{-1}x), \quad \rho_G(g)f(x) = \Delta^{1/2}(g)f(xg), \quad x \in G, \quad f \in L^2(G).$$

These representations are unitarily equivalent, via $U : L^2(G) \to L^2(G)$,

$$Uf(y) = \Delta^{-1/2}(y)f(y^{-1}), \quad y \in G, \quad f \in L^2(G). \tag{2.4}$$

We denote by $A(G)$ and $B(G)$ the *Fourier* and the *Fourier–Stieltjes algebra*, respectively. It is known that the elements of $A(G)$ are precisely the functions of the form $f = \xi * \widetilde{\eta} = \langle \xi, \lambda(\cdot)\eta \rangle$, for $\xi, \eta \in L^2(G)$; thus, $A(G)$ consists precisely of the matrix coefficients of the left regular representation (for further properties, consult [15]). By $\mathrm{VN}(G)$, we denote the von Neumann algebra generated by $\lambda(G)$ (equivalently, by $\lambda(L^1(G))$); $\mathrm{VN}(G) = \lambda(G)''$, it is the smallest von Neumann algebra which contains $\lambda(G)$, i.e.,

$$\mathrm{VN}(G) = \overline{\mathrm{span}\{\lambda(g) : g \in G\}}^{w^*},$$

where w^* denotes that the closure is taken with respect to w^*-topology (we note that it can be replaced by the closure in the weak operator topology), and its commutant $\mathrm{VN}(G)'$ is the von Neumann algebra generated by $\rho(G)$.

A *weight* on a von Neumann algebra $\mathcal{M}$ is a map $\varphi : \mathcal{M}^+ \to [0, +\infty]$ such that $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(\lambda x) = \lambda\varphi(x)$, for all $x, y \in \mathcal{M}_+$ and all $\lambda \geqslant 0$ (with convention $0(+\infty) = 0$); here, $\mathcal{M}^+$ denotes the set of positive operators in $\mathcal{M}$. A weight φ is called *semifinite* if the set $\{x \in \mathcal{M}^+ : \varphi(x) < +\infty\}$ generates $\mathcal{M}$, *faithful* if $\varphi(x) \neq 0$, for every non-zero $x \in \mathcal{M}^+$, *normal* if $\varphi(\sup x_i) = \sup \varphi(x_i)$ for every bounded increasing net $\{x_i\}$ in $\mathcal{M}^+$, and *tracial* if $\varphi(xx^*) = \varphi(x^*x)$ for all $x \in \mathcal{M}^+$; for more details consult, for instance, [26, VII]. Von Neumann algebra $\mathrm{VN}(G)$ is the left von Neumann algebra of $C_c(G)$, considered as a left Hilbert algebra (consult [26, VI. 1]); denote by φ_0 the canonical weight on $\mathrm{VN}(G)$, and let ψ_0 denote the corresponding weight on $\mathrm{VN}(G)' = J\mathrm{VN}(G)J$. Weight ψ_0 is given by

$$\psi_0(T^*T) = \begin{cases} \|\eta\|_2^2, & \text{if } T = \lambda'(\eta) \text{ for some } \eta \text{ which is right bounded} \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\lambda'(\eta) = \rho(\Delta^{-1/2}\check{\eta})$; for the details (including the definition of a right bounded element) and references, consult [24, p. 550]. For $\alpha \in \mathbb{R}$, an operator T on $L^2(G)$ is called *α-homogeneous* if

$$\rho(x)T \subseteq \Delta^{-\alpha}(x)T\rho(x), \quad \forall x \in G; \tag{2.5}$$

for further properties, we refer to [24, Remark 2.2]. By [7], any positive self-adjoint (-1)-homogeneous operator T can be expressed in the form of a *spatial derivative* $\frac{d\varphi}{d\psi_0}$, for some normal semifinite weight φ on $\mathrm{VN}(G)$, and the integral with respect to ψ_0 is defined by

$$\int T d\psi_0 := \varphi(1), \tag{2.6}$$

where 1 represents the identity operator on $L^2(G)$. For $p \in [1, +\infty\rangle$, *spatial* L^p*-spaces*, $L^p(\psi_0)$, are defined as the spaces of closed densely defined operators T, such that $|T|^p$ is (-1)-homogeneous and $\int |T|^p d\psi_0 < +\infty$. Moreover, $L^\infty(\psi_0)$ is identified with $\mathrm{VN}(G)$, equipped with the operator norm.

If G is unimodular, then $\Delta \equiv 1$, φ_0 is a trace (in fact, φ_0 is a trace only if G is unimodular), $L^p(\psi_0)$ spaces are simply L^p spaces over $\mathrm{VN}(G)$, which is a semifinite von Neumann algebra (consult [18]; for the Fourier transform on unimodular groups, consult also [16, 20, 23]).

Let Π be a dual integrable unitary representation of a locally compact group G; recall that Π is dual integrable if $F_{\varphi,\psi} \in A(G)$ for all $\varphi, \psi \in \mathbb{H}$. For any $\psi \in \mathbb{H}$, $[\psi,\psi] \in L^1(\psi_0)$; thus, it is a closed, densely defined operator on $L^2(G)$ (unbounded in general). The domain, kernel and range of a linear operator T will be denoted by $D(T)$, $N(T)$ and $R(T)$, respectively. The closure of a preclosed operator T will be denoted by $[T]$. For two positive operators F_1, F_2, we write $F_1 \leqslant F_2$ whenever $F_2 - F_1$ is positive. Throughout the paper, we shall use the following notation: $N_\psi = N([\psi,\psi])$; this is a closed subspace of $L^2(G)$ and, since $[\psi,\psi]$ is self-adjoint, we know that $N_\psi = R([\psi,\psi])^\perp$.

The *Plancherel transform* $\mathcal{P}$ is a unitary operator from $L^2(G)$ onto $L^2(\psi_0)$ given by

$$\mathcal{P}(f)\zeta = f * \Delta^{1/2}\zeta, \quad \zeta \in D(\mathcal{P}(f)), \quad f \in L^2(G),$$

where $D(\mathcal{P}(f)) = \{\zeta \in L^2(G) : f * \Delta^{1/2}\zeta \in L^2(G)\}$. Since $[\psi,\psi]^{1/2} \in L^2(\psi_0)$, there exists a unique $\sigma_\psi \in L^2(G)$ such that $\mathcal{P}(\sigma_\psi) = [\psi,\psi]^{1/2}$; for such σ_ψ we have $F_{\psi,\psi} = \overline{\sigma_\psi} * \check{\sigma_\psi}$. It was shown in [21, Proposition 3.13] that for any $\varphi \in \langle\psi\rangle$, there exists $\eta \in L^2(G)$ such that

$$F_{\varphi,\psi} = \overline{\sigma_\psi} * \widetilde{\eta}, \quad F_{\varphi,\varphi} = \eta * \widetilde{\eta}; \tag{2.7}$$

consequently (see [24, Corollary 5.7]),

$$[\psi,\psi] = |\mathcal{P}(\sigma_\psi)|^2, \quad [\varphi,\psi] = [\mathcal{P}(\overline{\eta})\mathcal{P}(\sigma_\psi)], \quad [\varphi,\varphi] = |\mathcal{P}(\overline{\eta})^*|^2. \tag{2.8}$$

To make the paper self-contained, we note that η is defined via isometry $\mathcal{U}_\psi$ defined on $\mathrm{span}\{\Pi(g)\psi : g \in G\}$ by $\mathcal{U}_\psi(\sum_{g\in\Omega} a_g \Pi(g)\psi) = \sum_{g\in\Omega} a_g \lambda(g)\sigma_\psi$ (Ω is a finite subset of G); we take $\eta = \overline{\mathcal{U}_\psi(\varphi)}$. Moreover, for any $\varphi, \varphi' \in \langle\psi\rangle$, for corresponding $\eta_\varphi, \eta_{\varphi'}$ we have

$$F_{\varphi,\varphi'} = \eta_{\varphi'} * \widetilde{\eta_\varphi}. \tag{2.9}$$

One of the most important consequences of dual integrability condition is the possibility of performing the analysis of cyclic subspaces $\langle\psi\rangle$ in terms of the corresponding weighted L^2 spaces, $L^2(\psi_0; [\psi, \psi])$. The latter is defined as the Hilbert space completion of the pre-Hilbert space ${}^{\mathrm{VN}(G)}/_{\mathcal{N}_\psi}$, where

$$\mathcal{N}_\psi = \{F \in \mathrm{VN}(G) : [F[\psi, \psi]^{1/2}] = 0\}, \tag{2.10}$$

equipped with norm

$$\|F\|_{2,\psi} = \|[F[\psi, \psi]^{1/2}]\|_{L^2(\psi_0)} = \int [F[\psi, \psi]F^*] d\psi_0. \tag{2.11}$$

The analysis relies upon the existence of the isometric isomorphism $S_\psi : \langle\psi\rangle \to L^2(\psi_0; [\psi, \psi])$ such that

$$S_\psi(\Pi(g)\psi) = \lambda(g), \tag{2.12}$$

for all $g \in G$ (consult [21, Section 4] for the general case or [2, Proposition 3.4] for the case of discrete groups).

3. The existence of (C–S)-pairs

Let G be a locally compact group. Recall that, in the abelian case, for each $\varphi, \psi \in \mathbb{H}$, $[\varphi, \psi]$ is a function in $L^1(\widehat{G})$. On the other hand, both in abelian and non-abelian case, we can treat $[\varphi, \psi]$ as an operator on $L^2(G)$ (though, generally, not everywhere defined), and the statements

$$[\psi, \varphi][\varphi, \psi] \leqslant [\varphi, \varphi][\psi, \psi], \quad \forall \varphi, \psi \in \mathbb{H}, \tag{3.1}$$

with

$$\varphi \in \langle\psi\rangle \quad \Leftrightarrow \quad [\psi, \varphi][\varphi, \psi] = [\varphi, \varphi][\psi, \psi] \text{ and } P_{N_\varphi^\perp} \leqslant P_{N_\psi^\perp}. \tag{3.2}$$

could be seen as the direct analogues of (1.4) and (1.5). Note however that, in the non-abelian case, $|[\varphi, \psi]|^2$ is, in general, not equal to $|[\psi, \varphi]|^2$; actually, we have $|[\varphi, \psi]| = |[\psi, \varphi]|$ precisely when $[\varphi, \psi]$ is a normal operator. Hence, besides (3.1), we may consider the other inequality:

$$[\varphi, \psi][\psi, \varphi] \leqslant [\varphi, \varphi][\psi, \psi], \quad \forall \varphi, \psi \in \mathbb{H}. \tag{3.3}$$

It was shown in [21] that, if G is unimodular, having all the equalities in (1.5) satisfied for only one injective element $[\psi, \psi] \in \mathrm{VN}(G)$ forces G to be abelian (and Cauchy–Schwarz property immediately holds for all elements). Still, there might exist elements of $\mathbb{H}$ which satisfy at least some of these conditions.

Question 1. What are the necessary and sufficient conditions on $\varphi, \psi \in \mathbb{H}$ such that the pair (φ, ψ) satisfies the Cauchy–Schwarz property (or at least some of the (in)equalities)?

Recall that, in general, $[\psi, \psi]$ is an unbounded operator and, more precisely, in (3.1)–(3.3), we should consider $[[\varphi, \varphi][\psi, \psi]]$ (note that the operators on the left-hand side are closed, since they are self-adjoint). For the sake of simplicity, we shall first focus on the case where both $[\psi, \psi]$ and $[\varphi, \varphi]$ are bounded (several difficulties which occur in the general case were pointed out in [21, Remark 2.35]). Boundedness of $[\psi,\psi]$ is a particularly nice property if we consider unimodular groups; in that case, $[\psi,\psi]$ is bounded precisely when $[\psi,\psi] \in \mathrm{VN}(G)$, which is equivalent to $\{\Pi(g)\psi : g \in G\}$ being a Bessel system, and such elements form a dense subset of $\mathbb{H}$ (consult [21, Section 5], and [2, Section 4] for discrete groups). On the other hand, if G is non-unimodular, the Bessel condition is not related to condition $[\psi,\psi] \in \mathrm{VN}(G)$ (note that this assumption requires for a (-1)-homogeneous operator to be 0-homogeneous). Thus, in this paper, we consider the unimodular group case (see also discussion in [21, p. 25-26]). In that case, φ_0 is a trace, $(-1/p)$–homogeneous operators are precisely the operators affiliated with the von Neumann algebra $\mathrm{VN}(G)$ and $L^p(\psi_0)$ spaces are simply $L^p(\mathrm{VN}(G);\varphi_0)$ spaces (see [18]). To avoid similarity in notation with elements $\varphi \in \langle\psi\rangle$, in the rest of the paper we shall denote the canonical trace on $\mathrm{VN}(G)$ by τ.

It was already observed in [21] that if both $[\varphi,\varphi]$ and $[\psi,\psi]$ are bounded, condition (3.1) implies that $[\varphi,\varphi][\psi,\psi]$ is a positive operator; consequently it is self-adjoint. Thus, if (3.1) holds, then

$$[\varphi,\varphi][\psi,\psi] = [\psi,\psi][\varphi,\varphi]. \tag{3.4}$$

For $\psi \in \mathbb{H}$, consider the set

$$\{\varphi \in \langle\psi\rangle : S_\psi(\varphi) \in \mathrm{VN}(G)\}. \tag{3.5}$$

Recall that S_ψ is the isometry from $\langle\psi\rangle$ onto $L^2(\psi_0;[\psi,\psi])$, given by (2.12). For such φ and $F = S_\psi(\varphi)$ we have

$$[\varphi,\varphi] = F[\psi,\psi]F^* \in \mathrm{VN}(G), \quad [\varphi,\psi] = F[\psi,\psi] \in \mathrm{VN}(G);$$

moreover, from the construction of η (which belongs to $\langle\sigma_\psi\rangle_\lambda$, i.e., to the closure of the linear span of $\{\lambda(g)\sigma_\psi : g \in G\}$; recall (2.2)), we have

$$\mathcal{P}(\overline{\eta}) = F[\psi,\psi]^{1/2} \in \mathrm{VN}(G). \tag{3.6}$$

Obviously, it follows that

$$N(\mathcal{P}(\overline{\eta})) \supseteq N_\psi, \quad \overline{R(\mathcal{P}(\overline{\eta})^*)} \subseteq \overline{R([\psi,\psi])} \tag{3.7}$$

and

$$N_\varphi^\perp = N(\mathcal{P}(\overline{\eta})^*). \tag{3.8}$$

We emphasize that, although we can choose $\mathcal{P}(\sigma_\psi)$ to be self-adjoint (and positive), $\mathcal{P}(\overline{\eta})$ (defined via (2.7)) is not self-adjoint; in fact, it is in general not even normal operator (see Remark 3.2 below). The following theorem shows that normality assumption plays an important part in the characterization of elements which satisfy the Cauchy–Schwarz property.

THEOREM 3.1. *Let G be a unimodular group and Π a dual integrable representation of G on a separable Hilbert space $\mathbb{H}$. Suppose that $[\psi,\psi] \in \mathrm{VN}(G)$ and let $\varphi \in \langle\psi\rangle$ such that $S_\psi(\varphi) \in \mathrm{VN}(G)$. The following are equivalent:*

(i) $|[\varphi,\psi]|^2 = [\varphi,\varphi][\psi,\psi]$ *and* $P_{N_\varphi^\perp} \leqslant P_{N_\psi^\perp}$,

(ii) $\mathcal{P}(\overline{\eta})$ *is normal and* $[\varphi,\varphi][\psi,\psi] = [\psi,\psi][\varphi,\varphi]$.

Moreover, if any of these two conditions is fulfilled, then

$$|[\psi,\varphi]|^2 = |[\varphi,\psi]|^2 = [\varphi,\varphi][\psi,\psi]$$

if and only if $\mathcal{P}(\overline{\eta})$ *and* $\mathcal{P}(\sigma_\psi)$ *commute.*

Proof. $(i) \Rightarrow (ii)$ We have already observed that self-adjointness of the operator on the right-hand side in (i) implies (3.4). By the property of the square root, it follows that $[\psi,\psi]^{1/2}$ and $[\varphi,\varphi]$ commute, i.e., we have

$$\mathcal{P}(\overline{\eta})\mathcal{P}(\overline{\eta})^*\mathcal{P}(\sigma_\psi) = \mathcal{P}(\sigma_\psi)\mathcal{P}(\overline{\eta})\mathcal{P}(\overline{\eta})^*. \tag{3.9}$$

Now, (i) and (3.9) imply that

$$\begin{aligned}\mathcal{P}(\sigma_\psi)\mathcal{P}(\overline{\eta})^*\mathcal{P}(\overline{\eta})\mathcal{P}(\sigma_\psi) = [\varphi,\psi]^*[\varphi,\psi] &= [\varphi,\varphi][\psi,\psi]\\ &= \mathcal{P}(\overline{\eta})\mathcal{P}(\overline{\eta})^*\mathcal{P}(\sigma_\psi)\mathcal{P}(\sigma_\psi)\\ &= \mathcal{P}(\sigma_\psi)\mathcal{P}(\overline{\eta})\mathcal{P}(\overline{\eta})^*\mathcal{P}(\sigma_\psi).\end{aligned}$$

It follows that

$$\mathcal{P}(\sigma_\psi)(\mathcal{P}(\overline{\eta})\mathcal{P}(\overline{\eta})^*\zeta - \mathcal{P}(\overline{\eta})^*\mathcal{P}(\overline{\eta})\zeta) = 0 \tag{3.10}$$

for all $\zeta \in \overline{R([\psi,\psi]^{\frac{1}{2}})} = N_\psi^\perp$. Observe that it follows from (i) that $N_\varphi \supseteq N_\psi$; by (3.7) and (3.8), we have

$$R(\mathcal{P}(\overline{\eta})\mathcal{P}(\overline{\eta})^* - \mathcal{P}(\overline{\eta})^*\mathcal{P}(\overline{\eta})) \subseteq N_\psi,$$

and, passing to the orthogonal complements, we conclude that $\mathcal{P}(\overline{\eta})\mathcal{P}(\overline{\eta})^* = \mathcal{P}(\overline{\eta})^*\mathcal{P}(\overline{\eta})$ on $N_\psi^\perp$. Now, using again the fact that N_ψ is contained both in $N(\mathcal{P}(\overline{\eta}))$ and $N(\mathcal{P}(\overline{\eta})^*)$, it follows that

$$\mathcal{P}(\overline{\eta})^*\mathcal{P}(\overline{\eta}) = 0 = \mathcal{P}(\overline{\eta})\mathcal{P}(\overline{\eta})^*$$

on N_ψ. Therefore, $\mathcal{P}(\overline{\eta})$ is normal.

$(ii) \Rightarrow (i)$ Conversely, (ii) implies (3.9), and we immediately prove the first condition in (i). Since $\mathcal{P}(\overline{\eta})$ is normal, we have $N(\mathcal{P}(\overline{\eta})) = N(\mathcal{P}(\overline{\eta})^*)$, and $P_{N_\varphi^\perp} \leqslant P_{N_\psi^\perp}$ follows from (3.7) and (3.8).

Suppose now that $\mathcal{P}(\overline{\eta})$ and $\mathcal{P}(\sigma_\psi)$ commute; then $\mathcal{P}(\overline{\eta})^*$ and $\mathcal{P}(\sigma_\psi)$ also commute. Hence, we have

$$\begin{aligned}|[\psi,\varphi]|^2 = \mathcal{P}(\overline{\eta})\mathcal{P}(\sigma_\psi)\mathcal{P}(\sigma_\psi)\mathcal{P}(\overline{\eta})^* &= \mathcal{P}(\sigma_\psi)\mathcal{P}(\overline{\eta})\mathcal{P}(\overline{\eta})^*\mathcal{P}(\sigma_\psi)\\ &= \mathcal{P}(\sigma_\psi)\mathcal{P}(\overline{\eta})^*\mathcal{P}(\overline{\eta})\mathcal{P}(\sigma_\psi)\\ &= |[\varphi,\psi]|^2.\end{aligned}$$

Suppose now that $|[\psi,\varphi]|^2 = [\varphi,\varphi][\psi,\psi]$; we have

$$\mathcal{P}(\overline{\eta})([\psi,\psi]\mathcal{P}(\overline{\eta})^* - \mathcal{P}(\overline{\eta})^*[\psi,\psi]) = 0.$$

It follows that

$$R([\psi,\psi]\mathcal{P}(\overline{\eta})^* - \mathcal{P}(\overline{\eta})^*[\psi,\psi]) \subseteq N(\mathcal{P}(\overline{\eta})),$$

and, hence, $N(\mathcal{P}(\overline{\eta})[\psi,\psi] - [\psi,\psi]\mathcal{P}(\overline{\eta})) \supseteq N(\mathcal{P}(\overline{\eta}))^\perp$, i.e.,

$$([\psi,\psi]\mathcal{P}(\overline{\eta}))(\zeta) = (\mathcal{P}(\overline{\eta})[\psi,\psi])(\zeta) \tag{3.11}$$

for all $\zeta \in N(\mathcal{P}(\overline{\eta}))^\perp$.

Equality (3.11) implies that $N(\mathcal{P}(\eta))^\perp$ is $[\psi,\psi]$-invariant. Indeed, if $\zeta \in N(\mathcal{P}(\overline{\eta}))^\perp = \overline{R(\mathcal{P}(\overline{\eta}))}$, then $\zeta = \lim_{n\to\infty} \mathcal{P}(\overline{\eta})\zeta_n$, for some sequence $\{\zeta_n\} \subseteq L^2(G)$, and for all $y \in N(\mathcal{P}(\overline{\eta}))$, we have

$$\begin{aligned}\langle [\psi,\psi]\zeta, y\rangle &= \lim_{n\to\infty} \langle \mathcal{P}(\overline{\eta})\zeta_n, [\psi,\psi]y\rangle \\ &= \lim_{n\to\infty} \langle \mathcal{P}(\overline{\eta})[\psi,\psi]\zeta_n, y\rangle \\ &= \lim_{n\to\infty} \langle [\psi,\psi]\zeta_n, \mathcal{P}(\overline{\eta})^* y\rangle = 0,\end{aligned}$$

i.e., $[\psi,\psi]\zeta \in N(\mathcal{P}(\overline{\eta}))^\perp$. Since $[\psi,\psi]$ is self-adjoint, $N(\mathcal{P}(\overline{\eta}))$ is also $[\psi,\psi]$-invariant, i.e. (3.11) holds also for all $\zeta \in N(\mathcal{P}(\overline{\eta}))$. Consequently, $\mathcal{P}(\overline{\eta})$ and $[\psi,\psi]^{1/2} = \mathcal{P}(\sigma_\psi)$ also commute. □

Remark 3.2. If $\varphi = \Pi(g)\psi$ for some $g \in G$, then the fact that $\mathcal{P}(\overline{\eta})$ is normal is equivalent to the fact that $\lambda(g)$ commutes with $[\psi,\psi]$. Indeed, since $\mathcal{P}(\overline{\eta}) = \lambda(g)[\psi,\psi]^{1/2}$, we have

$$\begin{aligned}\mathcal{P}(\overline{\eta})\mathcal{P}(\overline{\eta})^* = \mathcal{P}(\overline{\eta})^*\mathcal{P}(\overline{\eta}) \quad &\Leftrightarrow \quad \lambda(g)[\psi,\psi]\lambda(g)^* = [\psi,\psi] \\ &\Leftrightarrow \quad \lambda(g)[\psi,\psi] = [\psi,\psi]\lambda(g).\end{aligned}$$

Remark 3.3. Suppose that $\mathcal{P}(\overline{\eta})$ is normal and consider its polar decomposition; we have

$$\mathcal{P}(\overline{\eta}) = U|\mathcal{P}(\overline{\eta})| = U[\varphi,\varphi]^{1/2}, \tag{3.12}$$

since $|\mathcal{P}(\overline{\eta})| = |\mathcal{P}(\overline{\eta})^*|$. Moreover, U is a unitary operator on $N(\mathcal{P}(\overline{\eta}))^\perp$. We know that (see [8, Corollary 4]) $\mathcal{P}(\overline{\eta})$ commutes with $\mathcal{P}(\sigma_\psi) = [\psi,\psi]^{1/2}$ if and only if

$$[\varphi,\varphi]^{1/2}[\psi,\psi]^{1/2} = [\psi,\psi]^{1/2}[\varphi,\varphi]^{1/2} \text{ and } U[\psi,\psi]^{1/2} = [\psi,\psi]^{1/2}U. \tag{3.13}$$

Observe that if $\varphi_1, \varphi_2 \in \langle\psi\rangle$ satisfy the condition (i) of Theorem 3.1, the property is not necessarily fulfilled for $\varphi_1 + \varphi_2 \in \langle\psi\rangle$. Indeed, $\mathcal{P}(\overline{\eta_1}) + \mathcal{P}(\overline{\eta_2})$ need not be normal; this is, however, fulfilled if

$$\mathcal{P}(\overline{\eta_1})\mathcal{P}(\overline{\eta_2}) = \mathcal{P}(\overline{\eta_2})\mathcal{P}(\overline{\eta_1}).$$

On the other hand, from $\mathcal{P}(\sigma_\psi)\mathcal{P}(\overline{\eta}) = \mathcal{P}(\overline{\eta})\mathcal{P}(\sigma_\psi)$ we have

$$F[\psi,\psi]^{1/2}[\psi,\psi]^{1/2} = [\psi,\psi]^{1/2}F[\psi,\psi]^{1/2}; \tag{3.14}$$

observe that, since $F \in {}^{\mathrm{VN}(G)}/\mathcal{N}_\psi$, we may take a representative F such that $FP_{N_\psi^\perp} = F$, and it follows that $[\psi,\psi]^{1/2}F = F[\psi,\psi]^{1/2}$. If $\mathcal{P}(\overline{\eta})$ is normal, so is $\mathcal{P}(\overline{\eta})^*$; moreover, $\mathcal{P}(\overline{\eta})^*\mathcal{P}(\sigma_\psi) = \mathcal{P}(\sigma_\psi)\mathcal{P}(\overline{\eta})^*$. Since $N_\varphi^\perp = N(\mathcal{P}(\overline{\eta})^*)^\perp \subseteq N_\psi^\perp$, there exists $0 \neq \varphi' \in \langle\psi\rangle$ such that $S_\psi(\varphi') = F^*$; it follows that $\mathcal{P}(\overline{\eta_{\varphi'}}) = ([\psi,\psi]^{1/2}F)^* = \mathcal{P}(\overline{\eta})^*$. We have thus proved that condition (i) of Theorem 3.1 holds for $\varphi \in \langle\psi\rangle$ precisely when it holds for $\varphi' \in \langle\psi\rangle$ such that $\varphi' = S_\psi^{-1}(S_\psi(\varphi)^*)$.

Consider now the case where G is still unimodular and $[\psi,\psi] \in \mathrm{VN}(G)$, but we take general $\varphi \in \mathbb{H}$. Recall that we would like to answer whether the given pair (φ,ψ) satisfies the Cauchy–Schwarz inequality. Observe that on the left-hand side in (i) we have $|[\varphi,\psi]|^2 = |[\varphi_1,\psi]|^2$, where $\varphi_1 = P_{\langle\psi\rangle}(\varphi)$ and $\varphi_2 = P_{\langle\psi\rangle^\perp}(\varphi)$ ($P_{\langle\psi\rangle}$ denotes the projection onto $\langle\psi\rangle$), while on the right-hand side we have

$$[\varphi,\varphi][\psi,\psi] = [[\varphi_1,\varphi_1][\psi,\psi] + [\varphi_2,\varphi_2][\psi,\psi]].$$

Hence, it makes more sense to take into account both φ and φ_1.

For $\psi \in \mathbb{H}$, we shall call (φ,ψ) a *(C-S) pair* if

$$|[\varphi,\psi]|^2 \leqslant [\varphi,\varphi][\psi,\psi], \quad |[\psi,\varphi]|^2 \leqslant [\varphi,\varphi][\psi,\psi], \tag{3.15}$$

$$|[\varphi_1,\psi]|^2 = [\varphi_1,\varphi_1][\psi,\psi] = |[\psi,\varphi_1]|^2 \text{ and } P_{N_{\varphi_1}^\perp} \leqslant P_{N_\psi^\perp} \tag{3.16}$$

where $\varphi_1 = P_{\langle\psi\rangle}(\varphi)$. We easily obtain the following corollary.

Corollary 3.4. *Let Π be a dual integrable representation of a unimodular group G. Consider $\psi \in \mathbb{H}$ such that $[\psi,\psi] \in \mathrm{VN}(G)$, and let $\varphi \in \mathbb{H}$ such that $[\varphi,\varphi], S_\psi(\varphi) \in \mathrm{VN}(G)$. The following are equivalent:*

(i) (φ,ψ) is a (C-S) pair,

(ii) $\mathcal{P}(\overline{\eta_1})$ is normal, and $[\varphi,\varphi]$ and $\mathcal{P}(\overline{\eta_1})$ commute with $[\psi,\psi]$.

If $[\psi,\psi] \in \mathrm{VN}(G)$, for any $\varphi \in \langle\psi\rangle$ such that $S_\psi(\varphi) \notin \mathrm{VN}(G)$ we still have $F_{\varphi,\psi} \in L^2(G)$; thus, $[\varphi,\psi] = \mathcal{P}(\check{F}_{\varphi,\psi}) \in L^2(\mathrm{VN}(G),\tau)$. Therefore, $|[\varphi,\psi]|^2, [\varphi,\varphi][\psi,\psi],\ [[\psi,\psi][\varphi,\varphi]]$ belong to $L^1(\mathrm{VN}(G),\tau)$; specifically they are measurable with respect to trace τ (recall that for operators A, B which are measurable with respect to τ, $[A+B]$, $[AB]$ and A^* are also measurable with τ; consult, for instance [25, Proposition 24, p. 17] or [18, Theorem 4]). If we assume that $|[\varphi,\psi]|^2 = [\varphi,\varphi][\psi,\psi] = |[\psi,\varphi]|^2$ and $P_{N_\varphi^\perp} \leqslant P_{N_\psi^\perp}$, the fact that $[\varphi,\varphi][\psi,\psi]$ is positive implies that

$$[\varphi,\varphi][\psi,\psi] = ([\varphi,\varphi][\psi,\psi])^* \supseteq [\psi,\psi][\varphi,\varphi]. \tag{3.17}$$

Operator $[\varphi,\varphi][\psi,\psi]$ is closed, $[\psi,\psi][\varphi,\varphi]$ is preclosed and its closure belongs to $L^1(\mathrm{VN}(G),\tau)$; hence we have

$$[\varphi,\varphi][\psi,\psi] = [[\psi,\psi][\varphi,\varphi]].$$

We note that appropriate versions of the results can be formulated for such φ (for the arguments concerning unbounded operators which are used in the proof, we refer to [6] and [14, Section 6]); however, we skip the details, and turn to another, more important question.

4. The existence of (C–S) elements

Question 2. For the given unitary representation Π, does there exist $\psi \in \mathbb{H} \setminus \{0\}$ such that at least one of the inequalities (3.1) and (3.3) holds for all $\varphi \in \mathbb{H}$? Can we characterize such elements and in which cases the analogue of (1.5) can be used to characterize $\langle\psi\rangle$?

Recall that for an arbitrary von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathbb{H})$, the *center of* $\mathcal{M}$ is the abelian von Neumann algebra

$$Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}', \tag{4.1}$$

and $\mathcal{M}$ is a *factor* if $Z(\mathcal{M}) = \mathbb{C}1$, i.e., if the center of $\mathcal{M}$ contains only multiples of the operator 1. If p is a projection such that $p \in \mathcal{M}$, then $p\mathcal{M}p = \{pTp : T \in \mathcal{M}\}$ is a von Neumann algebra on $\mathcal{B}(p\mathbb{H})$, and its commutant is equal to $\mathcal{M}'p$. The following theorem (combined with the fact that the left regular representation is a cyclic dual integrable representation, and for the cyclic vector ψ, $\varphi \mapsto S_\psi(\varphi)$ maps $\langle\psi\rangle$ onto $L^2(\mathrm{VN}(G); [\psi,\psi])$) characterizes positive elements of $L^1(\mathrm{VN}(G);\tau)$ which are affiliated with $Z(\mathrm{VN}(G))$ as the operators for which (3.3) is fulfilled for all $\varphi \in \mathbb{H}$. Before the theorem, we emphasize one important fact separately in the following lemma.

Lemma 4.1. *Suppose that G is a unimodular group and Π is a dual integrable representation. Let $\psi \in \mathbb{H} \setminus \{0\}$ such that $[\psi,\psi] \in Z(\mathrm{VN}(G))$. Then*

$$P_{N_\varphi^\perp} \leqslant P_{N_\psi^\perp}, \forall \varphi \in \langle\psi\rangle. \tag{4.2}$$

Proof. Consider $[\psi,\psi] \in Z(\mathrm{VN}(G))$; then $[\psi,\psi]$ commutes with F, for any $F \in \mathrm{VN}(G)$. It follows from (3.6) that $N_\varphi^\perp \subseteq N_\psi^\perp$ (hence (4.2) holds) for any $\varphi \in \langle\psi\rangle$ such that $S_\psi(\varphi) \in \mathrm{VN}(G)$ (see also the discussion in [21, p. 28]; note that here we even have stronger assumption).

Consider now arbitrary $\varphi \in \langle\psi\rangle$. Since $P_{N_\psi^\perp}$ is both the left and the right support of $[\psi,\psi]$ (consult [2, 2.1]), we have $[\varphi,\varphi]P_{N_\varphi^\perp} = [\varphi,\varphi]$, and it is the minimal orthogonal projection with that property. Hence, it is enough to prove that $[\varphi,\varphi]P_{N_\psi^\perp} = [\varphi,\varphi]$. Take a sequence $\{\varphi_n\} \subseteq \mathrm{span}\{\Pi(g)\psi : g \in G\}$ such that $\varphi_n \to \varphi$. Then $[\varphi_n,\varphi_n] \to [\varphi,\varphi]$ in $L^1(\mathrm{VN}(G);\tau)$ (see [21, Lemma 4.12], or apply directly property (*iv*) on page 3). Specifically, we have

$$\begin{aligned}\varphi_0(\lambda(f)[\varphi,\varphi]P_{N_\psi^\perp}) &= \lim_{n\to\infty} \varphi_0(\lambda(f)[\varphi_n,\varphi_n]P_{N_\psi^\perp}) \\ &= \lim_{n\to\infty} \varphi_0(\lambda(f)[\varphi_n,\varphi_n]) = \varphi_0(\lambda(f)[\varphi,\varphi])\end{aligned}$$

for all $f \in L^1(G)$. It follows that $[\varphi,\varphi]P_{N_\psi^\perp} = [\varphi,\varphi]$. □

Theorem 4.2. *Suppose that Π is a dual integrable representation of a locally compact unimodular group G, and consider $\psi \in \mathbb{H} \setminus \{0\}$ such that $[\psi,\psi] \in \mathrm{VN}(G)$. The following are equivalent:*

(i) $|[\psi,\varphi]|^2 \leqslant [\varphi,\varphi][\psi,\psi]$, $\forall \varphi \in \mathbb{H}$,
(ii) $\lambda(g)[\psi,\psi] = [\psi,\psi]\lambda(g)$ *for all* $g \in G$, *i.e.,* $[\psi,\psi] \in Z(\mathrm{VN}(G))$.

Moreover, if either (i) *or* (ii) *is fulfilled, we have*

$$\varphi \in \langle\psi\rangle \quad \Leftrightarrow \quad |[\psi,\varphi]|^2 = [\varphi,\varphi][\psi,\psi] \text{ and } P_{N_\varphi^\perp} \leqslant P_{N_\psi^\perp},$$

and $|[\varphi,\psi]|^2 = |[\psi,\varphi]|^2$ *if and only if* $\mathrm{VN}(G)P_{N_\psi^\perp}$ *is abelian von Neumann algebra.*

Proof. Suppose that (i) holds. Specifically, for $\varphi = \Pi(g)\psi$, $g \in G$,

$$\lambda(g)[\psi,\psi][\psi,\psi]\lambda(g)^* \leqslant \lambda(g)[\psi,\psi]\lambda(g)^*[\psi,\psi].$$

It follows that for all $z \in L^2(G)$ we have

$$\langle\lambda(g)[\psi,\psi]\lambda(g)^*[\psi,\psi]z,z\rangle - \langle\lambda(g)[\psi,\psi][\psi,\psi]\lambda(g)^*z,z\rangle \geqslant 0;$$

consequently, for all $z' \in L^2(G)$ (consider $z = \lambda(g)z'$) we have

$$\langle[\psi,\psi]\lambda(g)^*[\psi,\psi]\lambda(g)z',z'\rangle - \langle[\psi,\psi][\psi,\psi]z',z'\rangle \geqslant 0.$$

Thus (also replacing g with g^{-1}), it follows that

$$[\psi,\psi][\psi,\psi] \leqslant [\psi,\psi]\lambda(g)[\psi,\psi]\lambda(g)^*$$

for all $g \in G$.

The latter implies

$$[\psi,\psi] \leqslant \lambda(g)[\psi,\psi]\lambda(g)^*, \quad \forall g \in G. \tag{4.3}$$

Indeed, observe first that, since $\lambda(g)[\psi,\psi]\lambda(g)^*$ commutes with $[\psi,\psi]$, it also commutes with $[\psi,\psi]^{1/2}$. We thus have

$$\langle[\psi,\psi]\lambda(g)[\psi,\psi]\lambda(g)^*z,z\rangle = \langle\lambda(g)[\psi,\psi]\lambda(g)^*([\psi,\psi]^{1/2}z),([\psi,\psi]^{1/2}z)\rangle.$$

for all $z \in L^2(G)$. Hence,

$$\langle\lambda(g)[\psi,\psi]\lambda(g)^*z,z\rangle \geqslant \langle[\psi,\psi]z,z\rangle$$

for all $z \in \overline{R([\psi,\psi]^{1/2})} = N_\psi^\perp$. On the other hand, since $\lambda(g)[\psi,\psi]\lambda(g)^*$ is positive,

$$\langle\lambda(g)[\psi,\psi]\lambda(g)^*z,z\rangle \geqslant 0 = \langle[\psi,\psi]z,z\rangle$$

for all $z \in N_\psi$. Hence, (4.3) holds.

Observe now that we actually have $[\psi,\psi]\lambda(g) = \lambda(g)[\psi,\psi]$ for all $g \in G$. Otherwise, there would exist $g \in G$ and $z \in L^2(G)$ is such that

$$\langle\lambda(g)[\psi,\psi]\lambda(g)^*z,z\rangle > \langle[\psi,\psi]z,z\rangle,$$

and for $z' = \lambda(g)^*z$ we would have

$$\langle[\psi,\psi]z',z'\rangle > \langle\lambda(g)^*[\psi,\psi]\lambda(g)z',z'\rangle,$$

which would derive a contradiction with (4.3). It follows that $[\psi,\psi]$ belongs to the center of $\mathrm{VN}(G)$.

Conversely, suppose that $[\psi,\psi] \in Z(\mathrm{VN}(G))$. Hence, $[\psi,\psi]$ commutes with every $F \in \mathrm{VN}(G)$. Moreover, since $[\varphi,\varphi]$ is a positive operator which belongs to $L^1(\mathrm{VN}(G);\tau)$, we have (see, for instance [25, Proposition 34, p. 54], or, using traciality of τ, note that $\tau([\varphi,\varphi][\psi,\psi]\lambda(g)^*) = \tau([[\psi,\psi][\varphi,\varphi]]\lambda(g)^*)$ for all $g \in G$)

$$[\varphi,\varphi][\psi,\psi] = [[\psi,\psi][\varphi,\varphi]], \quad \forall\varphi \in \mathbb{H},$$

and thus,

$$[\varphi,\varphi][\psi,\psi] = |[\varphi,\varphi]^{1/2}[\psi,\psi]^{1/2}|^2 \geqslant 0, \quad \forall\varphi \in \mathbb{H}. \tag{4.4}$$

Obviously, $[\psi,\psi]^{1/2}$ belongs to $Z(\mathrm{VN}(G))$, as well; hence, it commutes with $[\varphi,\varphi]^{1/2}$ for any $\varphi \in \langle\psi\rangle$ and $U \in \mathrm{VN}(G)$ appearing in the polar decomposition of $\mathcal{P}(\overline{\eta_\varphi})$. It follows that $\mathcal{P}(\overline{\eta_\varphi})\mathcal{P}(\sigma_\psi) = \mathcal{P}(\sigma_\psi)\mathcal{P}(\overline{\eta_\varphi})$. We now easily derive (i); apply also Lemma 4.1.

Suppose that $[\varphi,\psi][\psi,\varphi] = [\varphi,\varphi][\psi,\psi]$ and $P_{N_\varphi^\perp} \leqslant P_{N_\psi^\perp}$ hold. We would like to show that $\varphi \in \langle\psi\rangle$. Suppose to the contrary, that $\varphi \notin \langle\psi\rangle$. Observe that

$$|[\psi,\varphi]|^2 = [[\varphi_1,\varphi_1][\psi,\psi] + [\varphi_2,\varphi_2][\psi,\psi]],$$

(we consider again $\varphi_1 = P_{\langle\psi\rangle}(\varphi)$ and $\varphi_2 = P_{\langle\psi\rangle^\perp}(\varphi)$) while on the other hand we have

$$|[\psi,\varphi]|^2 = |[\psi,\varphi_1]|^2 = [\varphi_1,\varphi_1][\psi,\psi] \quad \text{and} \quad P_{N_{\varphi_1}^\perp} \leqslant P_{N_\psi^\perp}.$$

Therefore, it follows that $[\varphi_2,\varphi_2][\psi,\psi] = 0$, i.e. $\overline{R([\psi,\psi])} \subseteq N_{\varphi_2}$. Since $P_{N_{\varphi_2}^\perp} \leqslant P_{N_\psi^\perp} = P_{\overline{R([\psi,\psi])}}$, we have $\varphi_2 = 0$. Therefore, $\varphi \in \langle\psi\rangle$.

It remains to prove that $|[\varphi,\psi]|^2 = |[\psi,\varphi]|^2$ if and only if $\mathrm{VN}(G)P_{N_\psi^\perp}$ is abelian von Neumann algebra. Observe first that it follows from (ii) that $P_{N_\psi^\perp}$ is a central projection (and $N_\psi^\perp$ is left-invariant). Thus, $\mathrm{VN}(G)P_{N_\psi^\perp} = P_{N_\psi^\perp}\mathrm{VN}(G)P_{N_\psi^\perp}$ is a von Neumann algebra. If $\mathrm{VN}(G)P_{N_\psi^\perp}$ is abelian, then every $F \in \mathrm{VN}(G)P_{N_\psi^\perp}$ is normal, i.e., for such F we have

$$FF^* = F^*F.$$

If $F \in {}^{\mathrm{VN}(G)}/_{\mathcal{N}_\psi}$, then we can choose a representative which belongs to $\mathrm{VN}(G)P_{N_\psi^\perp}$. The result now follows by applying Theorem 3.1.

Now suppose that $|[\varphi,\psi|^2 = [\varphi,\varphi][\psi,\psi]$. This part is proved similarly as [21, Theorem 5.26]; we include the entire proof for the sake of completeness. From the assumption we get

$$[\psi,\psi]F^*F[\psi,\psi] = [\psi,\psi]F[\psi,\psi]F^* = [\psi,\psi]FF^*[\psi,\psi],$$

for all $F \in {}^{\mathrm{VN}(G)}/_{\mathcal{N}_\psi}$. It follows that

$$FF^* = F^*F$$

for all $F \in \mathrm{VN}(G)P_{N_\psi^\perp}$. If $F \in \mathrm{VN}(G)P_{N_\psi^\perp}$, then $\lambda(g)P_{N_\psi^\perp}F \in \mathrm{VN}(G)P_{N_\psi^\perp}$. It follows that for any $g \in G$ we have

$$\lambda(g)P_{N_\psi^\perp}FF^*\lambda(g)^*P_{N_\psi^\perp} = F^*F = FF^*;$$

hence, every positive element of $\mathrm{VN}(G)P_{N_\psi^\perp}$ belongs to its center. Since the latter is a C^*-algebra, any element can be written as a linear combination of its positive elements, and since the center is also the von Neumann algebra, it follows that any element of $\mathrm{VN}(G)P_{N_\psi^\perp}$ is contained in its center. Hence, $\mathrm{VN}(G)P_{N_\psi^\perp}$ is abelian. □

REMARK 4.3. Examining the proof above one more time, we can observe that condition $|[\varphi,\psi]|^2 \leqslant [\varphi,\varphi][\psi,\psi]$, $\forall\varphi \in \mathbb{H}$ implies (4.3); consequently, it follows that $[\psi,\psi] \in \mathrm{Z}(\mathrm{VN}(G))$, and, by Theorem 4.2 that $|[\psi,\varphi]|^2 \leqslant [\varphi,\varphi][\psi,\psi]$, $\forall\varphi \in \mathbb{H}$. Hence, inequality (3.3) (for the fixed ψ) is weaker than the inequality (3.1).

Consider now again the condition

$$|[\varphi,\psi]|^2 = [\varphi,\varphi][\psi,\psi] = |[\psi,\varphi]|^2 \text{ and } P_{N_\varphi^\perp} \leqslant P_{N_\psi^\perp}. \tag{4.5}$$

We have seen that if the latter is fulfilled for all $\varphi \in \langle\psi\rangle$ (note that actually we have the same conclusion if we consider only those for which $S_\psi(\varphi) \in \mathrm{VN}(G)$), this has strong implications on $\mathrm{VN}(G)$, in the sense of commutativity. On the other hand, we may observe that in some situations it might happen that elements $\psi \in \mathbb{H}$ satisfy neither stronger nor weaker inequality, whereas the Cauchy–Schwarz property in its full strength holds for φ belonging to a certain subset of $\mathbb{H}$. Therefore, it makes sense to weaken the requirement in the following sense. For $\psi \in \mathbb{H} \setminus \{0\}$, denote

$$CS_\psi = \{\varphi \in \langle\psi\rangle : (4.5) \text{ holds for } (\varphi,\psi)\}$$

and $\mathcal{M}_\psi = \{F \in {}^{\mathrm{VN}(G)}/_{\mathcal{N}_\psi} : S_\psi^{-1}(F) \in CS_\psi\}$. We have already observed that $\mathcal{M}_\psi$ is self-adjoint; note that $\mathcal{M}_\psi$ always contains an abelian von Neumann algebra, namely, the von Neumann algebra, $W^*(F)$, generated by the normal operator F.

THEOREM 4.4. *Suppose that G is a locally compact unimodular group and Π a dual integrable representation. Suppose that $[\psi,\psi] \in \mathrm{VN}(G)$ and $N_\psi = \{0\}$. The following are equivalent:*

(i) there exists a closed subgroup H of G such that CS_ψ contains a nontrivial closed $\Pi(H)$-invariant subspace of $\mathbb{H}_\Pi$,
(ii) $\mathcal{M}_\psi$ contains an invariant W^*-subalgebra of $\mathrm{VN}(G)$.

If either case, H is an abelian subgroup of G, and the invariant subalgebra in (b) is von Neumann algebra $\mathrm{VN}_H(G)$, which is the w^*-closure of *span*$\{\lambda_G(h) : h \in H\}$*, and it is abelian. Moreover, for any* $\varphi_1, \varphi_2 \in \langle\psi\rangle$ *such that* $\operatorname{supp}[\varphi_i, \varphi_i] \subseteq H$ *we have*

$$|[\varphi_1, \varphi_2]|^2 = [\varphi_1, \varphi_1][\varphi_2, \varphi_2]. \tag{4.6}$$

REMARK 4.5. For $T \in \mathrm{VN}(G)$, the *support* of T is defined as the set $\operatorname{supp} T$ of all $a \in G$ such that

$$u \in A(G), \quad u \cdot T = 0 \quad \Rightarrow \quad u(a) = 0;$$

for more details and equivalent formulations, consult [15, Proposition 2.5.3]. Here, $u \cdot T$ is defined by $\langle u \cdot T, v\rangle = \langle T, uv\rangle$, $v \in A(G)$. A W^*-subalgebra $\mathcal{M}$ of $\mathrm{VN}(G)$ is said to be *invariant* if for every $T \in \mathcal{M}$ and $u \in A(G)$ we have $u \cdot T \in \mathcal{M}$.

PROOF OF THEOREM 4.4. $(i) \Rightarrow (ii)$ Denote by S the closed nontrivial $\Pi(H)$-invariant subspace of $\mathbb{H}$ which is contained in CS_ψ. Obviously, $\Pi(H)$-invariance of S implies that $\{\lambda_G(h) : h \in H\} \subseteq \mathcal{M}_\psi$. Similarly as in Theorem 4.2, we conclude that $[\psi, \psi]\lambda_G(h) = \lambda_G(h)[\psi, \psi]$ for all $h \in H$. Thus, $[\psi, \psi]$ commutes with every $F \in \mathrm{VN}_H(G) = \overline{\mathrm{span}\{\lambda_G(h) : h \in H\}}^{w^*}$ (recall that w^* represents the closure with respect to w^*-topology). Moreover, any such F belongs to $\mathcal{M}_\psi$ and, again as in Theorem 4.2, it follows that any element $F \in \mathrm{VN}_H(G)$ is normal and that $\mathrm{VN}_H(G)$ is abelian von Neumann algebra. Consequently, since $\lambda_G(h)\lambda_G(h') = \lambda_G(h')\lambda_G(h)$ for all $h, h' \in H$, and λ_G is a faithful representation, it follows that H is abelian group.

$(ii) \Rightarrow (i)$ Suppose that $\mathcal{M}_\psi$ contains an invariant W^*-subalgebra $\mathcal{M}$ of $\mathrm{VN}(G)$. By [15, Corollary 3.4.7], it follows that

$$H = \{h \in G : \lambda_G(h) \in \mathcal{M}\}$$

is a closed subgroup of G and $\mathcal{M}$ equals $\mathrm{VN}_H(G)$. Since $\{\lambda_G(h) : h \in H\} \subseteq \mathcal{M}_\psi$, it follows that $\{\Pi(h)\psi : h \in H\} \subseteq CS_\psi$. Similarly as in $(i) \Rightarrow (ii)$, we prove that $\mathrm{VN}_H(G)$ is abelian and that H is abelian group.

Now, (4.6) follows easily from Theorem 3.1 and [15, Theorem 3.4.6]. □

5. CONSEQUENCES AND EXAMPLES

The results of the previous sections show that there certainly exist non-commutative groups and dual integrable representations such that the Cauchy-Schwarz property is fulfilled at least for some pairs of elements. In this section we provide the more detailed analysis according to the type of group and type of von Neumann algebra.

Results of Theorem 4.2 remain valid if consider general ψ (not necessarily the one such that $[\psi, \psi]$ is bounded). Observe first here that although $|[\varphi, \psi]|^2$ and $[\varphi, \varphi][\psi, \psi]$ might not belong to $L^1(\mathrm{VN}(G); \tau)$, they are at least measurable operators with respect to trace τ, preclosed and densely defined, whenever we take $\varphi \in \langle\psi\rangle$ such that $S_\psi(\varphi) \in \mathrm{VN}(G)$ (consult [25, Proposition

20 and Proposition 24, p. 17] or [18, Theorem 4]). We shall call $\psi \in \mathbb{H} \setminus \{0\}$ a *(C-S) element* if $|[\varphi,\psi]|^2 = [[\varphi,\varphi][\psi,\psi]] = [\psi,\varphi]^2$ is fulfilled for all $\varphi \in \langle\psi\rangle$ such that $S_\psi(\varphi) \in \mathrm{VN}(G)$. Observe that, for $\varphi = \Pi(g)\psi$, the latter implies that $D([\psi,\psi]\lambda(g)) = D([\psi,\psi])$, $D([\psi,\psi][\psi,\psi]\lambda(g)) = D([\psi,\psi][\psi,\psi])$ and

$$\lambda(g)[\psi,\psi]([\psi,\psi]\lambda(g)^* - \lambda(g)^*[\psi,\psi]) = 0.$$

It follows that $[\psi,\psi]\lambda(g)^* = \lambda(g)^*[\psi,\psi]$ on $N_\psi^\perp \cap D([\psi,\psi])$, since $D([\psi,\psi][\psi,\psi])$ is a core for $[\psi,\psi]$. The latter implies that $N_\psi^\perp \cap D([\psi,\psi])$ is λ-invariant. Hence, N_ψ is also λ-invariant. Thus, we have $\lambda(g)^*[\psi,\psi] = [\psi,\psi]\lambda(g)^*$ for all $g \in G$, i.e., $[\psi,\psi]$ is affiliated with $Z(\mathrm{VN}(G))$. Consequently, $[\psi,\psi]$ commutes with any $F \in \mathrm{VN}(G)$, and

$$FF^*[\psi,\psi][\psi,\psi] = F^*F[\psi,\psi][\psi,\psi] = 0$$

for all $F \in \mathrm{VN}(G)P_{N_\psi^\perp}$ (observe, since $P_{N_\psi^\perp}$ is central, F^* also belongs to $\mathrm{VN}(G)P_{N_\psi^\perp}$). Now, similarly as in the proof of Theorem 4.2 we conclude that $\mathrm{VN}(G)P_{N_\psi^\perp}$ is abelian von Neumann algebra.

We have thus observed that if $\psi \in \mathbb{H}$ is a (C-S) element, then $P_{N_\psi^\perp}$ is an abelian projection. Recall that a von Neumann algebra is of *type II* if it is semi-finite and has no non-zero abelian projections. Hence, we have an immediate corollary.

Corollary 5.1. *Suppose that* Π *is a dual integrable representation of a locally compact unimodular group and* $\psi \in \mathbb{H} \setminus \{0\}$ *is a (C-S) element. Then* $\mathrm{VN}(G)$ *is not of type II.*

Consequently, if we consider non-commutative countable discrete groups, then we should search for the (C-S) elements within the class of virtually abelian groups. The following result shows that among groups for which $\mathrm{VN}(G)$ is of type I, we can exclude factors.

Corollary 5.2. *Suppose that* G *is a locally compact unimodular group. If* $\mathrm{VN}(G)$ *is a type I factor and* Π *is a dual integrable representation, then there does not exist any* $\psi \in \mathbb{H} \setminus \{0\}$ *which is a (C-S) element unless* $\dim \mathbb{H}_\Pi = 1$.

Proof. Suppose to the contrary, that $0 \neq \psi \in \mathbb{H}$ is a (C-S) element. Since $\mathrm{VN}(G)$ is a factor, we have $Z(\mathrm{VN}(G)) = \mathbb{C}1_{\mathbb{H}_\Pi}$. It follows that $P_{N_\psi^\perp} = 1_{\mathbb{H}_\Pi}$; since it is an abelian projection, it follows that $P_{N_\psi^\perp}$ is minimal. Consequently, $P_{N_{\psi^\perp}}$ is a rank one projection, which implies that $\dim \mathbb{H}_\Pi = 1$. ∎

We have seen that the assumption $N_\psi = \{0\}$ combined with (C-S)-property has strong consequences on the underlying group; however, in general, given a unitary representation, there might not exist any $\psi \in \mathbb{H}$ such that $[\psi,\psi]$ is injective (this was already emphasized in the abelian case; consult [22, Section 3]). Note however that the existence of a (C-S)-element with

$[\psi,\psi]$ not injective may still imply that G is abelian; this is true as long as $(G,\lambda_G,N_\psi^\perp)$ is a faithful representation.

EXAMPLE 5.3. Consider the dihedral group D_4; it is a finite non-abelian group of order 8, $D_4=\{1,x,x^2,x^3,y,xy,x^2y,x^3y\}$, its presentation is given by $D_4=\langle x,y|x^4=1,y^2=1,yx=x^{-1}y\rangle$, and $Z(D_4)=\{1,x^2\}$. We may also view D_4 as the semidirect product $\mathbb{Z}_4\rtimes\mathbb{Z}_2$ with the group operation given by $(j,k)(j',k')=(j+(-1)^k j',k+k')$. Observe that G can be written as a disjoint union of N and Ny, where $N=\{1,x,x^2,x^3\}$ is abelian subgroup of D_4. Moreover, $L^2(N)$ and $L^2(Ny)$ are $\lambda(N)$-invariant subspaces of $L^2(G)$, and $L^2(G)=L^2(N)\oplus L^2(Ny)$.

Consider the left regular representation, λ_G; it is known that λ_G is dual integrable. Moreover, it is cyclic, and for any cyclic vector ψ', $\varphi\mapsto\mathcal{P}(\overline{\eta_\varphi})$ maps $\langle\psi'\rangle$ onto $L^2(\mathrm{VN}(G);\tau)$. Let T be any positive operator which belongs to $\mathrm{VN}_N(G)$. Since G is finite, $T^{1/2}\in L^2(\mathrm{VN}(G);\tau)$; hence, there exists $\psi\in\mathbb{H}\setminus\{0\}$ such that $[\psi,\psi]=T$. Since N is an abelian subgroup of G, we obviously have $\mathcal{M}_\psi\supseteq\mathrm{VN}_N(G)$. Moreover, if we take T which also belongs to $Z(\mathrm{VN}(G))$, then $|[\psi,\varphi]|^2\leqslant[\varphi,\varphi][\psi,\psi]$ for all $\varphi\in\mathbb{H}$.

Somewhat surprisingly, irreducibility condition, too, implies the lack of (C-S) elements (even satisfying only the weaker inequality); in connection with the result, see also [5, Proposition 7.A.4]. We first derive one easy, but useful consequence for the (C-S) pairs.

LEMMA 5.4. *Suppose that ψ is a (C-S) element such that $[\psi,\psi]\in\mathrm{VN}(G)$. Then for all $\varphi,\varphi'\in\langle\psi\rangle$ such that $S_\psi(\varphi),S_\psi(\varphi')\in\mathrm{VN}(G)$ we have*

$$[\varphi,\varphi']=0\quad\Leftrightarrow\quad N_\varphi^\perp\cap N_{\varphi'}^\perp=\emptyset.$$

PROOF. By Theorem 4.2, (φ,ψ) and (φ',ψ) are (C-S) pairs. By Theorem 3.1, $\mathcal{P}(\overline{\eta_\varphi})$ and $\mathcal{P}(\overline{\eta_{\varphi'}})$ are normal; hence, besides $\overline{R(\mathcal{P}(\overline{\eta_{\varphi'}})^*)}=N_{\varphi'}^\perp$, we have $N(\mathcal{P}(\overline{\eta_\varphi}))=N_\varphi$. Therefore,

$$[\varphi,\varphi']=0\quad\Leftrightarrow\quad\mathcal{P}(\overline{\eta_\varphi})\mathcal{P}(\overline{\eta_{\varphi'}})^*=0\quad\Leftrightarrow\quad N_{\varphi'}^\perp\subseteq N_\varphi.$$

□

PROPOSITION 5.5. *Suppose that Π is an irreducible dual integrable representation of a locally compact unimodular group G. Then there exists no $\psi\in\mathbb{H}\setminus\{0\}$ such that*

$$|[\psi,\varphi]|^2\leqslant[\varphi,\varphi][\psi,\psi],\quad\forall\varphi\in\mathbb{H}$$

unless $\dim\mathbb{H}_\Pi=1$.

PROOF. Suppose that Π is irreducible and that the weaker inequality is fulfilled. Since Π is irreducible, we have $\langle\psi\rangle=\mathbb{H}_\Pi$, $\{0\}$ and $\mathbb{H}_\Pi$ being the only closed Π-invariant subspaces. It follows from $[\psi,\psi]\in L^1(\mathrm{VN}(G);\tau)$ that

$P_{N_\psi^\perp} \in \mathrm{VN}(G)$ and $N_\psi^\perp$ is right-invariant. Moreover, $[\psi,\psi]$ is affiliated with $\mathrm{VN}(G)'$; consequently, $N_\psi^\perp$ is also left-invariant.

We shall prove that $(G,\rho,N_\psi^\perp)$ is an irreducible representation. Suppose to the contrary, that there exists a nontrivial closed right-invariant subspace M of $N_\psi^\perp$, $M \neq N_\psi^\perp$. Let $M' := M^\perp \cap N_\psi^\perp$. Then M' is also nontrivial right-invariant closed subspace of $N_\psi^\perp$ and $P_{M'} \in \mathrm{VN}(G)$. Observe that $P_M, P_{M'} \in L^2(\mathrm{VN}(G);[\psi,\psi])\setminus\{0\}$. Hence, there exist $0 \neq \varphi,\varphi' \in \langle\psi\rangle$ such that $S_\psi(\varphi) = P_M$ and $S_\psi(\varphi') = P_{M'}$. If we denote by η_φ and $\eta_{\varphi'}$ the elements of $L^2(G)$ which satisfy (2.9), since $[\psi,\psi]$ commutes with elements of $\mathrm{VN}(G)$, we have

$$[\varphi,\varphi'] = \mathcal{P}(\overline{\eta_\varphi})\mathcal{P}(\overline{\eta_{\varphi'}})^* = P_M|\mathcal{P}(\sigma_\psi)|^2 P_{M'} = P_M P_{M'}[\psi,\psi] = 0.$$

Hence, $\langle\varphi\rangle \subsetneqq \langle\psi\rangle$, which derives a contradiction with the assumption that Π is irreducible.

It now follows from irreducibility of $(G,\rho,N_\psi^\perp)$ that the only bounded operators on $N_\psi^\perp$ which commute with $\rho(g)$ are of form $\mathbb{C}P_{N_\psi^\perp}$. On the other hand, we know that the elements of $\mathrm{VN}(G)P_{N_\psi^\perp}$ commute with $\rho(g)$ for all $g \in G$. This is possible only if $\mathbb{H}_\Pi$ was one-dimensional. ∎

Recall that irreducible square integrable representations are dual integrable; for such representations we know that $F_{\varphi,\psi} \in L^2(G)$ for all $\varphi,\psi \in \mathbb{H}$, if the group G is unimodular. Could the latter in general imply the lack of (C-S) elements? The answer is negative; however, combination of these conditions characterizes the group is the following sense.

PROPOSITION 5.6. *Suppose that G is a locally compact unimodular group and Π a dual integrable representation of G on $\mathbb{H}$ such that there exists $\psi' \in \mathbb{H}$ with $N_{\psi'} = \{0\}$. The following are equivalent:*

(i) for every $\psi \in \mathbb{H}$, $[\psi,\psi]$ is bounded and there exists a (C-S) element ψ such that $[\psi,\psi]$ is injective,

(ii) G is a compact abelian group.

PROOF. $(ii) \Rightarrow (i)$ Suppose that G is a compact abelian group. It follows that the Haar measure on G is finite; hence, $F_{\varphi,\psi} \in L^\infty(G) \subseteq L^2(G)$ for all $\varphi,\psi \in \mathbb{H}$, which implies that every $[\psi,\psi]$ is bounded. Since G is abelian, (i) holds for every $\psi \in \mathbb{H}$.

$(i) \Rightarrow (ii)$ Consider $\psi \in \mathbb{H}$ such that $N_\psi = \{0\}$ and ψ is a (C-S)-element. By [21], it follows that G is abelian group. It remains to show that G is compact.

Observe first that, since $\mathcal{B}_\psi$ is Bessel, we have

$$\sigma_\psi * \eta \in L^2(G), \quad \forall \eta \in L^2(G). \tag{5.1}$$

Thus, we have $\widehat{\sigma_\psi} * \widehat{\eta} \in L^2(\widehat{G})$, for all $\eta \in L^2(G)$; here, $f \mapsto \widehat{f}$ is the Fourier transform on G. Since for any $h \in L^2(\widehat{G})$ such that $h \geqslant 0$ there exists $\varphi \in \langle\psi\rangle$

such that $p_\varphi^{1/2} = h$ (see [22, Corollary 3.5]), and since every $f \in L^2(\widehat{G})$ can be written as a linear combination of four non-negative functions from $L^2(\widehat{G})$, it follows that

$$\sigma * \eta \in L^2(G), \quad \forall \eta, \sigma \in L^2(G),$$

i.e., $L^2(G) * L^2(G) \subseteq L^2(G)$. This implies that G is a compact group (see e.g. [19]). □

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O VALJANOSTI CAUCHY–SCHWARZOVE NEJEDNAKOSTI ZA BRACKET PRESLIKAVANJE

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Sažetak. Zbog svojih svojstava, bracket preslikavanje pridruženo dualno integrabilnoj unitarnoj reprezentaciji lokalno kompaktne grupe može se promatrati kao izvjesni unutarnji produkt čije su vrijednosti operatori; međutim, u nekomutativnom okruženju, Cauchy-Schwarzovo svojstvo za bracket više nije prisutno u svojoj punoj snazi. U ovom radu pokazujemo da ispunjenje ovog svojstva, čak i u slabijim oblicima, ima jake posljedice na pripadnu grupu G i odgovarajuću von Neumannovu algebru $\mathrm{VN}(G)$. Konkretno, pokazujemo da za unimodularnu grupu G pozitivni elementi L^1 prostora nad $\mathrm{VN}(G)$ koji su pridruženi komutantu od $\mathrm{VN}(G)$ su upravo oni za koje je ispunjena slabija varijanta ove nejednakosti, te da valjanost Cauchy–Schwarzova svojstva za odgovarajući skup elemenata ukazuje na postojanje zatvorene abelove podgrupe ili abelove von Neumannove podalgebre od $\mathrm{VN}(G)$.