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MARKOV SET-VALUED FUNCTIONS ON COMPACT METRIC SPACES

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Abstract. We generalize the notion of Markov functions on closed intervals $[a, b]$ to Markov set-valued functions on compact metric spaces. We also introduce when two such Markov set-valued functions follow the same pattern and show that if the Markov set-valued functions $F : X \multimap X$ and $G : Y \multimap Y$ follow the same pattern, then the inverse limits $\varprojlim(X, F)$ and $\varprojlim(Y, G)$ are homeomorphic.

1. Introduction

In present paper, we generalize the notion of Holte's Markov single-valued functions [8] (which are often used in the theory of discrete topological dynamical systems and they allow the symbolic dynamics to be used in the study of such a dynamical system) to Markov upper semi-continuous set-valued functions on arbitrary compacta. Note that several papers on the topic of dynamical systems with (upper semi-continuous) set-valued functions have appeared recently, see [12, 14–16, 18], where more references may be found. However, there is not much known of such dynamical systems and, therefore, there are many properties of such set-valued dynamical systems that are yet to be studied.

The Markov partition of a closed interval $I = [0, 1]$ with respect to a continuous function $f : I \to I$ is usually given by finitely many points $0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$ in I such that all the restrictions $f|_{[x_{i-1},x_i]}$ of f to $[x_{i-1}, x_i]$ are homeomorphisms from $[x_{i-1}, x_i]$ onto some interval $[x_k, x_\ell]$. Since a Markov partition is usually given by a finite collection

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of points $A = \{x_0, x_1, \ldots, x_n\} \subseteq I$, we usually refer to A as a Markov partition for I. If a continuous function f has a Markov partition A, then we say that f is a Markov function with respect to A. A Markov partition of I with respect to f (if it exists) is a well-known tool in the dynamical system theory that allows the symbolic dynamics to be used in the study of the dynamical system (I, f). For more information about Markov partitions in dynamical systems and symbolic dynamics, see [4, 13].

Let the function f be a Markov function with respect to some Markov partition $A = \{a_0, a_1, a_2, \ldots, a_m\}$ for an interval I and let g be a Markov function with respect to some Markov partition $B = \{b_0, b_1, b_2, \ldots, b_m\}$ of an interval J. In [8], Holte defined when the Markov functions f and g follow the same pattern (with respect to A and B): $f(a_j) = a_k$ if and only if $g(b_j) = b_k$ for all j and k. Then she proved the following theorem, which is one of the main results of [8].

THEOREM 1.1. *Let I and J be closed intervals, let f be a Markov function with respect to $A \subseteq I$ and let g be a Markov function with respect to $B \subseteq J$. If f and g follow the same pattern then the inverse limits $\varprojlim(I, f_n)$ and $\varprojlim(J, g_n)$ are homeomorphic.*

Some generalizations of Holte's result have already been introduced, for examples see [2, 3, 5–7, 9, 10] where more references may be found. In all of the mentioned papers, the authors mostly generalized

1. the setting of such a Markov system from closed intervals to trees or graphs;
2. the notion of Markov partitions A of closed intervals I (we point out that they all have one thing in common — every generalization of Holte's Markov partition is closed in I);
3. the notion of Markov functions F on intervals — they are usually upper semi-continuous functions (or sometimes just set-valued functions) on closed intervals I that have a special structure on
   (a) the given Markov partition A on I (usually the boundary of the set $F(a)$ is a subset of A for any $a \in A$), as well as
   (b) on the complement of A, where it is usually assumed that on each connected component of the complement of A, F is a single-valued injective function (as in the original Holte's paper) or its inverse is single-valued;
4. the notion when two Markov functions follow the same pattern.

Then the main theorem is proven, saying that if Markov functions F and G follow the same pattern, then their inverse limits are homeomorphic. In present paper we generalize Holte's result in the following ways:

1. the setting of such a Markov system from closed intervals, trees or graphs to any compact metric spaces;

2. the notion of Markov partitions A of closed intervals I to Markov partitions on any compact metric spaces;
3. the notion of Markov functions on intervals to Markov functions F on any compact metric spaces X in such a way that
   (a) for a given Markov partition A on X, the boundary of the set $F(a)$ is a subset of A for any $a \in A$, and
   (b) on each connected component of $X \setminus A$, the function F is mimicking a single-valued injective function;
4. the notion when two Markov functions follow the same pattern.

Then we prove our main theorem which says that, if Markov functions F and G follow the same pattern, then their inverse limits are homeomorphic.

2. Definitions and notation

Definition 2.1. *Let X be a metric space and let $A \subseteq X$. We use* $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$ *and* $\mathrm{Bd}(A)$ *to denote the closure, the interior and the boundary, respectively, of the set A in X.*

Definition 2.2. A continuum *is a non-empty connected compact metric space. A continuum is* degenerate *if it consists of only a single point. Otherwise it is* non-degenerate. A subcontinuum *is a subspace of a continuum which itself is also a continuum.*

Definition 2.3. *Let X be a continuum and let $a, b \in X$. We say that* X is irreducible between points a and b, *if for any subcontinuum Y of X the following holds:*

$$a, b \in Y \Longrightarrow Y = X.$$

We say that X is irreducible *if there are points $a, b \in X$ such that X is irreducible between points a and b.*

The following theorem is a well-known result.

Theorem 2.4. *Let X be a continuum and let $a, b \in X$. Then there is a subcontinuum Y of X such that Y is irreducible between a and b.*

Proof. See [17, page 68]. □

Definition 2.5. *Let $f : X \to Y$ be a function. We use $\Gamma(f)$ to denote the graph of the function f.*

Definition 2.6. An inverse sequence of compact metric spaces and continuous bonding functions *is any double sequence (X_n, f_n) of compact metric spaces X_n and continuous functions $f_n : X_{n+1} \to X_n$.* The inverse limit of such an inverse sequence (X_n, f_n) *is defined to be the subspace of $\prod_{n=1}^{\infty} X_n$ of all points*

$$\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \prod_{n=1}^{\infty} X_n,$$

such that $x_n = f_n(x_{n+1})$ *for each positive integer* n. *The inverse limit is denoted by* $\varprojlim(X_n, f_n)$.

DEFINITION 2.7. *If* X *is a metric space, then* 2^X *denotes the family of all non-empty compact subspaces of* X.

DEFINITION 2.8. *Let* X *be a compact metric space and let* (A_n) *be a sequence of subsets of* X. *Then we use* $\limsup A_n$ *and* $\liminf A_n$ *to denote the limes superior and the limes inferior of the sequence* (A_n), *respectively, where*

$$\limsup A_n = \{x \in X \mid \text{for each open set } U \text{ in } X,\ x \in U \Longrightarrow U \cap A_n \neq \emptyset \text{ for infinitely many } n\}$$

and

$$\liminf A_n = \{x \in X \mid \text{for each open set } U \text{ in } X,\ x \in U \Longrightarrow U \cap A_n \neq \emptyset \text{ for all but finitely many } n\}$$

If $\limsup A_n = \liminf A_n$, *then we define the limit of the sequence of sets* (A_n) *as follows:*

$$\lim_{n\to\infty} A_n = \limsup A_n = \liminf A_n.$$

OBSERVATION 2.9. *Let* X *be a compact metric space, let* $x \in X$ *and* $A \subseteq X$, *let* (x_n) *be a sequence of points in* X *such that* $\lim_{n\to\infty} x_n = x$ *and let* (A_n) *be a sequence of subsets of* X *such that* $\lim_{n\to\infty} A_n = A$. *Then* $A \in 2^X$ *and if for each positive integer* n, $x_n \in A_n$, *then* $x \in A$.

DEFINITION 2.10. *Let* X *and* Y *be metric spaces. The function* $F : X \to 2^Y$ *is called* a set-valued function from X to Y *and is denoted by* $F : X \multimap Y$. The graph of a set-valued function F *is defined to be the subset of* $X \times Y$, *which is defined by*

$$\Gamma(F) = \{(x, y) \mid y \in F(x),\ x \in X\}.$$

DEFINITION 2.11. *A set-valued function* $F : X \multimap Y$ *is* an upper semi-continuous set-valued function *if for any* $x_0 \in X$ *and for any open set* U *in* Y, *it holds that if* $F(x_0) \subseteq U$, *then there is an open set* V *in* X *such that*

1. $x_0 \in V$ *and*
2. *for each* $x \in V$, $F(x) \subseteq U$.

There is a simple characterization of upper semicontinuous set-valued functions ([1, Proposition 11, p. 128] and [11, Theorem 1.2, p. 3]).

THEOREM 2.12. *Let* X *and* Y *be compact metric spaces and* $F : X \multimap Y$ *a set-valued function. Then* F *is upper semicontinuous if and only if its graph* $\Gamma(F)$ *is closed in* $X \times Y$.

OBSERVATION 2.13. *Let* X *and* Y *be any compact metric spaces.*

1. *Let $F : X \multimap Y$ be a set-valued function such that for each $x \in X$ there is exactly one $y_x \in Y$ such that*
$$F(x) = \{y_x\}$$
and let $f : X \to Y$ be defined by $f(x) = y_x$ for any $x \in X$. Then $\Gamma(f) = \Gamma(F)$.
2. *Let $f : X \to Y$ be a function and let $F : X \multimap Y$ be defined by $F(x) = \{f(x)\}$ for each $x \in X$. Then $\Gamma(F) = \Gamma(f)$.*

Note that in both cases,
$$f \text{ is continuous } \Longleftrightarrow F \text{ is upper semicontinuous.}$$

Definition 2.14. *Let $F : X \multimap Y$ be a set-valued function and let $A \subseteq X$. Then*
$$F|_A : A \multimap Y,$$
defined by $F|_A(x) = F(x)$ for each $x \in A$, is the restriction of F to A.

Definition 2.15. *Let $F : X \multimap Y$ be a set-valued function such that for each $x \in X$ there is exactly one $y_x \in Y$ such that $F(x) = \{y_x\}$. Then we always use $\overline{F}$ to denote the function $\overline{F} : X \to F(X)$, defined by $\overline{F}(x) = y_x$ for any $x \in X$.*

Definition 2.16. A generalized inverse sequence of compact metric spaces and set-valued bonding functions *is any double sequence (X_n, F_n) of compact metric spaces X_n and set-valued functions $F_n : X_{n+1} \multimap X_n$.* The generalized inverse limit of such a generalized inverse sequence (X_n, F_n) *is defined to be the subspace of $\prod_{n=1}^{\infty} X_n$ of all points $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \prod_{n=1}^{\infty} X_n$, such that $x_n \in F_n(x_{n+1})$ for each positive integer n. The generalized inverse limit is denoted by $\varprojlim(X_n, F_n)$.*

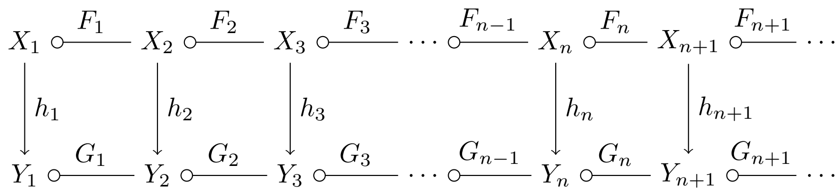
Inverse limits with upper semicontinuous set-valued bonding functions were first introduced in 2004 by Mahavier and later by Ingram and Mahavier. Since their introduction many authors have been interested in this area and many papers appeared (for more details and other references see [11]).

Definition 2.17. *Let $f : X \to Y$, $G : Y \multimap Z$ and $h : Z \to W$. Then $G \circ f$ and $h \circ G$ are defined by*
$$(G \circ f)(x) = G(f(x))$$
for any $x \in X$ and
$$(h \circ G)(y) = h(G(y))$$
for any $y \in Y$.

We also use the following well-known result in the proofs of our main theorems.

LEMMA 2.18. *Let (X_n, F_n) and (Y_n, G_n) be two generalized inverse sequences of compact metric spaces and upper semicontinuous set-valued functions. If for each positive integer n, there is a homeomorphism $h_n : X_n \to Y_n$ such that $h_n \circ F_n = G_n \circ h_{n+1}$, then the inverse limits $\varprojlim (X_n, F_n)$ and $\varprojlim (Y_n, G_n)$ are homeomorphic.*

(2.1)



PROOF. For any $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \varprojlim (X_n, F_n)$, we define

$$h(\mathbf{x}) = (h_1(x_1), h_2(x_2), h_3(x_3), \ldots).$$

Obviously, $h : \varprojlim (X_n, F_n) \to \varprojlim (Y_n, G_n)$, since for any positive integer n,

$$h_n \circ F_n = G_n \circ h_{n+1}$$

and therefore $h_n(x_n) \in G_n(h_{n+1}(x_{n+1}))$. It also follows that $h : \varprojlim (X_n, F_n) \to \varprojlim (Y_n, G_n)$ is a continuous function since each h_n is a continuous function.

Next, for any $\mathbf{y} = (y_1, y_2, y_3, \ldots) \in \varprojlim (Y_n, G_n)$ we define

$$g(\mathbf{y}) = (h_1^{-1}(y_1), h_2^{-1}(y_2), h_3^{-1}(y_3), \ldots).$$

Since for any positive integer n, $h_n \circ F_n = G_n \circ h_{n+1}$, it follows that $h_n^{-1} \circ G_n = F_n \circ h_{n+1}^{-1}$. Therefore $h_n^{-1}(y_n) \in F_n(h_{n+1}^{-1}(y_{n+1}))$. This means that

$$g : \varprojlim (Y_n, G_n) \to \varprojlim (X_n, F_n)$$

is a continuous function. Note that

$$g(h(\mathbf{x})) = (h_1^{-1}(h_1(x_1)), h_2^{-1}(h_2(x_2)), h_3^{-1}(h_3(x_3)), \ldots) = \mathbf{x}$$

for each $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \varprojlim (X_n, F_n)$ and

$$h(g(\mathbf{y})) = (h_1(h_1^{-1}(y_1)), h_2(h_2^{-1}(y_2)), h_3(h_3^{-1}(y_3)), \ldots) = \mathbf{y}$$

for each $\mathbf{y} = (y_1, y_2, y_3, \ldots) \in \varprojlim (Y_n, G_n)$. Therefore, h is a homeomorphism.

□

3. Markov set-valued functions on arbitrary compact metric spaces

In this section we introduce the concept of Markov set-valued functions on compact metric spaces and prove the main result of the paper.

Definition 3.1. *Let C be a continuum, let A be a totally disconnected closed subset of C, let $a, b \in A$, and let $\mathcal{K}_{a,b}^C$ be the family of subcontinua K of C such that*

1. *K is irreducible between a and b and*
2. *$K \cap A = \{a, b\}$.*

Then we use $K_{A,C}[a, b]$ to denote the set

$$K_{A,C}[a, b] = C$$

if $|A| = 2$, and

$$K_{A,C}[a, b] = \bigcup_{K \in \mathcal{K}_{a,b}^C} K$$

to denote the union of all subcontinua of C that are irreducible between a and b, if $|A| > 2$. We also use

$$K_{A,C}(a, b) = K_{A,C}[a, b] \setminus \{a, b\} \text{ and } K_{A,C}[a, b) = K_{A,C}[a, b] \setminus \{b\}.$$

Observation 3.2. *Let C be a continuum, let A be a totally disconnected closed subset of C, and let $a, b \in A$. Then the following holds.*

1. *$K_{A,C}[a, b] = K_{A,C}[b, a]$,*
2. *If $K_{A,C}[a, b] \neq \emptyset$, then $a, b \in K_{A,C}[a, b]$.*

Also, note that $K_{A,C}[a, b]$ is not necessarily open or closed in C; see the following example and Figure 1 for an idea of how to construct such a $K_{A,C}[a, b]$.

Example 3.3. Let C be the continuum in Figure 1 (it is the union of the following line segments: the black line segment from a to b, the black line segment from c to d, the black line segment from a to c, and the union of blue arcs C_n all from a to b such that $\lim_{n\to\infty} C_n = Y$, where Y is the union of the following arcs: the black line segment from a to c, the black line segment from c to d and the black line segment from v to b) and let $A = \{a, b, c, d\}$. Then $K_{A,C}[a, b]$ is not open or closed in C.

Definition 3.4. *Let C be a continuum, let A be a totally disconnected closed subset of C and let $a, b \in A$. We say that (a, b) is an* admissible pair in C with respect to A, *if $a \neq b$ and if*

1. *either $|A| = 2$*
2. *or $|A| > 2$ and there is a subcontinuum K of C which is irreducible between a and b such that*

$$K \cap A = \{a, b\}.$$

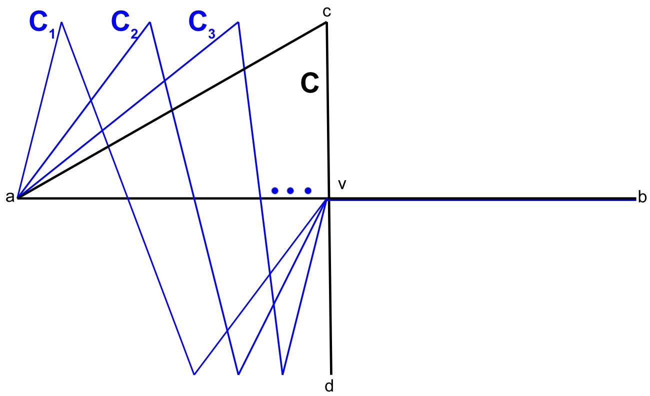


Figure 1. $K_{A,C}[a,b]$ that is not closed or open in C.

We use

$$\mathcal{A}(A,C) = \{(a,b) \in A \times A \mid (a,b) \text{ is an admissible pair in } C \text{ with respect to } A\}$$

to denote the set of all admissible pairs in C with respect to A.

Definition 3.5. *Let X be a compact metric space, let A be a totally disconnected closed subset of X and let $\mathcal{C}$ be the family of all connected components of X. We define*

$$\mathcal{A}(A,X) = \bigcup_{C \in \mathcal{C}} \mathcal{A}(C \cap A, C).$$

Definition 3.6. *Let X be a compact metric space. For each $x \in X$, we use C_X^x to denote the connected component of X that contains the point x.*

Definition 3.7. *Let C be a non-degenerate continuum and let A be a totally disconnected closed subset of C. We say that A takes over C if for each $x \in C \setminus A$ there is a unique admissible pair $(a,b) \in \mathcal{A}(A,C)$ such that $x \in K_{A,C}(a,b)$.*

Definition 3.8. *Let C be a non-degenerate continuum and let A be a totally disconnected closed subset of C such that A takes over C. For each $x \in C \setminus A$, we define a_x, $b_x \in A$ to be the points such that $(a_x, b_x) \in \mathcal{A}(A,C)$ and*

$$x \in K_{A,C}(a_x, b_x).$$

Definition 3.9. *Let X be a compact metric space and let A be a totally disconnected closed subset of X. We say that A* is a Markov partition for *X, if*

1. *for each degenerate connected component C of X, $C \subseteq A$;*
2. *for each non-degenerate connected component C of X, $|C \cap A| \geq 2$ and $C \cap A$ takes over C;*
3. *for each non-degenerate connected component C of X and for each $(a,b) \in \mathcal{A}(C \cap A, C)$, the set $K_{C \cap A, C}(a,b)$ is open in X;*
4. *for any sequence (x_n) in $X \setminus A$ such that for all positive integers k and ℓ,*

$$k \neq \ell \Longrightarrow K_{C_X^{x_k} \cap A, C_X^{x_k}}(a_{x_k}, b_{x_k}) \neq K_{C_X^{x_\ell} \cap A, C_X^{x_\ell}}(a_{x_\ell}, b_{x_\ell}),$$

it holds that if $\lim_{n\to\infty} x_n$ exists, then the limit of sets

$$\lim_{n\to\infty} K_{C_X^{x_n} \cap A, C_X^{x_n}}(a_{x_n}, b_{x_n})$$

exists and there is $c \in A$ such that

$$\lim_{n\to\infty} K_{C_X^{x_n} \cap A, C_X^{x_n}}(a_{x_n}, b_{x_n}) = \{c\}.$$

Observation 3.10. *Note that if A is a Markov partition of X, then for each non-degenerate connected component C of X,*

$$C \cap A \neq \emptyset$$

and

$$\mathrm{Cl}\Big(\bigcup_{(a,b)\in\mathcal{A}(C\cap A, C)} K_{C\cap A, C}[a,b] \Big) = C.$$

Also, note that there may be connected components C of X such that

$$\bigcup_{(a,b)\in\mathcal{A}(C\cap A, C)} K_{C\cap A, C}[a,b] \neq C;$$

see the following example.

Example 3.11. Let $X = [0,1]$ and let $A = \{\frac{1}{n} \mid n \text{ is a positive integer}\} \cup \{0\}$. Then $C = X$ is the only connected component of X and

$$\bigcup_{(a,b)\in\mathcal{A}(C\cap A, C)} K_{C\cap A, C}[a,b] = (0,1] \neq C.$$

In Figure 2, another Markov partition of a continuum is presented.

Definition 3.12. *Let X be a compact metric space, let A be a Markov partition for X, and let $F : X \multimap X$ be an upper semi-continuous set-valued function. We say that F* is Markov with respect to *A, if*

1. *for each $a \in A$, there are uniquely determined sets $\mathcal{A}_{a,X} \subseteq \mathcal{A}(A, X)$ and $A_{a,X} \subseteq A$ such that*

$$A_{a,X} \cap \{b \in A \mid \text{there is } c \in A \text{ such that } (b,c) \in \mathcal{A}_{a,X}\} = \emptyset$$

and

$$F(a) = A_{a,X} \cup \bigcup_{(c,d)\in\mathcal{A}_{a,X}} K_{C_X^c \cap A, C_X^c}[c,d];$$

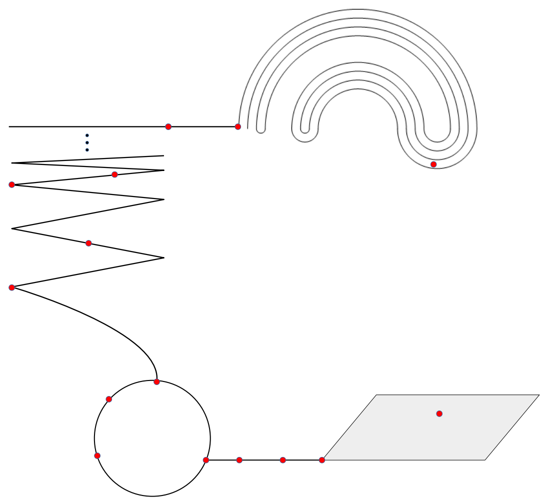


FIGURE 2. A Markov partition of a continuum.

2. *for each non-degenerate connected component C of X and for all $(a,b) \in \mathcal{A}(C \cap A, C)$, it holds that for each $x \in K_{C\cap A,C}(a,b)$ there is $y_x \in X$ such that $F(x) = \{y_x\}$ and the function*
$$\overline{F|_{K_{C\cap A,C}(a,b)}} : K_{C\cap A,C}(a,b) \to F(K_{C\cap A,C}(a,b))$$
is a homeomorphism;
3. *for each non-degenerate connected component C of X, and for all $(a,b) \in \mathcal{A}(C \cap A, C)$, there are sets $\mathcal{O}, \mathcal{N} \subseteq \mathcal{A}(A, X)$, such that*
$$F(K_{C\cap A,C}(a,b)) = \left(\bigcup_{(c,d)\in\mathcal{O}} K_{C_X^c\cap A,C_X^c}(c,d)\right) \cup \left(\bigcup_{(c,d)\in\mathcal{N}} K_{C_X^c\cap A,C_X^c}[c,d)\right);$$
4. *for each non-degenerate connected component C and for each $(a,b) \in \mathcal{A}(C \cap A, C)$, the limits $\lim_{x\to a} \overline{F|_{K_{C\cap A,C}(a,b)}}(x)$ and $\lim_{x\to b} \overline{F|_{K_{C\cap A,C}(a,b)}}(x)$ exist and*
$$\lim_{x\to a} \overline{F|_{K_{C\cap A,C}(a,b)}}(x), \lim_{x\to b} \overline{F|_{K_{C\cap A,C}(a,b)}}(x) \in A;$$
5. *for each non-degenerate connected component C and for each $(a,b) \in \mathcal{A}(C \cap A, C)$, let*
$$L_a = \lim_{x\to a} \overline{F|_{K_{C\cap A,C}(a,b)}}(x) \text{ and } L_b = \lim_{x\to b} \overline{F|_{K_{C\cap A,C}(a,b)}}(x).$$
Then
$$\lim_{x\to L_a} \left(\overline{F|_{K_{C\cap A,C}(a,b)}}\right)^{-1}(x) = a \text{ and } \lim_{x\to L_b} \left(\overline{F|_{K_{C\cap A,C}(a,b)}}\right)^{-1}(x) = b.$$

We say that F is Markov*, if there is a Markov partition A for X such that F is Markov with respect to A.*

LEMMA 3.13. *Let X and Y be any compact metric spaces, let $F : X \multimap X$ and $G : Y \multimap Y$ be Markov set-valued functions with respect to A and B, respectively, and let $\tau : X \to Y$ be any homeomorphism such that $\tau(A) = B$. Then*

$$(a, b) \in \mathcal{A}(A, X) \Longleftrightarrow (\tau(a), \tau(b)) \in \mathcal{A}(B, Y).$$

for any $a, b \in A$.

PROOF. Let $a, b \in A$. Let C be the connected component of X such that $a, b \in C$. If $|C \cap A| = 2$, then the statement is obvious. Suppose that $|C \cap A| > 2$ and that $(a, b) \in \mathcal{A}(A, X)$. Then, since τ is a homeomorphism,

1. $\tau(a) \neq \tau(b)$ (since $a \neq b$) and
2. for any subcontinuum K, which is irreducible between a and b such that $K \cap A = \{a, b\}$, $\tau(K)$ is irreducible between $\tau(a)$ and $\tau(b)$, and

$$\tau(K) \cap B = \{\tau(a), \tau(b)\}.$$

Therefore, $(\tau(a), \tau(b)) \in \mathcal{A}(B, Y)$. The proof that $(a, b) \in \mathcal{A}(A, X)$ follows from $(\tau(a), \tau(b)) \in \mathcal{A}(B, Y)$ is analogous. □

Next we introduce when two Markov set-valued functions follow the same pattern.

DEFINITION 3.14. *Let X and Y be any compact metric spaces, let $F : X \multimap X$ and $G : Y \multimap Y$ be Markov set-valued functions with respect to A and B, respectively. We say that F and G* follow the same pattern with respect to A and B, *if there is a homeomorphism $\tau : X \to Y$ such that*

1. *$\tau(A) = B$;*
2. *for each non-degenerate connected component C and for all $(a, b) \in \mathcal{A}(C \cap A, C)$,*

$$\tau(K_{C \cap A, C}(a, b)) = K_{\tau(C) \cap B, \tau(C)}(\tau(a), \tau(b));$$

3. *for each $a \in A$,*

$$F(a) = A_{a,X} \cup \bigcup_{(c,d) \in \mathcal{A}_{a,X}} K_{C_X^c \cap A, C_X^c}[c, d]$$

if and only if

$$G(\tau(a)) = \tau(A_{a,X}) \cup \bigcup_{(c,d) \in \mathcal{A}_{a,X}} K_{C_Y^{\tau(c)} \cap B, C_Y^{\tau(c)}}[\tau(c), \tau(d)];$$

4. *for each non-degenerate connected component C, for each $(a,b) \in \mathcal{A}(C \cap A, C)$, and for all $\mathcal{O}, \mathcal{N} \subseteq \mathcal{A}(C \cap A, C)$,*

$$F(K_{C\cap A,C}(a,b)) = \left(\bigcup_{(c,d)\in\mathcal{O}} K_{C_X^c\cap A, C_X^c}(c,d)\right) \cup \left(\bigcup_{(c,d)\in\mathcal{N}} K_{C_X^c\cap A, C_X^c}[c,d)\right)$$

if and only if

$$G(K_{\tau(C)\cap B,\tau(C)}(\tau(a),\tau(b))) = \left(\bigcup_{(c,d)\in\mathcal{O}} K_{C_Y^{\tau(c)}\cap B, C_Y^{\tau(c)}}(\tau(c),\tau(d))\right) \cup \left(\bigcup_{(c,d)\in\mathcal{N}} K_{C_Y^{\tau(c)}\cap B, C_Y^{\tau(c)}}[\tau(c),\tau(d))\right);$$

5. *for each $L \in A$, for each non-degenerate connected component C and for all $(a,b) \in \mathcal{A}(C \cap A, C)$, it holds that*

$$\lim_{x\to a} \overline{F|_{K_{C\cap A,C}(a,b)}}(x) = L \Longleftrightarrow \lim_{y\to\tau(a)} \overline{G|_{K_{\tau(C)\cap B,\tau(C)}(\tau(a),\tau(b))}}(y) = \tau(L)$$

and

$$\lim_{x\to b} \overline{F|_{K_{C\cap A,C}(a,b)}}(x) = L \Longleftrightarrow \lim_{y\to\tau(b)} \overline{G|_{K_{\tau(C)\cap B,\tau(C)}(\tau(a),\tau(b))}}(y) = \tau(L);$$

6. *for each $L \in A$, for each non-degenerate connected component C and for all $(a,b) \in \mathcal{A}(C \cap A, C)$, it holds that*

$$\lim_{x\to L} \left(\overline{F|_{K_{C\cap A,C}(a,b)}}\right)^{-1}(x) = a \Longleftrightarrow \lim_{y\to\tau(L)} \left(\overline{G|_{K_{\tau(C)\cap B,\tau(C)}(\tau(a),\tau(b))}}\right)^{-1}(y) = \tau(a)$$

and

$$\lim_{x\to L} \left(\overline{F|_{K_{C\cap A,C}(a,b)}}\right)^{-1}(x) = b \Longleftrightarrow \lim_{y\to\tau(L)} \left(\overline{G|_{K_{\tau(C)\cap B,\tau(C)}(\tau(a),\tau(b))}}\right)^{-1}(y) = \tau(b);$$

7. *For any $c \in A$ and for any sequence (x_n) in $X \setminus A$ such that for all positive integers k and ℓ,*

$$k \neq \ell \Longrightarrow K_{C_X^{x_k}\cap A, C_X^{x_k}}(a_{x_k}, b_{x_k}) \neq K_{C_X^{x_\ell}\cap A, C_X^{x_\ell}}(a_{x_\ell}, b_{x_\ell}),$$

it holds that

$$\lim_{n\to\infty} K_{C_X^{x_n}\cap A, C_X^{x_n}}(a_{x_n}, b_{x_n}) = \{c\}$$

$$\Updownarrow$$

$$\lim_{n\to\infty} K_{\tau(C_X^{x_n})\cap B, \tau(C_X^{x_n})}(\tau(a_{x_n}), \tau(b_{x_n})) = \{\tau(c)\}.$$

We say that the Markov set-valued functions F and G follow the same pattern, *if F and G follow the same pattern with respect to some Markov partitions A and B.*

The following theorem is our main result.

THEOREM 3.15. *Let X and Y be any compact metric spaces, and let $F : X \multimap X$ and $G : Y \multimap Y$ be Markov set-valued functions. If F and G follow the same pattern, then the inverse limits $\varprojlim(X, F)$ and $\varprojlim(Y, G)$ are homeomorphic.*

PROOF. Using mathematical induction, we construct the following commutative diagram – Diagram (3.1). Then, we use Lemma 2.18 to show that the inverse limits $\varprojlim(X, F)$ and $\varprojlim(Y, G)$ are homeomorphic.

(3.1)

$$\begin{array}{ccccccccccccc}
X & \overset{F}{\circ\!\!-\!\!-} & X & \overset{F}{\circ\!\!-\!\!-} & X & \overset{F}{\circ\!\!-\!\!-} & \cdots & \overset{F}{\circ\!\!-\!\!-} & X & \overset{F}{\circ\!\!-\!\!-} & X & \overset{F}{\circ\!\!-\!\!-} & \cdots \\
\downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & & & \downarrow h_n & & \downarrow h_{n+1} & & \\
Y & \overset{G}{\circ\!\!-\!\!-} & Y & \overset{G}{\circ\!\!-\!\!-} & Y & \overset{G}{\circ\!\!-\!\!-} & \cdots & \overset{G}{\circ\!\!-\!\!-} & Y & \overset{G}{\circ\!\!-\!\!-} & Y & \overset{G}{\circ\!\!-\!\!-} & \cdots
\end{array}$$

Let A be a Markov partition of X and let B be a Markov partition of Y such that F and G follow the same pattern with respect to A and B. For each $x \in X \setminus A$, let a_x, $b_x \in C_X^x \cap A$ be such that

$$x \in K_{C_X^x \cap A, C_X^x}(a_x, b_x).$$

Also, for each $y \in Y \setminus B$, let c_y, $d_y \in C_Y^y \cap B$ be such that

$$y \in K_{C_Y^y \cap B, C_Y^y}(c_y, d_y).$$

Let $\tau : X \to Y$ be a homeomorphism satisfying Definition 3.14 and let $h_1 : X \to Y$ be defined by $h_1 = \tau$. Let n be a positive integer and suppose that for each $k \in \{1, 2, 3, \ldots, n\}$, $h_k : X \to Y$ is a homeomorphism such that

1. for each $a \in A$, $h_k(a) = \tau(a)$ for each $k \in \{1, 2, 3, \ldots, n\}$,
2. for each connected component C and for all $(a, b) \in \mathcal{A}(C \cap A, C)$,

$$h_k(K_{C \cap A, C}[a, b]) = \tau(K_{C \cap A, C}[a, b]),$$

   for each $k \in \{1, 2, 3, \ldots, n\}$, and
3. $h_k \circ F = G \circ h_{k+1}$ for each $k \in \{1, 2, 3, \ldots, n-1\}$.

Then we define the function $h_{n+1} : X \to Y$ by

- $h_{n+1}(x) = \tau(x)$ for each $x \in A$, and
- $h_{n+1}(x) = \left(\overline{G|_{K_{C_Y^{\tau(a_x)} \cap B, C_Y^{\tau(a_x)}}(\tau(a_x), \tau(b_x))}}\right)^{-1}(h_n(\overline{F|_{K_{C_X^{a_x} \cap A, C_X^{a_x}}(a_x, b_x)}}(x)))$ for each $x \in X \setminus A$.

Note that by 2. of Definition 3.12, $\overline{G|_{K_{C_Y^{\tau(a_x)} \cap B, C_Y^{\tau(a_x)}}(\tau(a_x), \tau(b_x))}}$ is a homeomorphism, therefore, its inverse does exist. Also, note that since A and $X \setminus A$

are disjoint sets and h_{n+1} is a single valued function on each of these sets, it is a well-defined function. Let $\varphi : Y \to X$ be defined by

- $\varphi(y) = \tau^{-1}(y)$ for each $y \in B$, and
- $$\varphi(y) = \left(\overline{F|_{K_{C_X^{\tau^{-1}(c_y)}\cap A, C_X^{\tau^{-1}(c_y)}}(\tau^{-1}(c_y),\tau^{-1}(d_y))}}\right)^{-1} \left(h_n^{-1}(\overline{G|_{K_{C_Y^{c_y}\cap B, C_Y^{c_y}}(c_y,d_y)}}(y))\right)$$
  for each $y \in Y \setminus B$

To show that h_{n+1} is bijective, we show that $\varphi = h_{n+1}^{-1}$. We treat the following possible cases.

1. Let $x \in A$.Then
$$\varphi(h_{n+1}(x)) = \varphi(\tau(x)) = \tau^{-1}(\tau(x)) = x.$$
2. Let $y \in B$. Then
$$h_{n+1}(\varphi(y)) = h_{n+1}(\tau^{-1}(y)) = \tau(\tau^{-1}(y)) = y.$$
3. Let $x \in X \setminus A$. Also, let
$$\overline{G|_{K_{C_Y^{\tau(a_x)}\cap B, C_Y^{\tau(a_x)}}(\tau(a_x),\tau(b_x))}} = G_x \text{ and } \overline{F|_{K_{C_X^{a_x}\cap A, C_X^{a_x}}(a_x,b_x)}} = F_x.$$
Then
$$\varphi(h_{n+1}(x)) = F_x^{-1}(h_n^{-1}(G_x(G_x^{-1}(h_n(F_x(x)))))) = x.$$
4. Let $y \in Y \setminus B$. Also, let
$$\overline{G|_{K_{C_Y^{c_y}\cap B, C_Y^{c_y}}(c_y,d_y)}} = G_y \text{ and } \overline{F|_{K_{C_X^{\tau^{-1}(c_y)}\cap A, C_X^{\tau^{-1}(c_y)}}(\tau^{-1}(c_y),\tau^{-1}(d_y))}} = F_y.$$
Then
$$h_{n+1}(\varphi(y)) = G_y^{-1}(h_n(F_y(F_y^{-1}(h_n^{-1}(G_y(y)))))) = y.$$

Therefore, h_{n+1} is bijective.

Next, we prove that h_{n+1} is continuous. Let $x \in X$ be any point. We show that h_{n+1} is continuous at the point x. We treat the following possible cases.

1. $x \in X \setminus A$. Let (x_m) be a sequence in X such that $\lim_{m\to\infty} x_m = x$. Since $K_{C_X^{a_x}\cap A, C_X^{a_x}}(a_x, b_x)$ is open in X (by 3. of Definition 3.9) it follows that there is a positive integer m_0 such that for each positive integer m,
$$m \geq m_0 \Longrightarrow x_m \in K_{C_X^{a_x}\cap A, C_X^{a_x}}(a_x, b_x)$$

since $x \in K_{C_X^{a_x} \cap A, C_X^{a_x}}(a_x, b_x)$. Then

$$\begin{aligned}
&\lim_{\substack{m\to\infty\\ m\geq m_0}} h_{n+1}(x_m)\\
&= \lim_{m\to\infty} \left(\overline{G|_{K_{C_Y^{\tau(a_x)} \cap B, C_Y^{\tau(a_x)}}(\tau(a_x),\tau(b_x))}}\right)^{-1} (h_n(\overline{F|_{K_{C_X^{a_x} \cap A, C_X^{a_x}}(a_x,b_x)}}(x_m)))\\
&= \left(\overline{G|_{K_{C_Y^{\tau(a_x)} \cap B, C_Y^{\tau(a_x)}}(\tau(a_x),\tau(b_x))}}\right)^{-1} (h_n(\overline{F|_{K_{C_X^{a_x} \cap A, C_X^{a_x}}(a_x,b_x)}}(x)))\\
&= h_{n+1}(x)
\end{aligned}$$

since $\left(\overline{G|_{K_{C_Y^{\tau(a_x)} \cap B, C_Y^{\tau(a_x)}}(\tau(a_x),\tau(b_x))}}\right)^{-1} \circ h_n \circ \overline{F|_{K_{C_X^{a_x} \cap A, C_X^{a_x}}(a_x,b_x)}}$ is continuous.

2. $x \in A$. Let (x_m) be a sequence in X such that $\lim_{m\to\infty} x_m = x$. For any subsequence (x_{i_m}) of the sequence (x_m), it follows that if $x_{i_m} \in A$ for each positive integer m, then

$$\lim_{m\to\infty} h_{n+1}(x_{i_m}) = \lim_{m\to\infty} \tau(x_{i_m}) = \tau(x) = h_{n+1}(x).$$

Next, let (x_{i_m}) be any subsequence of the sequence (x_m) in $X \setminus A$. We treat the following possible cases.

Case 1. Suppose that there are a connected component C of X and $(a,b) \in \mathcal{A}(C \cap A, C)$ such that for some positive integer m_0 it holds that for each positive integer m

$$m \geq m_0 \Longrightarrow x_{i_m} \in K_{C\cap A, C}(a,b).$$

Fix such C, a, b and m_0. It follows that $x \in \{a, b\}$. Without any loss of generality suppose that $x = a$. Let $L \in A$ such that

$$\lim_{x\to a} \overline{F|_{K_{C\cap A,C}(a,b)}}(x) = L.$$

It follows from 4. of Definition 3.12 that this limit does exist. Then, using 4. and 5. of Definition 3.12, and 5. and 6. of Definition 3.14, we get

$$\begin{aligned}
&\lim_{\substack{m\to\infty\\ m\geq m_0}} h_{n+1}(x_{i_m})\\
&= \lim_{m\to\infty} \left(\overline{G|_{K_{\tau(C)\cap B,\tau(C)}(\tau(a),\tau(b))}}\right)^{-1} (h_n(\overline{F|_{K_{C\cap A,C}(a,b)}}(x_{i_m})))\\
&= \tau(a) = h_{n+1}(a) = h_{n+1}(x).
\end{aligned}$$

Case 2. Suppose that it is not true that there are a connected component C of X and $(a,b) \in \mathcal{A}(C \cap A, C)$ such that for some positive integer m_0 it holds that for each positive integer m

$$m \geq m_0 \Longrightarrow x_{i_m} \in K_{C\cap A, C}(a,b).$$

Without any loss of generality suppose that for all positive integers ℓ and k,

$$\ell \neq k \Longrightarrow K_{C_X^{a_{x_{i_\ell}}} \cap A, C_X^{a_{x_{i_\ell}}}}(a_{x_{i_\ell}}, b_{x_{i_\ell}}) \neq K_{C_X^{a_{x_{i_k}}} \cap A, C_X^{a_{x_{i_k}}}}(a_{x_{i_k}}, b_{x_{i_k}}).$$

For each positive integer m, let

$$K_m = K_{C_X^{a_{x_{i_m}}} \cap A, C_X^{a_{x_{i_m}}}}(a_{x_{i_m}}, b_{x_{i_m}})$$

and

$$H_m = K_{\tau(C_X^{a_{x_{i_m}}}) \cap B, \tau(C_X^{a_{x_{i_m}}})}(\tau(a_{x_{i_m}}), \tau(b_{x_{i_m}})),$$

and let $\mathcal{N}_m, \mathcal{O}_m \subseteq \mathcal{A}(A, X)$, such that

$$F(K_m) = \left(\bigcup_{(c,d) \in \mathcal{O}_m} K_{C_X^c \cap A, C_X^c}(c, d) \right) \cup \left(\bigcup_{(c,d) \in \mathcal{N}_m} K_{C_X^c \cap A, C_X^c}[c, d) \right),$$

$F(K_m)$ is equal to the union above by 3. of Definition 3.12. Note that for each positive integer m,

$$G(H_m) = \left(\bigcup_{(c,d) \in \mathcal{O}_m} K_{C_Y^{\tau(c)} \cap B, C_Y^{\tau(c)}}(\tau(c), \tau(d)) \right) \cup \left(\bigcup_{(c,d) \in \mathcal{N}_m} K_{C_Y^{\tau(c)} \cap B, C_Y^{\tau(c)}}[\tau(c), \tau(d)) \right)$$

by 4. of Definition 3.14. Therefore,

$$\begin{aligned}
&h_{n+1}(x_{i_m}) \\
&= \left(\overline{G|_{H_m}}\right)^{-1} (h_n(\overline{F|_{K_m}}(x_{i_m}))) \in \left(\overline{G|_{H_m}}\right)^{-1} (h_n(F(K_m))) \\
&= \left(\overline{G|_{H_m}}\right)^{-1} \\
&\quad \left(h_n \left(\left(\bigcup_{(c,d) \in \mathcal{O}_m} K_{C_X^c \cap A, C_X^c}(c, d) \right) \cup \left(\bigcup_{(c,d) \in \mathcal{N}_m} K_{C_X^c \cap A, C_X^c}[c, d) \right) \right) \right) \\
&= \left(\overline{G|_{H_m}}\right)^{-1} \\
&\quad \left(\left(\bigcup_{(c,d) \in \mathcal{O}_m} h_n(K_{C_X^c \cap A, C_X^c}(c, d)) \right) \cup \left(\bigcup_{(c,d) \in \mathcal{N}_m} h_n(K_{C_X^c \cap A, C_X^c}[c, d)) \right) \right) \\
&= \left(\overline{G|_{H_m}}\right)^{-1}
\end{aligned}$$

$$\left(\left(\bigcup_{(c,d)\in\mathcal{O}_m}\tau(K_{C_X^c\cap A,C_X^c}(c,d))\right)\cup\left(\bigcup_{(c,d)\in\mathcal{N}_m}\tau(K_{C_X^c\cap A,C_X^c}[c,d))\right)\right)$$

$$= \left(\overline{G|_{H_m}}\right)^{-1}$$

$$\left(\left(\bigcup_{(c,d)\in\mathcal{O}_m}K_{C_Y^{\tau(c)}\cap B,C_Y^{\tau(c)}}(\tau(c),\tau(d))\right)\right.$$

$$\left.\cup\left(\bigcup_{(c,d)\in\mathcal{N}_m}K_{C_Y^{\tau(c)}\cap B,C_Y^{\tau(c)}}[\tau(c),\tau(d))\right)\right)$$

$$= \left(\overline{G|_{H_m}}\right)^{-1}(G(H_m)) = H_m.$$

It follows from 4. of Definition 3.9 that $\lim_{m\to\infty} K_m = \{x\}$. Thus, $\lim_{m\to\infty} H_m = \{\tau(x)\}$ by 7. of Definition 3.14. Therefore,

$$\lim_{m\to\infty} h_{n+1}(x_{i_m}) = \tau(x) = h_{n+1}(x).$$

Since h_{n+1} is a continuous bijection from a compact metric space to a metric space, it follows that it is a homeomorphism.

Obviously, for each $a \in A$, $h_{n+1}(a) = \tau(a)$. Next, let C be a connected component of X and let $(a,b) \in \mathcal{A}(C\cap A, C)$. Since h_{n+1} and τ are both homeomorphisms such that for each $a \in A$, $h_{n+1}(a) = \tau(a)$ it follows that

$$h_{n+1}(K_{C\cap A,C}[a,b]) = \tau(K_{C\cap A,C}[a,b]).$$

Finally, we show that $h_n \circ F = G \circ h_{n+1}$. Let $x \in X$. We treat the following possible cases.

1. $x \in A$. Then by 1. of Definition 3.12,

$$F(x) = A_{x,X} \cup \bigcup_{(c,d)\in\mathcal{A}_{x,X}} K_{C_X^c\cap A,C_X^c}[c,d].$$

Then by 3. of Definition 3.14,

$$G(\tau(x)) = \tau(A_{x,X}) \cup \bigcup_{(c,d)\in\mathcal{A}_{x,X}} K_{C_Y^{\tau(c)}\cap B,C_Y^{\tau(c)}}[\tau(c),\tau(d)] = \tau(F(x)).$$

It follows that

$$(G\circ h_{n+1})(x) = G(h_{n+1}(x)) = G(\tau(x)) = \tau(F(x))$$

and

$$\begin{aligned}(h_n \circ F)(x) &= h_n(F(x)) = h_n(A_{x,X} \cup \bigcup_{(c,d)\in\mathcal{A}_{x,X}} K_{C_X^c \cap A, C_X^c}[c,d]) \\ &= h_n(A_{x,X}) \cup \bigcup_{(c,d)\in\mathcal{A}} h_n(K_{C_X^c \cap A, C_X^c}[c,d]) \\ &= \tau(A_{x,X}) \cup \bigcup_{(c,d)\in\mathcal{A}} \tau(K_{C_X^c \cap A, C_X^c}[c,d]) \\ &= \tau(A_{x,X}) \cup \bigcup_{(c,d)\in\mathcal{A}} K_{C_Y^{\tau(c)} \cap B, C_Y^{\tau(c)}}[\tau(c),\tau(d)] = \tau(F(x)).\end{aligned}$$

Therefore,

$$(G \circ h_{n+1})(x) = (h_n \circ F)(x).$$

2. $x \in X \setminus A$. Then

$$h_{n+1}(x) = \left(\overline{G|_{K_{C_Y^{\tau(a_x)} \cap B, C_Y^{\tau(a_x)}}(\tau(a_x),\tau(b_x))}}\right)^{-1}(h_n(\overline{F|_{K_{C_X^{a_x} \cap A, C_X^{a_x}}(a_x,b_x)}}(x)))$$

and

$$(G \circ h_{n+1})(x) = (h_n \circ F)(x)$$

follows.

We have constructed the following commutative diagram – Diagram (3.2).

(3.2)

$$\begin{array}{ccccccccccccc} X & \overset{F}{\circ\!\!-\!\!-} & X & \overset{F}{\circ\!\!-\!\!-} & X & \overset{F}{\circ\!\!-\!\!-} & \cdots & \overset{F}{\circ\!\!-\!\!-} & X & \overset{F}{\circ\!\!-\!\!-} & X & \overset{F}{\circ\!\!-\!\!-} & \cdots \\ \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & & & \downarrow h_n & & \downarrow h_{n+1} & & \\ Y & \overset{G}{\circ\!\!-\!\!-} & Y & \overset{G}{\circ\!\!-\!\!-} & Y & \overset{G}{\circ\!\!-\!\!-} & \cdots & \overset{G}{\circ\!\!-\!\!-} & Y & \overset{G}{\circ\!\!-\!\!-} & Y & \overset{G}{\circ\!\!-\!\!-} & \cdots \end{array}$$

By Lemma 2.18, the inverse limits $\varprojlim(X,F)$ and $\varprojlim(Y,G)$ are homeomorphic. □

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MARKOVLJEVE SKUPOVNE FUNKCIJE NA KOMPAKTNIM METRIČKIM PROSTORIMA

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Sažetak. U ovom radu poopćujemo pojam Markovljevih funkcija na zatvorenim intervalima $[a, b]$ na Markovljeve skupovne funkcije na kompaktnim metričkim prostorima. Također opisujemo kada dvije takve Markovljeve skupovne funkcije slijede isti obrazac i pokazujemo da, ako Markovljeve skupovne funkcije $F : X \multimap X$ i $G : Y \multimap Y$ slijede isti obrazac, tada su inverzni limesi $\varprojlim(X, F)$ i $\varprojlim(Y, G)$ homeomorfni.