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ON THE THREE-NUCLEON GROUND STATE WAVE FUNCTION

I. *The construction of the wawe function and the derivation of the one-particle matrix elements contributions*

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Abstract: The group-theoretical arguments are used to write the three nucleon ground state wave function. The obtained expressions are rewritten in the spherical coordinates convenient for the application of the angular momentum algebra and for the spatial integration. The formulae for different interference terms in the case of one particle Hamiltonian are given explicitly.

1. Introduction

Among many approaches to form, without solving the Schrödinger equation, the three-nucleon bound state wave function with the total angular momentum $J = \frac{1}{2}$, the even parity and the total isotopic spin $T = \frac{1}{2}$, the two approaches are often used. One is due to Gerjouy and Schwinger[1] and Sachs[2] who gave the systematics of the components of the total wave function. Recently, Schiff and Gibson[3] renewed the method and Bolsterli and Jezak[4] tried to give to this approach the group-theoretical basis. In the framework of this approach the wave function for a three-nucleon system was not written in a complete form. The other approach is due to Derick and Blatt[5]. Although complete, their metod seems not to be well suited for practical uses.*

The aim of this article is twofold. Firstly, we will show on the group-theoretical basis how one can rewrite the Rarita-Schwinger approach, which will be (in spherical coordinates and using angular momentum algebra), adjusted to the practical uses. On this ground we shall discuss the questions concerning

* A survey of the present situation can be found in many review articles. See for exemple ref.[6]

the S-state spatial antisymmetric wave function and the P-state function containing the antisymmetric spatial scalar function, (chapter 2). Secondly, as a consequence of the form of the wave function obtained in chapter 2, we shall derive the formulae for the complete set of matrix elements for the electric form factor for a three-nucleon system to which many other physically important quantities of one-particle nature can be connected (chapter 3).

2. *The eight components of the three-nucleon bound state wave function*

Owing to the fact that the Schrödinger equation for the three body bound state cannot be solved using reasonable two--body potentials, one usually proceeds by writing the general form of the wave function which satisfies the antisymmetry conditions for spatial, spin and isotopic spin coordinates and is subject of the following constraints:

total angular momentum $J = \frac{1}{2}$,

parity even

total isotopic spin $T = \frac{1}{2}$.

These constraints are the physical characteristics of the ground state of ^{3}He and ^{3}H nuclei.

The above conditions and the antisymmetry requirement, as was shown in ref.[5] can be satisfied by ten independent components, whose linear combination gives the total wave function. It was also shown that the components can be classified in accordance with the properties of the irreducible representations of the permutation group of three objects (S_3) and that of the rotation group in three dimensions (R_3).

We proceed by rewriting the properties of the S_3 group. The irreducible representations of the S_3 group are symmetric and antisymmetric (both one dimensional) which we shall further denote by the indices »S« and »A«, respectively, and a mixed representation (two-dimensional) denoted by the index »M«. The objects which undergo the transformations under the mixed representation will be denoted by indices »1« and »2«. Having formed representations in a given basis one constructs the representation in the union of the basis by applying the rules for the direct product, which for the S_3 group reads:

$$R_S(\alpha) \otimes R_S(\beta) = R_S(\alpha\beta)\,,$$

$$R_S(\alpha) \otimes R_M(\beta\,;\,\gamma) = R_M(\alpha\beta\,;\,\alpha\gamma)\,,$$

$$R_S(\alpha) \otimes R_A(\beta) = R_A(\alpha\beta)\,, \tag{1}$$

$$R_M(\alpha\,;\,\beta) \otimes R_M(\gamma\,;\,\delta) = R_S(\alpha\gamma+\beta\delta) \oplus R_M(\beta\delta-\alpha\gamma\,;\,\alpha\delta+\beta\gamma) \oplus R_A(\alpha\delta-\beta\gamma)\,,$$

$$R_A(\alpha) \otimes R_M(\beta\,;\,\gamma) = R_M(\alpha\gamma\,;\,-\alpha\beta)\,,$$

$$R_A(\alpha) \otimes R_A(\beta) = R_S(\alpha\beta)\,,$$

where α, β, γ and δ are one of the realisation of the representation of the group S_3 [4]. These rules are directly applicable to the spin and isospin functions for three particles.

Supposing the exchange moments equal zero, and by coupling firstly the spins of the particles denoted by 2 and 3 into the spin S_{23} with the projection M_{23} and after by coupling the spin S_{23} and that of particle 3 into the total spin S and its projection S_z one gets the function:

$$X^{S}_{S_z}(S_{23}) = \sum_{\sigma_1\sigma_2\sigma_3} (-)^{M_{23}-\frac{1}{2}+S_{23}-S_z} [(2S_{23}+1)(2S+1)]^{1/2} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & S_{23} \\ \sigma_2 & \sigma_3 & -M_{23} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & S_{23} & S \\ \sigma_1 & M_{23} & -S_z \end{pmatrix} (1\,|\sigma_1)(2\,|\sigma_2)(3\,|\sigma_3)\,. \tag{2}$$

Two objects which immediately follow from (2).

$$X^{\frac{1}{2}}_{S_z}(1) \equiv X_1\,,$$

$$X^{\frac{1}{2}}_{S_z}(0) \equiv X_2\,, \tag{3}$$

undergo the rules for the mixed representation and the functions (3) are one of its realizations.

The isotopic spin functions have the same properties and they have been formed in the complete analogy with the spin functions. We will denote them by $\eta^{T}_{T_z}(T_{23})$, η_1 and η_2.

For the spatial coordinates to span the representation of the S_3 group one chooses the centre of mass system by applying the following transformations:

$$\vec{r}_1 = \vec{R}_0 + \frac{1}{\sqrt{3}} \vec{R}_1 , \qquad \vec{R}_0 = \frac{1}{3} (\vec{r}_1 + \vec{r}_2 + \vec{r}_3),$$

$$\vec{r}_2 = \vec{R}_0 + \frac{1}{2} \vec{R}_2 - \frac{1}{2\sqrt{3}} \vec{R}_1 , \qquad \vec{R}_1 = \frac{2}{\sqrt{3}} \left[\vec{r}_1 - \frac{1}{2} (\vec{r}_2 + \vec{r}_3) \right] , \tag{4}$$

$$\vec{r}_3 = \vec{R}_0 - \frac{1}{2} \vec{R}_2 - \frac{1}{2\sqrt{3}} \vec{R}_1 , \qquad \vec{R}_2 = \vec{r}_2 - \vec{r}_3 .$$

$\vec{R}_0$ is the symmetric representation of the position vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$. $\vec{R}_1$ and $\vec{R}_2$ form the mixed representation. The Jacobian of the transformation is $\frac{3}{8}\sqrt{3}$. Notice, however, that the parity of these objects is odd. In order to get the even parity equations (1) must be used. Again, by applying equations (1) combined with the direct inspection, one gets the scalar, vector and tensor representations. The scalar representation follows directly and reads:

$$S_S = R_1^2 + R_2^2 , \qquad S_1 = R_2^2 - R_1^2 , \tag{5}$$

$$S_A = 0 , \qquad S_2 = 2 \vec{R}_1 \cdot \vec{R}_2 .$$

One gets the vector representation by noting that the unique vector with even parity is the vector product of $\vec{R}_1$ and $\vec{R}_2$. With respect to the permutation properties this vector is antisymmetric. The symmetric vector does not exist and the vectors which span the mixed representation follow from equations (1),

$$\vec{P}_S = 0 , \qquad \vec{P}_1 = \vec{P}_A S_2 ,$$

$$\vec{P}_A = \vec{R}_1 \times \vec{R}_2 , \qquad \vec{P}_2 = - \vec{P}_A S_1 .$$

It is convenient to rewrite these objects in the spherical coordinates. Defining the spherical components of a unit vector by:

$$\xi_1 = - \frac{1}{\sqrt{2}} (\vec{e}_1 + i \vec{e}_2) ,$$

$$\xi_0 = \vec{e}_3 , \tag{7}$$

$$\xi_{-1} = \frac{1}{\sqrt{2}} (\vec{e}_1 - i \vec{e}_2) ,$$

where $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ are the Cartesian unit vectors, a vector $\vec{A}$ is then

$$\vec{A} = \Sigma (-)^{\mu} A_{\mu} \xi_{-\mu} , \tag{8}$$

and the scalar product of two vectors $\vec{A}$ and $\vec{B}$ is

$$\vec{A} \cdot \vec{B} = \sum_{\mu} (-)^{\mu} A_{\mu} B_{-\mu} . \tag{9}$$

The vector components A_{μ} are:

$$A_{\mu} = \left(\frac{4\pi}{3}\right)^{1/2} A\, Y^{1}{}_{\mu} (\hat{A}), \tag{10}$$

where by $\hat{A}$ the angular orientation of the vector is defined. Then, $\vec{P}_A = \vec{R}_1 \times \vec{R}_2$ reads

$$\vec{R}_1 \times \vec{R}_2 = \frac{4\pi}{3} R_1 R_2 \sum_{\mu \mu'} Y^{1}_{\mu} (\hat{R}_1)\, Y^{1}_{\mu'} (\hat{R}_2) (-)^{\mu + \mu'} (\xi_{-\mu} \times \xi_{-\mu'}) \tag{11}$$

and since:

$$\zeta_{\mu} \times \zeta_{\mu'} = i \sqrt{6} (-)^{m} \begin{pmatrix} 1 & 1 & 1 \\ -\mu & \mu' & -m \end{pmatrix} \zeta_{m} , \tag{12}$$

finally one has:

$$\vec{R}_1 \times \vec{R}_2 = i \sqrt{2}\, \frac{4\pi}{3} R_1 R_2 \sum_{\mu \mu'} Y^{1}_{\mu} (\hat{R}_1)\, Y^{1}_{\mu'} (\hat{R}_2) \sqrt{3} \begin{pmatrix} 1 & 1 & 1 \\ -\mu & -\mu' & -m \end{pmatrix} \zeta_{m} . \tag{13}$$

In order to insure the total angular momentum $J = \frac{1}{2}$, and even parity, the quantities of the vector representation must be multiplied by the spin quantities following equations (1). That was done in ref.[4]. Here, we prefer, for practical purposes, to use more direct way, and to multiply $\vec{P}_A$ by the quantities $\vec{\Pi}_S$, $\vec{\Pi}_1$ and $\vec{\Pi}_2$ defined as follows

$$\vec{\Pi}_S = \left[\vec{\sigma}_{23} - \frac{1}{2} i (\vec{\sigma}_1 \times \vec{\sigma}_{23}) \right] X^{\frac{1}{2}}_{S_z} (0) \; ;$$

$$\vec{\Pi}_1 = \frac{1}{\sqrt{12}} \left[\vec{\sigma}_{23} + i (\vec{\sigma}_1 \times \vec{\sigma}_{23}) \right] X^{\frac{1}{2}}_{S_z} (0) \; ; \quad \vec{\Pi}_2 = \vec{\sigma}_1 \, X^{\frac{1}{2}}_{S_z} (0) \; . \tag{14}$$

The quantities (14) in addition to the obvious properties respect to the S_3 group multiplied by $\vec{P}_A$ are the realisation of an irreducible representation of the SU_2 group which authomatically insures the total angular momentum $J = \frac{1}{2}$.

Applying the standard formulae for the addition of angular momenta, one gets in the spherical coordinates:

$$\vec{\Pi}_S \cdot \vec{P}_A = i\, 6 \sqrt{6} \frac{4\pi}{3} R_1 R_2 \sum_{\mu\,\mu'} \begin{pmatrix} 1 & 1 & 1 \\ -\mu & -\mu' & -\varrho \end{pmatrix} \begin{pmatrix} 1 & 3/2 & 1/2 \\ \varrho & -S_z & J_z \end{pmatrix} (-)^{\frac{1}{2}+S_z} Y^1_\mu (\hat{R}_1) Y^1_{\mu'} (\hat{R}_2) X^{3/2}_{S_z}$$

$$\vec{\Pi}_1 \cdot \vec{P}_A = i\, 6 \frac{4\pi}{3} R_1 R_2 \sum_{\mu\,\mu'} \begin{pmatrix} 1 & 1 & 1 \\ -\mu & -\mu' & -\varrho \end{pmatrix} \begin{pmatrix} & 1/2 & 1/2 \\ \varrho & -S_z & J_z \end{pmatrix} (-)^{\frac{1}{2}+S_z} Y^1_\mu (\hat{R}_1) Y^1_{\mu'} (\hat{R}_2) X^{\frac{1}{2}}_{S_z} (0) \tag{15}$$

$$\vec{\Pi}_2 \cdot \vec{P}_A = i \frac{4\pi}{3} R_1 R_2 \sum_{\mu\,\mu'} \begin{pmatrix} 1 & 1 & 1 \\ -\mu & -\mu' & -\varrho \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/2 \\ \varrho & -S_z & J_z \end{pmatrix} (-)^{\frac{1}{2}-S_z} Y^1_\mu (\hat{R}_1) Y^1_{\mu'} (\hat{R}_2) X^{1/2}_{S_z} (0) \, .$$

The tensor representation follows directly from equations (1) since the tensor product for the R_3 group span the S_3 group representation. One has

$$\mathcal{D}_S = \{ [\vec{R}_1, \vec{R}_1]^2_m + [\vec{R}_2, \vec{R}_2]^2_m \} \, ,$$

$$\mathcal{D}_1 = \{ [\vec{R}_2, \vec{R}_2]^2_m - [\vec{R}_1, \vec{R}_1]^2_m \} \, , \tag{16}$$

$$\mathcal{D}_2 = 2 [\vec{R}_1, \vec{R}_2]^2_m \, .$$

In the spherical coordinates these formulae have simple form

$$\mathcal{D}_S = \left(\frac{8\pi}{15} \right)^{1/2} \left\{ R^2_1 \, Y^2_m (\hat{R}_1) + R^2_2 \, Y^2 (\hat{R}_2) \right\} ,$$

$$\mathcal{D}_1 = \left(\frac{8\pi}{15} \right)^{1/2} \left\{ R^2_2 \, Y^2_m (\hat{R}_2) - R^2_1 \, Y^2_m (\hat{R}_1) \right\} , \tag{17}$$

$$\mathcal{D}_2 = 2\,\frac{4\pi}{3} R_1 R_2 \sum_{\mu\,\mu'} (-)^m \sqrt{5} \begin{pmatrix} 1 & 1 & 2 \\ \mu & \mu' & -m \end{pmatrix} Y^1_\mu (\hat{R}_1) Y^1_{\mu'} (\hat{R}_2)\,.$$

Notice, however that these formulae differ with respect to the Schiff's formulae[3)] by the factor $\frac{1}{\sqrt{5}}$.

The total angular momentum $J = \frac{1}{2}$, even parity, functions are obtained from the functions (16) or (17) and quartet spin functions by applying the addition rules for angular momenta. One has:

$$D_S = \left(\frac{8\pi}{15}\right)^{1/2} \frac{\sqrt{5}}{2} \sum_{m\,\mu} \sqrt{2}\,(-)^{1/2 + J_z} \begin{pmatrix} 2 & 3/2 & 1/2 \\ m & \mu & -J_z \end{pmatrix} \left[R_2^2\, Y^2_m (\hat{R}_2) + R_1^2\, Y^2_m (\hat{R}_1) \right] X^{3/2}_\mu\,,$$

$$D_1 = \left(\frac{8\pi}{15}\right)^{1/2} \frac{\sqrt{5}}{2} \sum_{m\,\mu} \sqrt{2}\,(-)^{1/2 + J_z} \begin{pmatrix} 2 & 3/2 & 1/2 \\ m & \mu & -J_z \end{pmatrix} \cdot \tag{18}$$

$$\cdot \left[R_2^2\, Y^2_m (\hat{R}_2) - R_1^2\, Y^2_m (\hat{R}_1) \right] X^{3/2}_\mu\,,$$

$$D_2 = 2\sqrt{5}\,\frac{4\pi}{3} \sum_{m\,\mu\,\nu\,\nu'} \sqrt{10}\,(-)^{m_z + \frac{1}{2} + J_z} \begin{pmatrix} 1 & 1 & 2 \\ \nu & \nu' & -m \end{pmatrix} \begin{pmatrix} 2 & 3/2 & 1/2 \\ m & \mu & -J_z \end{pmatrix} R_1 R_2\, Y^1_\nu (\hat{R}_1)\, Y^1_{\nu'} (\hat{R}_2)\, X^{3/2}_\mu\,.$$

The total wave function is obtained from the formulae (3), (5), (15) and (18) using equations (1). It is a linear combination of the eight components:

$$\Psi^{\frac{1}{2}\,\frac{1}{2}}_{J_z\,T_z} (1,\,2,\,3) = \sum_{i=1}^{8} a_i\, \psi_i{}^{\frac{1}{2}\,\frac{1}{2}}_{J_z\,T_z} (1,\,2,\,3)\,, \tag{19}$$

where the index »*i*« refers to the properties not included in the constraints: $J = \frac{1}{2}$, $T = \frac{1}{2}$ and even parity. So, in the explicit forms one has

S-state:

$$\Psi_1 = \Psi (S_S)\, \Phi_A\, (\sigma_1,\, \sigma_2,\, \sigma_3;\, \tau_1,\, \tau_2,\, \tau_3)\,,$$

$$\Phi_A (\sigma_1 \ldots \tau_3) = \frac{1}{\sqrt{2}} \left(X^{1/2}_{S_z} (0)\, \eta^{1/2}_{T_z} (1) - X^{1/2}_{S_z} (1)\, \eta^{1/2}_{T_z} (0) \right); \tag{20}$$

S'-state:

$$\Psi_2 = \frac{1}{\sqrt{2}}\left[S_2\,\Phi_1(\sigma_1\ldots\tau_3) - S_1\,\Phi_2(\sigma_1\ldots\tau_3)\right]\Psi_2(S_s)\,,$$

$$\Phi_1(\sigma_1\ldots\tau_3) = \frac{1}{\sqrt{2}}\left(X^{1/2}_{S_z}(0)\,\eta^{1/2}_{T_z}(0) - X^{1/2}_{S_z}(1)\,\eta^{1/2}_{T_z}(1)\right), \tag{21}$$

$$\Phi_2(\sigma_1\ldots\tau_3) = \frac{1}{\sqrt{2}}\left(X^{1/2}_{S_z}(0)\,\eta^{1/2}_{T_z}(1) + X^{1/2}_{S_z}(1)\,\eta^{1/2}_{T_z}(0)\right);$$

P-States:

$$\Psi_3 = \left[\vec{\Pi}_2\,\eta^{1/2}_{T_z}(0) + \vec{\Pi}_1\,\eta^{1/2}_{T_z}(1)\right]\vec{P}_A\;\Psi_3(S_s)\,, \tag{22}$$

$$\Psi_4 = \left[(\vec{\Pi}_2 S_1 + \vec{\Pi}_1 S_2)\,\eta^{1/2}_{T_z}(0) + (\vec{\Pi}_2 S_2 - \vec{\Pi}_1 S_1)\,\eta^{1/2}_{T_z}(1)\right]\Psi_4(S_s)\,\vec{P}_A\,, \tag{23}$$

$$\Psi_5 = \left[S_2\,\eta^{1/2}_{T_z}(0) + S_1\,\eta^{1/2}_{T_z}(1)\right]\vec{\Pi}_S\vec{P}_A\,\Psi_6(S_s)\,, \tag{24}$$

and *D-states:*

$$\Psi_6 = \left[(5\,D_S S_2 - 2D_2 S_s)\,\eta^{1/2}_{T_z}(1) - (5\,D_S S_1 - 2\,D_1 S_s)\,\eta^{1/2}_{T_z}(0)\right]\Psi_6(S_s)\,, \tag{25}$$

$$\Psi_7 = \left[D_2 S_S\,\eta^{1/2}_{T_z}(1) - D_1 S_S\,\eta^{1/2}_{T_z}(0)\right]\Psi_7(S_s)\,, \tag{26}$$

$$\Psi_8 = \left[(D_2 S_1 + D_1 S_2)\,\eta^{1/2}_{T_z}(1) - (D_2 S_2 - D_2 S_1)\,\eta^{1/2}_{T_z}(0)\right]\Psi_8(S_s)\,. \tag{27}$$

The determination of the function $\psi_i\,(S_s)$ lies outside this approaches. The choice of this function is subject to specific phenomenological suppositions concerning the radial dependence and the numerical values of the parameters, and it has always to be confronted with the experiment.

In the framework of the above described approach the S-state component of the wave function whose spatial part is antisymetric as well as the P-state funotion whose spatial part depends on an antisymmetric scalar function cannot be formed (eqs. 5. and 6.). In the ref.[5] these functions are different from zero owing to the special choice of the spatial coordinates. To this end they have used as the spatial variables the absolute values of the mutual distances between two nucleons $x_1 = |\vec{r}_{23}|$, $x_2 = |\vec{r}_{13}|$, $x_3 = \vec{r}_{12}|$ which form a triangle, and three Euleri angles to describe the orientation of the triangle. One is than able to construct the spatial antisymmetric S-function as a determinant formed from nine scalar functions $f_i\ (x_j)$, $(i,\ j = 1, 2, 3)$, multipled by the symmetric product of spin and isospin functions. The spatially antisymmetric P-state function is formed by multiplying the determinant by the symmetric combination of $\vec{\Pi}_1$, $\vec{\Pi}_2$, $\eta_1\ \eta$, and $\vec{P}_A$ which is, following equations (1):

$$(\vec{\Pi}_2\ \eta_1 - \vec{\Pi}_1\ \eta_2)\ \vec{P}_A.$$

We have two reasons not to take into consideration these components. The calculation of the binding energy for ^{3}He and ^{3}H nuclei have shown that their contributions are negligible. This seems understandable since the determinant formed by nine $f_i\ (x_j)$ functions implies the existence of an average potential for the bound state of three nucleons which is physically hardly permissible.

Further, the procedure is not suitable for numerical calculations. Since the three distances do not always form a triangle one is obliged to control the machine program by the additional condition $x_i + x_j \geqslant x_k$. Moreover, in the collinear case (three nucleons form a line) the functions dependent on the Euleri angles are not defined, which introduces new ambiguities.

3. *One-particle matrix elements*

The description of many physically interesting processes by which one gains informations about the properties of three-nucleon bound state wave function goes via a Hamiltonian of one particle nature. These are: electron elastic or quasi-elastic scattering, μ-meson particle capture, photodesintegration, radiative capture etc.

We shall illustrate on the example of the electric form factor of the ^{3}He nucleus how the wave function obtained in chapter 2. facilitates the calculations of various integrals by which one takes into account different interference terms.

The electric charge form-factor for ^{3}He nucleus is defined by:

$$2\ F_{ch}\,(^3H_e) = \int e^{i\,\vec{q}\,\vec{r}}\ \Psi^*\ \varrho_c\,(\vec{r}, \vec{r}_i)\ \Psi\ d\tau_1\, d\tau_2\, d\tau_3\,, \tag{28}$$

where q is the momentum transfer, r the electron coordinate, r_i the nucleon coordinates, Ψ is the complet 3-nucleon wave function (19) and the electric charge density operator $\varrho_c\ (\vec{r},\ \vec{r}_i)$ defined by

$$\varrho_c(\vec{r}, \vec{r}_i) = \sum_{i=1}^{3} \left[\frac{1}{2} (1+\tau_z^i) f_{ch}^{p} (\vec{r} - \vec{r}_i) + \frac{1}{2} (1-\tau_z^i) f_{ch}^{n} (\vec{r} - \vec{r}_i), \right. \tag{29}$$

where τ_z^i is the z-component isotopic spin operator for the i-th particle and f_{ch}^{p} and f_{ch}^{n} the proton and neutron charge form-factor, respectively.

Since eq. (29) is symmetric to the particle permutation, it is allwed to calculate the contribution of a particle, say 1, and the total contribution is obtained simply by multiplying one particle expression by the factor 3. The one particle spin (and isotopic spin) matrix element is calculated from the well-known expression

$$\left\langle X_{S_z}^{S} (S_{23}) \left| \sigma_{\varrho}^{1} (1) \right| X_{S'_z}^{S'} (S'_{23}) \right\rangle = \left[(2S+1)(2S'+1) \right]^{1/2} \times$$
$$\times (-)^{1/2 - S_z + S_{23}} \begin{Bmatrix} S & S' & 1 \\ 1/2 & 1/2 & S_{23} \end{Bmatrix} \begin{pmatrix} S & S' & 1 \\ -S & S'_z & \varrho \end{pmatrix} \sqrt{\frac{3}{2}} \, \delta_{S_{23} S'_{23}} , \tag{30}$$

where $\sigma_{\varrho}^{1}(1)$ are the spin components of the particle 1 in the spherical coordinates.

The remaining spatial integrals can be classified into the *S-S, S-S', S-P, S-D, S'-D, P-D* and *D-D* contributions.

S-S contributions

After performing the coordinate transformation (4) the spatial integral for this contribution is

$$\int | \Psi_1 (S_s) |^2 e^{i \frac{1}{\sqrt{3}} \vec{q} \vec{R}_1} d^3 R_1 d^3 R_2 . \tag{31}$$

The angular integration over $d\,\Omega_1$ and $d\,\Omega_2$ gives

$$(4\pi)^2 \int | \Psi_1 (S_s) |^2 \frac{\sin\left(\frac{1}{\sqrt{3}} q R_1\right)}{\frac{1}{\sqrt{3}} q R_1} R_1^2 R_2^2 \, dR_1 \, dR_2 . \tag{32}$$

By the transformation

$$R_2 = R\cos\Theta\ ,$$
$$R_1 = R\sin\Theta\ ,$$
$$\int_0^\infty dR_1 \int_0^\infty dR_2 = \int_0^\infty RdR \int_0^{\pi/2} d\Theta\ , \tag{33}$$

with the help of the formulae

$$[1-(-1)^n] \int_0^{\pi/2} \sin(z\sin x)\sin(nx)\,dx = [1-(-1)^n]\frac{\pi}{2} J_n(Z)\ ,$$
$$[1+(-1)^n] \int_0^{\pi/2} \cos(z\sin x)\cos(nx)\,dx = [1+(-1)^n]\frac{\pi}{2} J_n(Z)\ , \tag{34}$$

one obtains

$$J_{S-S} = \int_0^\infty |\Psi_1(R)|^2 R^k J_n(QR)\,dR\ . \tag{35}$$

One-dimensional integral (35) can be calculated either analitically or numerically depending on the choice of a phenomenological function.

S-S' contributions

There are two integrals for this contribution:

$$J^{(1)}_{S-S'} = \int \Psi_1(S_S)\,\Psi_2(S_S)\,S_1 \sin\left(\frac{1}{\sqrt{3}}\,q\,R_1\right)\frac{\sqrt{3}}{q\,R_1}\,R_1^2\,R_2^2\,dR_1\,dR_2\ , \tag{36}$$

$$J^{(2)}_{S-S'} = \int \Psi_1(S_S)\,\Psi_2(S_S)\,S_2\,e^{i\frac{1}{\sqrt{3}}\vec{q}\cdot\vec{R}_1}\,d^3R_1\,d^3R_2\ . \tag{37}$$

The substitution (33), by noting that

$$S_1 = R^2\cos 2\Theta,$$

gives for $J^{(1)}_{S-S'}$ an integral of the type (35). The $J^{(2)}_{S-S'}$ integral is equal zero since the integration over $d\,\Omega_2$ gives zero.

S-P contribution

The matrix element for this contribution is composed of 5 components. They are those which contain the factors, $\vec{\Pi}_S\vec{P}_A$, $S_1\vec{\Pi}_1\vec{P}_A$, $S_1\vec{\Pi}^2\vec{P}_A$, $S_2\vec{\Pi}_1\vec{P}_A$, $S_2\vec{\Pi}_2\vec{P}_A$. One sees that the contributions of the spin functions. The components with $S_1\vec{\Pi}_1\vec{P}_A$ and $S_1\vec{\Pi}_2\vec{P}_A$ give again zero since the integration over $d\,\Omega_2$ in these cases gives zero. The two remaining components give zero too, which can be seen by the following argument. In the explicit form the factor, say, $S_2\,\vec{\Pi}_1\,\vec{P}_A$ is

$$S_2\,\vec{\Pi}_1\,\vec{P}_A = i\cdot 6\left(\frac{4\pi}{3}\right)^2 2R_1^2\,R_2^2 \sum_{\mu\,\mu'\,\gamma} \begin{pmatrix} 1 & 1 & 1 \\ -\mu & -\mu' & -\varrho \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/2 \\ \varrho & -S_z & J_z \end{pmatrix} (-)^{1/2+S_z}$$

$$Y^1_{\mu}(\hat{R}_1)\,Y^1_{\gamma}(\hat{R}_1)\,Y^1_{\mu'}(\hat{R}_2)\,Y^{1}_{-\gamma}(\hat{R}_2)\,. \qquad (38)$$

By the use of the addition theorem

$$Y^1_{\mu}(\hat{R}_1)\,Y^1_{\gamma}(\hat{R}_1) = \sum_j (-)^m \left[\frac{9\,(2j+1)}{4\pi}\right]^{1/2} \begin{pmatrix} 1 & 1 & j \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & j \\ \mu & \gamma & -m \end{pmatrix} Y^j_m(\hat{R}_1), \qquad (39)$$

one gets j = 0,2 and j' = 0,2. But the sum over $3-j$ symbols gives

$$\sum_{\mu\,\mu'\,\gamma} \begin{pmatrix} 1 & 1 & j' \\ \mu' & -\gamma & -m' \end{pmatrix} \begin{pmatrix} 1 & 1 & j \\ \gamma & -\mu & -m \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \mu & -\mu' & -\varrho \end{pmatrix} (-)^{3+\mu+\mu'+\gamma} =$$

$$= \begin{Bmatrix} j & 1 & j' \\ 1 & 1 & 1 \end{Bmatrix} \begin{pmatrix} j & 1 & j' \\ -m & -\varrho & -m' \end{pmatrix},$$

and the 6-j symbol differs from zero only for $j = j' = 2$. In the integral expression this insures the presence of the function $Y^2_{\mu}(\hat{R}_2)$ which causes the integration over $d\,\Omega_2$ to be equal zero. By similar arguments the same result holds for the *S'-P* terms.

The *S-D* and *S'-D* contributions are equal to zero owing to the orthogonality of the spin functions and the *P-D* contributions can be calculated like the *D-D* contributions (see bellow), but we shall not write the explicit expression since they are of less physical interest.

D-D contributions

The spatial part of *D-D* matrix element is of the type:

$$J_{D-D} = \int \Psi_i (S_s) \Psi_j (S_s) e^{i \frac{1}{\sqrt{3}} \vec{q} \vec{R}_1} D_S D_1^{*} S_S S_1 \, d^3 R_1 \, d^3 R_2 , \quad (40)$$

where »*i*« and »*j*« refer to the Ψ_6 (S_s), Ψ_7 (S_s), Ψ_8 (S_s), (eqs. 25, 26, 27), and D_s, D_1 to the expressions (18). The integration over $d\,\Omega_2$ gives:

$$J_{D-D} = 4\pi \int \Psi_i (S_s) \Psi_j (S_S) \Big[R_2^4 \, j_0 \left(\frac{1}{\sqrt{3}} q R_1 \right) - R_1^4 \sum_L i^L \, Y_M^L (q) \left(\frac{25 \, (2L+1)}{4\pi} \right)^{1/2} \begin{pmatrix} 2 & 2 & L \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & L \\ m & -m' & M \end{pmatrix} \begin{pmatrix} 2 & 3/2 & 1/2 \\ m & \mu & -J_z \end{pmatrix} \begin{pmatrix} 2 & 3/2 & 1/2 \\ m' & \mu & -J_z \end{pmatrix} j_L \left(\frac{1}{\sqrt{3}} q R_1 \right) \Big] \quad (41)$$

$$S_S S_1 R_1^2 R_2^2 \, d R_1 \, d R_2 .$$

The sumation over m, m', μ gives the 6-j symbol $\begin{Bmatrix} 1/2 & 1/2 & L \\ 2 & 2 & 3/2 \end{Bmatrix}$.

It follows then $L = O$, where from one has

$$J_{D-D} \sim \int \Psi_i (S_S) \Psi_j (S_S) \, j_0 \left(\frac{1}{\sqrt{3}} q R_1 \right) \left(R_2^4 - R_1^4 \right) \times$$
$$\times S_S S_1 R_1^2 R_2^2 \, dR_1 \, dR_2 . \quad (42)$$

The substitution (33) with the help of (34) gives for this expression and integral with the same structure as the expression (35).

Depending on the choice of the fenomenological function, the remaining one-dimensional integration is carried but either analytically or numerically.

In subsequent papers this mathematical approach will be applied to studies of radiative proton and neutron capture on deutron and triton at various energies.

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O TALASNOJ FUNKCIJI OSNOVNOG STANJA TROČESTIČNIH NUKLEARNIH SISTEMA

I. *Konstrukcija talasne funkcije i prilozi izvodu jednostaničnih matričnih elemenata*

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Sadržaj

Upotrebljene su metode teorije grupa za konstrukciju talasne funkcije osnovnog stanja jezgra ^3H i ^3He. Pokazano je kako se komponente funkcije mogu klasificirati prema svojstvima ireducibilnih reprezentacija permutacione grupe (S_3) i rotacione grupe u trodimezionalnom prostoru (R_3). Upotrebom specijalne koordinantne transformacije (jed. 4.) prostorne koordinate postaju elementi grupe (S_3), čime se omogućuje primena pojednostavljenog postupka za ispunjenje uslova antisimetričnosti talasne funkcije u odnosu na prostorne, spinske i izospinske koordinate. Za to je dovoljno uvek koristiti pravila direktnog proizvoda formule (1.). Upotrebom ovih formula dobivene su skalarne, vektorska i tenzorske komponente. U cilju homogenizacije i pojednostavljenja računa ove su komponente pisane u sfernim koordinatima upotrebom formula (6–13). Sistematizacija komponenata, čija linearna kombinacija daje ukupnu talasnu funkciju data je formulama (20) do (27).

Diskutovana je razlika ovog prilaza od do sada poznatih konstrukcija talasne funkcije i navedeni su razlozi zbog kojih određene komponente nisu uzete u obzir.

Na primeru električnog form-faktora za jezgru ^3He izračunati su interferencioni članovi svih komponenata talasne funkcije. Lako se pokazuje da se oni fizički procesi, koji se opisuju jednočestičnim hamiltonijanima (fotodezintegracija, radiacioni zahvat, rasejanje elektrona itd.) svode na tipove matričnih elemenata: formule: (35), (37) i (40). Funkcije Ψ_i (S_s) koje u ovim izrazima figurišu određuju se poređenjem sa eksperimentom.