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APPLICATION OF URSELL-MAYER METHOD IN THE MODEL OF SEMI-FREE GAS OF LIQUID ^{4}He

S. KILIĆ

University of Sarajevo, Sarajevo

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1. K. Ljolje[1] has shown that short range correlations of each pair of particles secure a bound state of liquid ^{4}He in low density limit. Evaluation of relevant integrals were performed by making use of hard-sphere method. Here we want to apply Ursell-Mayer method in the evaluation of these integrals, that means of the expectation value of the energy, in order to obtain exact expansion with respect to density, and to compare the result with that of K. Ljolje in the low density limit.

2. The expression of expectation value of the energy of liquid ^{4}He in the stated paper reads

$$E_0 = n(n-1)(n-2)\frac{\int \psi^2 \Phi(123)\, d\vec{r}_1 \dots d\vec{r}_n}{\int \psi^2\, d\vec{r}_1 \dots d\vec{r}_n} + n(n-1)\frac{\int \psi^2 f(12)\, d\vec{r}_1 \dots d\vec{r}_n}{\int \psi^2\, d\vec{r}_1 \dots d\vec{r}_n}. \tag{1}$$

The notations are the same as in the paper[1].
We define the functions:

$$G_2 = V^2\, e^{-2Be^{-\frac{\alpha}{2} r_{12}}} \cdot \frac{1}{I_n} \cdot \int \psi^2\, d\vec{r}_3 \cdots d\vec{r}_n, \tag{2}$$

$$G_3 = V^3 \, e^{-2Be^{-\frac{\alpha}{2}r_{12}} - 2Be^{-\frac{\alpha}{2}r_{13}} - 2Be^{-\frac{\alpha}{2}r_{23}}} \cdot \frac{1}{I_n} \cdot \int \psi^2 \, d\vec{r}_4 \cdots d\vec{r}_n \,, \qquad (3)$$

where

$$I_n = \int \psi^2 \, d\vec{r}_1 \ldots d\vec{r}_n \; ; \qquad (4)$$

and similarly we introduce

$$I_{n-2} = \int \psi^2 (3 \ldots n) \, d\vec{r}_3 \ldots d\vec{r}_n \, , \qquad (5)$$

$$I_{n-3} = \int \psi^2 (4 \ldots n) \, d\vec{r}_4 \ldots d\vec{r}_n \, . \qquad (6)$$

After the Mayer's supstitution[2)]

$$f' \left(r_{ij} \right) \equiv f'_{ij} = e^{-2Be^{-\frac{\alpha}{2} r_{ij}}} - 1 \, , \qquad (7)$$

we have

$$\psi^2 = \sum_{\nu_{kl} = 0,1} \prod_{k = 1,2}^{n \geqslant 1 \geqslant 3} f'^{\nu_{kl}}_{kl} \, e^{-2Be^{-\frac{\alpha}{2} r_{12}}} \psi^2 (3 \ldots n) \, . \qquad (8)$$

Let us first consider the function G_2. After supstitution of (8) in (2), let us concentrate our attention to one term in the sum. Let be the term in which the function f' with m particles have $\nu_{kl} \neq 0$ (the particles 1 and 2 are excluded here).

Integral over other particles multiplied with V^m/I_{n-2} is

$$F_m = \frac{V^m}{I_{n-2}} \int \psi^2 \; (3 \ldots n) \; d\vec{r}_{m+3} \ldots d\vec{r}_n \, . \qquad (9)$$

Since the integrals of each set of m particles are equal, and

$$\frac{(n-2-m)! \; m!}{(n-2)!} \sim \frac{n^m}{m!} \; (\text{for } m \ll n),$$

we obtain:

$$G_2 = \frac{V^2 I_{n-2}}{I_n} \left\{ 1 + \sum_{n \gg m \geqslant 1} \frac{\rho^m}{m!} \int \prod_{l=3}^{m+2} \left| e^{-\sum\limits_{k=1}^{2} 2Be^{-\frac{\alpha}{2} r_{kl}}} - 1 \right] \cdot F_m \, d\vec{r}_3 \cdots d\vec{r}_{m+2} + \cdots \right\} . \qquad (10)$$

Performing similar procedures one obtains

$$G_3 = \frac{V^3 I_{n-3}}{I_n}\left\{1 + \sum_{n \geqslant m \geqslant 1} \frac{\rho^m}{m!} \int \prod_{l=4}^{m+3} \left[e^{-\sum_{k=1}^{3} 2Be^{-\frac{\alpha}{2} r_{kl}}} - 1\right] \cdot F'_m \, d\vec{r}_4 \cdots d\vec{r}_{m+3} + \cdots \right\}. \tag{11}$$

$$\frac{I_n}{I_{n-2}} = \int e^{-2Be^{-\frac{\alpha}{2} r_{12}}} \left\{1 + \sum_{n \geqslant m \geqslant 1} \frac{\rho_m}{m!} \int \prod_{l=4}^{m+3} \cdot \left[e^{-\sum_{k=1}^{2} 2Be^{-\frac{\alpha}{2} r_{kl}}} - 1\right] \cdot F_m \, d\vec{r}_3 \cdots d\vec{r}_{m+2} + \cdots \right\} d\vec{r}_1 \, d\vec{r}_2 \, , \tag{12}$$

$$\frac{I_n}{I_{n-3}} = \int e^{-2Be^{-\frac{\alpha}{2} r_{12}} - 2Be^{-\frac{\alpha}{2} r_{13}} - 2Be^{-\frac{\alpha}{2} r_{23}}} \left\{1 + \sum_{n \geqslant m \geqslant 1} \frac{\rho^m}{m!} \int \prod_{l=4}^{m+3} \left[e^{-\sum_{k=1}^{3} 2Be^{-\frac{\alpha}{2} r_{kl}}} - 1\right] F'_m \, d\vec{r}_4 \cdots d\vec{r}_{m+3} + \cdots \right\} d\vec{r}_1 \, d\vec{r}_2 \, d\vec{r}_3 \, , \tag{13}$$

with

$$F'_m = \frac{V^m}{I_{n-3}} \int \psi^2 \, (4 \ldots n) \, d\vec{r}_{m+4} \ldots d\vec{r}_n \, . \tag{14}$$

The supstitution of (10) and (11) in Eq. (1) gives the expansion of the expectation value of the energy with respect to the density.

3. In the limit of low density ($\varrho \to 0$) it follows

$$\frac{E_0}{n} = \varrho I_1 + \varrho^2 \left(I_2 + I'_2 + \gamma I_1\right) , \tag{15}$$

where

$$I'_2 = V^{-1} \int e^{-2Be^{-\frac{\alpha}{2} r_{12}}} f(12) \left[e^{-2Be^{-\frac{\alpha}{2} r_{13}}-2Be^{-\frac{\alpha}{2} r_{23}}} - 1 \right] d\vec{r}_1 \, d\vec{r}_2 \, d\vec{r}_3 \,,$$

$$\gamma = -2 \int f'(r) \, d\vec{r} \,.$$

Other notations are the same as in the paper[1].

We see that the first term of Eqs. (15) and (16) in the paper[1] are equal. For real densities of liquid ^{4}He the first two powers of ϱ are not enough.

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References

1) K. Ljolje, Fizika, 1 (1968) 11;
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