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Degree 6 hyperbolic polynomials and orders of moduli

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Abstract. We consider real univariate degree d real-rooted polynomials with non-vanishing coefficients. Descartes' rule of signs implies that such a polynomial has $\tilde{c}$ positive and $\tilde{p}$ negative roots counted with multiplicity, where $\tilde{c}$ and $\tilde{p}$ are the numbers of sign changes and sign preservations in the sequence of its coefficients, $\tilde{c}+\tilde{p}=d$. For $d=6$, we give an exhaustive answer to the question: When the moduli of all 6 roots are distinct and arranged on the real positive half-axis, in which positions can the moduli of the negative roots depend

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1. Introduction

A real univariate polynomial is *hyperbolic* if all its roots are real. We consider hyperbolic polynomials with all coefficients non-vanishing. For such a degree d polynomial, the classical Descartes' rule of signs implies that the number of its positive (resp. negative) roots counted with multiplicity is equal to the number $\tilde{c}$ of sign changes (resp. $\tilde{p}$ of sign preservations) in the sequence of its coefficients, see [1, 2, 3, 4, 6, 8, 9, 17, 18]; $\tilde{c}+\tilde{p}=d$. This fact, however, does not answer the following more subtle question:

Question 1. *For fixed degree d, consider the set of hyperbolic polynomials with given signs of the coefficients and with distinct moduli of roots. Suppose that these moduli are arranged on the real positive half-axis. In which positions can the moduli of the negative roots depend on the signs of the coefficients?*

We give an exhaustive answer to the question for $d=6$. For $d\leq 5$, its answer can be found in [15], see Example 1.1 and Section 3 therein. In order to recall some other results directly related to Question 1 we recall the following definition:

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Definition 1. *(1) A real polynomial* $Q := \sum_{j=0}^{d} q_j x^j$ *is said to define the* sign pattern $\sigma(Q) := (\mathrm{sgn}(q_d), \ldots, \mathrm{sgn}(q_0))$. *Formally, a sign pattern of length* $d+1$ *is a string of* $d+1$ *signs* $+$ *and/or* $-$. *We operate mainly with sign patterns beginning with a* $+$. *Thus a sign pattern is completely defined by the corresponding* change-preservation pattern *(and vice versa), which is a* d*-vector whose components are the letters* p *and* c*; when* $q_j q_{j-1} > 0$ *(resp.* $q_j q_{j-1} < 0$*), in the* j*th position from the right there is a* p *(resp. a* c*).*

(2) The order of moduli *defined by the roots of a given hyperbolic polynomial* Q *is denoted as follows. (The general definition should be clear from this example.) Suppose that* $d = 6$ *and that there are three negative roots* $-\gamma_3 < -\gamma_2 < -\gamma_1$ *and three positive roots* $\alpha_1 < \alpha_2 < \alpha_3$ *(so* $\tilde{c} = \tilde{p} = 3$*), where*

$$\alpha_1 < \gamma_1 < \gamma_2 < \alpha_2 < \gamma_3 < \alpha_3.$$

Then we say that the roots define the order of moduli $PNNPNP$*, i. e. the letters* P *and* N *denote the relative positions of the moduli of positive and negative roots.*

(3) For a given degree d*, a couple (change-preservation pattern, order of moduli) (or a* couple *for short) is* compatible *if the number of letters* c *(resp.* p*) of the former is equal to the number of letters* P *(resp.* N*) of the latter. A compatible couple is* realizable *if there exists a hyperbolic polynomial whose coefficients (resp. moduli of roots) define the change-preservation pattern (resp. the order of moduli) of the couple.*

We can give now a more precise formulation of Question 1:

Question 2. *For a given degree* d*, which compatible couples are realizable?*

There are two extremal situations with regard to Question 2.

Definition 2. *For a given change-preservation pattern (or, equivalently, a sign pattern) one defines the corresponding* canonical order of moduli *as follows. One reads the pattern from the right and one writes the order from the left. To each letter* c *(resp.* p*) one puts in correspondence the letter* P *(resp.* N*).*

Each sign pattern (or equivalently a change-preservation pattern) is realizable with its corresponding canonical order, see [11, Proposition 1].

Definition 3. *A change-preservation pattern (or a sign pattern) is* canonical *if it is realizable only with the corresponding canonical order of moduli.*

It is shown in [14, Theorem 7] that a sign pattern is canonical if and only if it does not contain any of the 4-tuples $(+,+,-,-)$, $(+,-,-,+)$, $(-,-,+,+)$ or $(-,+,+,-)$. Hence, a change-preservation pattern is canonical if and only if it contains no string *cpc* or *pcp*. Canonical sign patterns are exceptional in the sense that the ratio of their number and the number of all sign patterns tends to 0 as d tends to ∞, see [14, Proposition 10].

The second extremal situation is the one of *rigid* orders of moduli.

Definition 4. *An order of moduli is* rigid *if all hyperbolic polynomials with this order of moduli define one and the same sign pattern.*

It is proved that (see [12, Theorem 8]) rigid are exactly the orders of moduli of the form $PNPNPN\cdots$, $NPNPNP\cdots$, $PP\cdots P$ or $NN\cdots N$. The corresponding change-preservation patterns are of the form $\cdots pcpcpc$, $\cdots cpcpcp$, $cc\cdots c$ or $pp\cdots p$. Hence, rigid orders of moduli are also exceptional.

We introduce now a $\mathbb{Z}_2\times\mathbb{Z}_2$-action:

Definition 5. *(1) For a given degree d, there are two commuting involutions which act on the set of couples. These are*

$$i_m\ :\ Q(x)\mapsto(-1)^dQ(-x)\quad\text{ and }\quad i_r\ :\ Q(x)\mapsto x^dQ(1/x)/Q(0).$$

The role of the factors $(-1)^d$ and $1/Q(0)$ is to preserve the set of monic polynomials. The involution i_m exchanges the letters P and N in the order of moduli, the letters c and p in the change-preservation pattern and the quantities $\tilde{c}$ and $\tilde{p}$. The involution i_r reads orders, patterns and polynomials (modulo the factor $1/Q(0)$) from the right. It preserves the quantities $\tilde{c}$ and $\tilde{p}$.

(2) The *orbits* of couples under the $\mathbb{Z}_2\times\mathbb{Z}_2$-action are of length 4 or 2. One can also consider orbits of only sign patterns or of orders of moduli.

Remark 1. *(1) For any given sign pattern σ, its orbit can be of length 2 only if either $i_r(\sigma)=\sigma$ or $i_ri_m(\sigma)=\sigma$. Indeed, one always has $i_m(\sigma)\neq\sigma$. All couples of a given orbit are simultaneously (non-)realizable.*

(2) In the text, we use the following notation: if a sign pattern consists of m_1 pluses followed by m_2 minuses followed by m_3 pluses etc., then we denote this sign pattern by $\Sigma_{m_1,m_2,m_3,\ldots}$. For $d=6$, an example of an orbit of a sign pattern of length 2 is the one of $\Sigma_{3,1,3}$ with $\tilde{c}=2$, $\tilde{p}=4$ and $i_r(\Sigma_{3,1,3})=\Sigma_{3,1,3}$. The other sign pattern of the orbit is $i_m(\Sigma_{3,1,3})=i_mi_r(\Sigma_{3,1,3})=\Sigma_{1,1,3,1,1}$, with $\tilde{c}=4$, $\tilde{p}=2$.

The involution i_m exchanging the quantities $\tilde{c}$ and $\tilde{p}$ when studying the realizability of the couples with $d=6$, it suffices to consider the cases $\tilde{c}=0$, 1, 2 and 3. The first three of them have been thoroughly analyzed in [16] (we recall the corresponding results in Section 2), so we concentrate on the case $\tilde{c}=3$.

Lemma 1. *For $d=6$, there are 7 orbits of sign patterns with three sign changes:*

$$\begin{array}{ll}A:\{\Sigma_{3,1,2,1},\ \Sigma_{1,2,1,3},\ \Sigma_{2,3,1,1},\ \Sigma_{1,1,3,2}\}\ , & D:\{\Sigma_{4,1,1,1},\ \Sigma_{1,1,1,4}\},\\ B:\{\Sigma_{1,4,1,1},\ \Sigma_{1,1,4,1},\ \Sigma_{3,1,1,2},\ \Sigma_{2,1,1,3}\}, & E:\{\Sigma_{2,2,2,1},\ \Sigma_{1,2,2,2}\},\\ C:\{\Sigma_{2,1,2,2},\ \Sigma_{2,2,1,2},\ \Sigma_{1,2,3,1},\ \Sigma_{1,3,2,1}\}, & F:\{\Sigma_{3,2,1,1},\ \Sigma_{1,1,2,3}\}\\ \quad\text{and} & G:\{\Sigma_{1,3,1,2},\ \Sigma_{2,1,3,1}\}.\end{array}$$

Out of these, the canonical are exactly B, D and G.

Remark 2. *(1) For $\sigma=\Sigma_{4,1,1,1}$, $\Sigma_{2,2,2,1}$, $\Sigma_{3,2,1,1}$ and $\Sigma_{1,3,1,2}$, one has $i_mi_r(\sigma)=\sigma$.*

(2) The canonical orbits B, D and G give rise to the following realizable couples and only to them:

$$\begin{aligned} B\ &: (\Sigma_{1,4,1,1},\ PPNNNP), \qquad (\Sigma_{1,1,4,1},\ PNNNPP),\\ &\ (\Sigma_{3,1,1,2},\ NPPPNN)\ \text{ and } (\Sigma_{2,1,1,3},\ NNPPPN);\\ D\ &: (\Sigma_{4,1,1,1},\ PPPNNN)\ \text{ and } (\Sigma_{1,1,1,4},\ NNNPPP);\\ G\ &: (\Sigma_{1,3,1,2},\ NPPNNP)\ \text{ and } (\Sigma_{2,1,3,1},\ PNNPPN). \end{aligned}$$

Proof of Lemma 1. Among the sign patterns of the form Σ_{m_1,m_2,m_3,m_4} with $m_1 + m_2 + m_3 + m_4 = 7$ and $\tilde{c} = \tilde{p} = 3$, all components m_i must be ≤ 4. Hence, there are exactly four such sign patterns in which exactly one component equals 4 (the other components equal 1), exactly twelve in which one component equals 3, and exactly four in which three components equal 2. These are all the 20 sign patterns listed in the lemma. The last statement of the lemma is checked straightforwardly. □

Part (2) of Remark 2 settles the cases B, D and G. Theorem 1 below finishes the study of realizability of couples with $d = 6$, $\tilde{c} = 3$. We remind that by Definition 5 and part (1) of Remark 1 it suffices to give the answer only for one sign pattern from each of the cases A, C, E and F.

Theorem 1. *(1) The sign pattern $\Sigma_{3,1,2,1}$ is realizable by and only by the following orders of moduli: $PPPNNN$, $PPNPNN$, $PPNNPN$, $PNPPNN$, and $NPPPNN$.*

(2) The sign pattern $\Sigma_{2,1,2,2}$ is realizable by and only by the following orders of moduli: $PNNPPN$, $NPPPNN$, $NPPNPN$, $NPPNNP$, $NPNPPN$ and $NNPPPN$.

(3) The sign pattern $\Sigma_{2,2,2,1}$ is not realizable by and only by the following compatible orders of moduli: $NPNPNP$, $NPNNPP$, $NNPPNP$, $NNPNPP$, and $NNNPPP$.

(4) The sign pattern $\Sigma_{3,2,1,1}$ is realizable by and only by the following orders of moduli: $PPPNNN$, $PPNPNN$, $PPNNPN$ and $PNPPNN$.

Theorem 1 is proved in Section 3. The method of its proof and some comments are given in Section 2.

2. Comments and the method of proof of Theorem 1

2.1. Systems of linear differential equations

The characteristic polynomials of linear systems of ordinary differential equations are often hyperbolic. Consider such a system $dX/dt = AX$, where A is an $n \times n$-matrix with real entries. Suppose that all its eigenvalues $\lambda_1, \ldots, \lambda_n$ are real. This is true, in particular, for symmetric matrices. Suppose also that they are distinct. Then any component of any solution is of the form $\sum c_j e^{\lambda_j t}$, $c_j \in \mathbb{R}$. For a generic solution, all coefficients c_j are non-zero.

If the characteristic polynomial of A defines a canonical sign pattern, then one knows whether the eigenvalue of the largest modulus is positive or negative. Hence, one knows (without computing the eigenvalues) whether a generic solution grows faster in modulus as $t \to +\infty$ or as $t \to -\infty$.

2.2. The results for $\tilde{c} < 3$

We begin by reminding that for $d = 6$, there is just one change-preservation pattern with $\tilde{c} = 0$. This is *pppppp* and it is realizable with the only compatible order of moduli $NNNNNN$.

Notation 1. *For $d = 6$ and $\tilde{c} = 1$ (resp. $\tilde{c} = 2$), we denote by u_1 and u_2 (resp. u_1, u_2 and u_3) the number of moduli of negative roots belonging to the respective intervals $(0, \alpha_1)$ and $(\alpha_1, +\infty)$ (resp. to $(0, \alpha_1)$, (α_1, α_2) and $(\alpha_2, +\infty)$). For $d = 6$ and $\tilde{c} = 3$, we denote by analogy the quantities u_1, u_2, u_3 and u_4 with respect to the intervals $(0, \alpha_1)$, (α_1, α_2), (α_2, α_3) and $(\alpha_3, +\infty)$. Example: the order of moduli $NNPNNN$ corresponds to $[u_1, u_2] = [2, 3]$, while $NPNNPN$ corresponds to $[u_1, u_2, u_3] = [1, 2, 1]$ and $NPPNNP$ corresponds to $[u_1, u_2, u_3, u_4] = [1, 0, 2, 0]$. There are 6 couples $[u_1, u_2]$, $u_1 + u_2 = 5$, 15 triples $[u_1, u_2, u_3]$, $u_1 + u_2 + u_3 = 4$, and 20 quadruples $[u_1, u_2, u_3, u_4]$, $u_1 + u_2 + u_3 + u_4 = 3$.*

For $d = 6$, $\tilde{c} = 1$, we list the orders of moduli with which the sign patterns Σ_{m_1,m_2}, $m_1 + m_2 = 7$, are realizable (see [10]):

$$\begin{gathered}\text{for } 1 \leq m_1 < m_2,\ 0 \leq u_2 \leq 2m_1 - 2;\\ \text{for } 1 \leq m_2 < m_1,\ 0 \leq u_1 \leq 2m_2 - 2.\end{gathered} \tag{1}$$

For $d = 6$, $\tilde{c} = 2$, realizability of couples has been studied in [16]. There are two cases of canonical sign patterns. The corresponding couples are:

$$(\Sigma_{1,5,1},\ [0,4,0]) \quad \text{and} \quad (\Sigma_{m,1,q},\ [q-1,0,m-1]), \qquad m + q = 6. \tag{2}$$

We give the remaining results in a table in which the first column contains the sign pattern, the second contains the realizable and the third the non-realizable triples $[u_1, u_2, u_3]$:

SP	Y	N
$\Sigma_{2,4,1}$	$[0,2,2]$, $[0,3,1]$, $[0,4,0]$	all other cases
$\Sigma_{3,3,1}$	$[1,0,3]$, $[0,0,4]$, $[0,1,3]$, $[0,2,2]$, $[0,3,1]$, $[0,4,0]$	all other cases
$\Sigma_{4,2,1}$	$[1,0,3]$, $[0,0,4]$, $[0,1,3]$ $[0,2,2]$	all other cases
$\Sigma_{2,3,2}$	all possible cases	no cases
$\Sigma_{3,2,2}$	all other cases	$[4,0,0]$, $[3,1,0]$ $[2,2,0]$, $[1,3,0]$.

2.3. The ratio between the numbers of realizable and all possible couples

The number of realizable couples with $d = 6$ and $\tilde{c} = 3$ can be found using Theorem 1 and Remark 2. It equals

$$5 \times 4 + 6 \times 4 + 15 \times 2 + 4 \times 2 + 1 \times 4 + 1 \times 2 + 1 \times 2 = 90.$$

These products correspond to the orbits A, C, E, F, B, D and G, respectively. The second factor corresponds to the number of sign patterns in the given orbit.

At the same time, the number of compatible orders of moduli with 3 letters P and 3 letters N equals 20. So the number all couples with $d = 6$ and $\tilde{c} = 3$ equals

$$(4 + 4 + 2 + 2 + 4 + 2 + 2) \times 20 = 400.$$

For $\tilde{c} = 0$ and 6, the only couples $K := (\Sigma_7, NNNNNN)$ and $i_m(K)$ are realizable. For $\tilde{c} = 1$, the numbers of realizable and of all couples are (see (1))

$$1 + 3 + 5 + 5 + 3 + 1 = 18 \quad \text{and} \quad 6 \times 6 = 36, \quad \text{respectively.}$$

The same numbers apply to the case $\tilde{c} = 5$ as well (one has to use the involution i_m). Factor 6 stands for the number of orders of moduli with $\tilde{c} = 1$ or 5.

For $\tilde{c} = 2$ and 4, we use the information at the end of the previous subsection to find these numbers. The last table shows that there are 4 orbits of sign patterns of length 4 and 1 of length 2, each with 15 compatible orders of moduli (half of which correspond to the case $\tilde{c} = 2$ and the other half to $\tilde{c} = 4$). This makes 270 couples. To these one has to add the canonical sign patterns (see (2)), which brings another $6 \times 15 = 90$ couples with $\tilde{c} = 2$ and 90 with $\tilde{c} = 4$. So there are 450 couples, 12 of which are realizable in the case of canonical sign patterns and $3 \times 4 + 6 \times 4 + 4 \times 4 + 15 \times 2 + 11 \times 4 = 126$ in other cases.

Thus the ratio between the numbers of realizable and all couples is

$$r(6) = (90 + 2 + 36 + (12 + 126))/(400 + 2 + 72 + 450) = 19/66.$$

The numbers $r(d)$, $d \leq 5$ are computed in [15]. For $d \leq 6$, the sequence of numbers $r(d)$ looks like this: 1, 2/3, 3/5, 3/7, 47/126, 19/66. One could conjecture that this sequence (defined for $d \in \mathbb{N}^*$) is decreasing. The sequence $r(d+1)/r(d)$, $d = 1, \ldots,$ 5, equals

$$2/3 = 0.66\ldots, \quad 9/10 = 0.9, \quad 5/7 = 0.71\ldots, \quad 47/54 = 0.87\ldots, \quad 399/517 = 0.77\ldots.$$

It seems that when the ratio $r(d+1)/r(d)$ is defined for $d \in \mathbb{N}^*$, this gives two adjacent sequences.

2.4. The methods used in the proof of Theorem 1

We use four methods in the proof of Theorem 1. Three of them can be qualified as analytic and the fourth as computational. In the next subsection, we explain how realizability of certain couples for degree $d+1$ hyperbolic polynomials can be deduced from realizability of couples for degree d. The second method consists of proving that the inequalities between the moduli of roots do not allow certain coefficients of a hyperbolic polynomial to have certain signs. In Subsection 2.6, we describe another method used to prove that certain couples are not realizable. The method is based on properties of the set E_d of hyperbolic polynomials having a couple of non-zero opposite real roots.

Finally, in order to quickly obtain examples of realizability, we use a Python program which generates uniformly distributed random numbers. For a given degree

d, a given sign pattern and a given order of moduli, the program generates d real numbers to create roots in a given order of moduli. Then the code calculates the coefficients of the polynomial and checks whether they match the sign pattern. If this is the case, the code stops and returns the result, i.e. the polynomial. If not, it continues and repeats the simulation until it finds one or stops if the assigned number of simulations is reached. Finding concrete examples of realizability when analytic methods fail turns out to be indispensable in the context of a problem closely related to Question 2, see [5]. The problem asks to describe real, but not necessarily hyperbolic polynomials, for which triples (sign pattern, number of positive roots, number of negative roots) compatible with Descartes' rule of signs are realizable.

2.5. Concatenation of couples

Consider a hyperbolic degree d polynomial V with distinct moduli of roots and non-vanishing coefficients. Denote by Ω the order of the moduli of its roots, where Ω is a string of letters P and/or N. Then for $\varepsilon > 0$ small enough, the first $d+1$ coefficients of the degree $d+1$ hyperbolic polynomials $W_- := V(x)(x-\varepsilon)$ and $W_+ := V(x)(x+\varepsilon)$ are perturbations of the respective coefficients of V. Hence, they are of the same signs as the latter coefficients. The three polynomials realize the couples

$$V := (\sigma(V), \Omega), \quad W_- : (\sigma(W_-), P\Omega) \quad \text{and} \quad W_+ : (\sigma(W_+), N\Omega).$$

Denote by α the last component of the sign pattern $\sigma(V)$, where $\alpha = +$ or $-$. Hence $\sigma(W_-)$ (resp. $\sigma(W_+)$) is obtained from $\sigma(V)$ by adding the component $-\alpha$ (resp. α) to the right. We say that the couples W_- and W_+ are obtained by *concatenation* of the couple V with the couples $((+,-), P)$ and $((+,+), N)$, respectively. The method of concatenation is explained in a broader context in [5] and within the framework of the problem mentioned at the end of Subsection 2.4.

2.6. The set E_d and neighbours of quadruples

Notation 2. For a given sign pattern σ of length $d+1$, we denote by $\Pi_d(\sigma)$ the set of monic hyperbolic degree d polynomials with distinct roots defining the sign pattern σ. For an order of moduli Ω compatible with σ, we denote by $\Pi_d(\sigma, \Omega) \subset \Pi_d(\sigma)$ the set of monic hyperbolic degree d polynomials defining the sign pattern σ the order of moduli of their roots is Ω. We denote by $E_d(\sigma)$ the subset of $\Pi_d(\sigma)$ on which a positive and a negative root have an equal modulus.

It is proved in [13, Theorem 2] that all sets of the form $\Pi_d(\sigma)$ are open and contractible. It is shown in [7, Theorem 1.5] that at a generic point the set $E_d(\sigma)$ is locally a smooth hypersurface, and at a point, where there are s distinct couples (positive root, negative root) of equal modulus, $E_d(\sigma)$ is the transversal intersection of s smooth hypersurfaces.

Definition 6. *Two quadruples* $[u_1, u_2, u_3, u_4]$ *are* neighbours *if they are obtained from one another by shifting a unit one position to the left or to the right. E.g. all neighbours of* $[0, 2, 0, 1]$ *are* $[1, 1, 0, 1]$, $[0, 1, 1, 1]$ *and* $[0, 2, 1, 0]$.

Proposition 1. *Suppose that for given degree d and sign pattern σ, two couples (σ,Ω_1) and (σ,Ω_2) are realizable, with $\Omega_1 \neq \Omega_2$. Then there is a continuous path connecting two points $A_i \in \Pi_d(\sigma,\Omega_i)$, $i = 1, 2$, passing through a point $A' \in \Pi_d(\sigma,\Omega')$, where Ω' is a neighbour of Ω_1.*

Proof. Indeed, one can assume that the path γ is smooth and avoids the non-generic points of $E_d(\sigma)$, i.e. the points at which there is more than one pair of opposite real non-zero roots, see [7, Theorem 1.5]. On the other hand, as $\Omega_2 \neq \Omega_1$, the path γ intersects $E_d(\sigma)$. The first time when this occurs, the path passes from $\Pi_d(\sigma,\Omega_1)$ to $\Pi_d(\sigma,\Omega')$, where Ω' is a neighbour of Ω_1. □

Remark 3. *As the path from the proof of Proposition 1 avoids the non-generic points of E_d, when it intersects the common boundary of the sets $\Pi_d(\sigma,\Omega_1)$ and $\Pi_d(\sigma,\Omega')$, this point corresponds to a polynomial having two opposite real roots (and this is the only equality between moduli of its roots). After a linear change of the variable, x this polynomial can be given the form $Q = (x^2-1)R$, where R is a monic hyperbolic polynomial of degree $d-2$.*

3. Proof of Theorem 1

Part (1). The couple $(\Sigma_{3,1,2,1},\ PNPPNN)$ is realizable, because $PNPPNN$ is the canonical order of moduli (see [11, Proposition 1]). We prove by examples realizability of the remaining 4 couples mentioned in part (1) of the theorem. Our examples involve polynomials having a positive and a negative root of equal moduli. After perturbing these roots so that they become of distinct moduli (the perturbation does not change the signs of the coefficients), one obtains polynomials realizing the given order of moduli with the sign pattern $\Sigma_{3,1,2,1}$. For the first polynomial of the list below the perturbed roots equal -9, -1.01, $-1-\varepsilon$, 0.39, 0.4 and $1-\varepsilon$, $0 < \varepsilon \ll 0.1$:

$$
\begin{array}{ll}
PPPNNN & (x-0.39)(x-0.4)(x-1)(x+1)(x+1.01)(x+9) \\
& = x^6+8.23x^5+0.1902x^4-13.26928x^3 \\
& +0.08276x^2+5.03928x-1.27296, \\
PPNPNN & (x-0.2)(x-1)(x+1)(x-3.1)(x+5)(x+10) = \\
& = x^6+11.7x^5+0.12x^4-167.4x^3+29.88x^2+155.7x-31, \\
PPNNPN & (x-0.39)(x-0.4)(x+0.99)(x+1)(x-1)(x+9) \\
& = x^6+9.2x^5+0.1739x^4-14.68046x^3 \\
& +0.21606x^2+5.48046x-1.38996, \\
NPPPNN & (x+1)(x-1)(x-2)(x-2.1)(x+5)(x+20) \\
& = x^6+20.9x^5+0.7x^4-325.9x^3+418.3x^2+305x-420.
\end{array}
$$

Now we prove non-realizability of the rest of the orders of moduli with the sign pattern $\Sigma_{3,1,2,1}$. Part of the results also concern the sign pattern $\Sigma_{3,2,1,1}$.

Proposition 2. *The 10 orders of moduli with $u_4 = 0$ are not realizable with any of the sign patterns $\Sigma_{3,1,2,1}$ or $\Sigma_{3,2,1,1}$.*

Proof. Below σ denotes any of the sign patterns $\Sigma_{3,1,2,1}$ or $\Sigma_{3,2,1,1}$. The couples $(\sigma, PPNNPN)$ are realizable, see the examples at the beginning of the proofs of parts (1) and (4) of the theorem. For these couples one has $u_4 = 1$. This implies that the set $\Pi_6(\sigma, PPNNPN)$ is open and non-empty, see Notation 2. Denote by Ω an order of moduli compatible with the sign pattern σ and $u_4 = 0$. If the set $\Pi_6(\sigma, \Omega)$ is non-empty, then there exists a continuous path $\gamma \subset \Pi_6(\sigma)$ leading from a point of $\Pi_6(\sigma, \Omega)$ to a point of $\Pi_6(\sigma, PPNNPN)$ (see Proposition 1 and its proof). Hence, this path contains a polynomial Q defining the sign pattern σ and having a positive and a negative roots of equal modulus. Moreover, its other roots are of smaller modulus.

Set $Q := x^6 + \sum_{j=1}^{5} q_j x^j$. After a linear change of the variable x, Q takes the form $Q := (x^2-1)R$, where $R := x^4 + \sum_{j=0}^{3} a_j x^j$ is a degree 4 hyperbolic polynomial the moduli whose roots are < 1.

Lemma 2. *Suppose that the polynomials $Q := x^6 + \sum_{j=1}^{5} q_j x^j$ and $R := x^4 + \sum_{j=0}^{3} a_j x^j$ are hyperbolic, $Q := (x^2 - 1)R$ and Q defines one of the sign patterns $\Sigma_{3,1,2,1}$ and $\Sigma_{3,2,1,1}$. Then R defines the sign pattern $\Sigma_{3,1,1}$.*

Proof. It is clear that

$$Q = x^6 + a_3x^5 + (a_2 - 1)x^4 + (a_1 - a_3)x^3 + (a_0 - a_2)x^2 - a_1x - a_0.$$

If Q defines the sign pattern $\Sigma_{3,1,2,1}$ or $\Sigma_{3,2,1,1}$, then $a_0 > 0$, $a_1 < 0$ and $a_3 > 0$. If $a_2 \leq 0$, then $a_2 - 1 < 0$, which contradicts each of the two sign patterns, so one must have $a_2 > 0$ and R defines the sign pattern $\Sigma_{3,1,1}$. □

The sign pattern $\Sigma_{3,1,1}$ is canonical. Hence, the order of moduli defined by the roots of R is $PPNN$. We denote these roots by $0 < a < b$ and $-g < -f < 0$, where $a < b < f < g$. Thus

$$q_4 = ab - af - ag - bf - bg + fg - 1.$$

If $g \leq 1$, then $q_4 = (fg - 1) + a(b - f) - ag - bf - bg < 0$, which is a contradiction. Thus none of the orders of moduli with $u_4 = 0$ are realizable.

□

We give the proof of non-realizability of the remaining 5 couples. The couples $(\Sigma_{3,1,2,1}, PNPNPN)$ and $(\Sigma_{3,2,1,1}, PNPNPN)$ (the order of moduli corresponds to the quadruple $[0, 1, 1, 1]$) are not realizable, because the order of moduli $PNPNPN$ is rigid and hence realizable only with the sign pattern $\Sigma_{2,2,2,1}$, see Definition 4 and the text that follows.

Suppose that the order of moduli $[1, 1, 0, 1]$ is realizable with the sign pattern $\Sigma_{3,1,2,1}$ or $\Sigma_{3,2,1,1}$, meaning that the following inequalities are satisfied:

$$\gamma_1 < \alpha_1 < \gamma_2 < \alpha_2 < \alpha_3 < \gamma_3.$$

In this case, we have

$$q_4 = (\alpha_1\alpha_2 - \alpha_2\gamma_2) + (\alpha_1\alpha_3 - \alpha_1\gamma_3) + (\alpha_2\alpha_3 - \alpha_2\gamma_3) + (\gamma_1\gamma_2 - \alpha_1\gamma_2) \\ +(\gamma_1\gamma_3 - \alpha_3\gamma_3) + (\gamma_2\gamma_3 - \alpha_3\gamma_2) - \alpha_1\gamma_1 - \alpha_2\gamma_1 - \alpha_3\gamma_1. \quad (3)$$

However, this is a sum of negative terms, which leads to a contradiction.

Suppose that the order of moduli $[1, 0, 1, 1]$ is realizable with the sign pattern $\Sigma_{3,1,2,1}$ or $\Sigma_{3,2,1,1}$, which means that the following inequalities are satisfied:

$$\gamma_1 < \alpha_1 < \alpha_2 < \gamma_2 < \alpha_3 < \gamma_3.$$

In this case, we obtain the following expression for q_4:

$$q_4 = (\alpha_1\alpha_2 - \alpha_2\gamma_2) + (\alpha_1\alpha_3 - \alpha_2\gamma_3) + (\alpha_2\alpha_3 - \alpha_3\gamma_2) + (\gamma_1\gamma_2 - \alpha_1\gamma_2) \\ +(\gamma_1\gamma_3 - \alpha_1\gamma_3) + (\gamma_2\gamma_3 - \alpha_3\gamma_3) - \alpha_1\gamma_1 - \alpha_2\gamma_1 - \alpha_3\gamma_1. \quad (4)$$

It is clear that q_4 is a sum of negative quantities, which is a contradiction.

The orders of moduli $[2, 0, 0, 1]$ and $[0, 2, 0, 1]$ are not realizable with the sign pattern $\Sigma_{3,1,2,1}$, because neither of their neighbours is, see Proposition 1. For $[2, 0, 0, 1]$, these neighbours are $[1, 1, 0, 1]$ and $[2, 0, 1, 0]$. For $[0, 2, 0, 1]$, they are $[1, 1, 0, 1]$, $[0, 1, 1, 1]$ and $[0, 2, 1, 0]$.

□

Part (2). The couple $(\Sigma_{2,1,2,2}, NPNPPN)$ is realizable, because $NPNPPN$ is the canonical order. The other 5 couples mentioned in part (2) of the theorem are also realizable:

$PNNPPN$ $\quad (x - 4.52)(x + 5.02)(x + 5.32)(x - 7.002)(x - 8.003)(x + 9.32)$
$= x^6 + 0.135x^5 - 136.926694x^4 + 27.6529548x^3$
$+5404.574382x^2 - 344.273285x - 63044.12478,$

$NPPPNN$ $\quad (x + 2.5)(x - 4.95)(x - 6.47)(x - 8.19)(x + 8.57)(x + 9.05)$
$= x^6 + 0.51x^5 - 147.3884x^4 + 73.049286x^3$
$+6188.991502x^2 - 7552.653247x - 50858.41147,$

$NPPNPN$ $\quad (x + 1.49)(x - 1.87)(x - 5.77)(x + 5.96)(x - 7.58)(x + 8.07)$
$= x^6 + 0.3x^5 - 98.5114x^4 + 5.90954x^3$
$+2380.426651x^2 - 720.0363792x - 5861.282963,$

$NPPNNP$ $\quad (x + 1.34)(x - 3.43)(x - 5.34)(x + 7.86)(x + 9)(x - 9.4)$
$= x^6 + 0.03x^5 - 136.6074x^4 + 60.496052x^3$
$+4547.732428x^2 - 6518.600281x - 16320.4859,$

$NNPPPN$ $\quad (x + 2.5)(x + 3.03)(x - 4.28)(x - 4.4)(x - 5.6)(x + 9.4),$
$x^6 + 0.65x^5 - 86.2034x^4 + 122.15104x^3$
$+1425.210824x^2 - 1478.768374x - 7509.222336.$

We prove that the remaining 14 cases are not realizable. Assume for the first 8 of them that they are realizable by a polynomial $Q := x^6 + \sum_{j=0}^{5} q_j x^j$. There are 4 cases in which one obtains that

$$q_5 := (\gamma_1 - \alpha_1) + (\gamma_2 - \alpha_2) + (\gamma_3 - \alpha_3) < 0,$$

which contradicts the sign pattern. These are

$$\begin{aligned}
&\gamma_1 < \gamma_2 < \gamma_3 < \alpha_1 < \alpha_2 < \alpha_3 : [3,0,0,0],\\
&\gamma_1 < \gamma_2 < \alpha_1 < \gamma_3 < \alpha_2 < \alpha_3 : [2,1,0,0],\\
&\gamma_1 < \alpha_1 < \gamma_2 < \gamma_3 < \alpha_2 < \alpha_3 : [1,2,0,0],\\
&\gamma_1 < \gamma_2 < \alpha_1 < \alpha_2 < \gamma_3 < \alpha_3 : [2,0,1,0].
\end{aligned}$$

There are 4 cases in which

$$q_1 := \alpha_1\alpha_2\alpha_3\gamma_1\gamma_2\gamma_3(1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 - 1/\gamma_1 - 1/\gamma_2 - 1/\gamma_3) > 0,$$

which also contradicts the sign pattern. The cases are:

$$\begin{aligned}
&\alpha_1 < \alpha_2 < \alpha_3 < \gamma_1 < \gamma_2 < \gamma_3 : [0,0,0,3],\\
&\alpha_1 < \gamma_1 < \alpha_2 < \alpha_3 < \gamma_2 < \gamma_3 : [0,1,0,2],\\
&\alpha_1 < \alpha_2 < \gamma_1 < \alpha_3 < \gamma_2 < \gamma_3 : [0,0,1,2],\\
&\alpha_1 < \alpha_2 < \gamma_1 < \gamma_2 < \alpha_3 < \gamma_3 : [0,0,2,1].
\end{aligned}$$

The orders of moduli $[0,1,1,1]$ and $[1,1,1,0]$, i.e. $PNPNPN$ and $NPNPNP$ are rigid, see Definition 4 and the lines after it, hence non-realizable with the sign pattern $\Sigma_{2,1,2,2}$.

In the four remaining cases of orders of moduli there exists at least one realizable neighbour and one can apply Proposition 1 and Remark 3. We list these cases to the left and their neighbours to the right; non-realizable neighbours are marked by the subscript $_0$:

1) $[0,0,3,0]$	$[0,1,2,0]$, $[0,0,2,1]_0$
2) $[0,1,2,0]$	$[1,0,2,0]$, $[0,2,1,0]$, $[0,1,1,1]_0$, $[0,0,3,0]$
3) $[0,2,1,0]$	$[1,1,1,0]_0$, $[0,1,2,0]$, $[0,2,0,1]$, $[0,3,0,0]$
4) $[0,3,0,0]$	$[1,2,0,0]_0$, $[0,2,1,0]$.

The four cases and their neighbours realizable with $\Sigma_{2,1,2,2}$ are all with $u_4 = 0$ or $u_4 = 1$. Hence, if one applies Proposition 1 and Remark 3, one concludes that realizability of one of the cases 1–4 implies the existence of a polynomial $Q = (x^2-1)R$, where all roots of R are of modulus < 1. Consider the orders of moduli $[0,0,3,0]$ and $[0,1,2,0]$. For the roots $-g < -f < 0 < a < b$ of R, one should have

$$a < b < f < g < 1 \quad \text{or} \quad a < f < b < g < 1, \quad \text{respectively.}$$

However, this would imply $q_1 = abcd((1/a - 1/f) + (1/b - 1/g)) > 0$, which is a contradiction. So the orders $[0,0,3,0]$ and $[0,1,2,0]$ are not realizable.

Set $\sigma := \Sigma_{2,1,2,2}$. Consider the sets $S := S_1 \cup S_2$, $S_1 := \Pi_6(\sigma, [0,2,1,0])$, $S_2 := \Pi_6(\sigma, [0,3,0,0])$, see Notation 2. If at least one of the set S_1 and S_2 is non-empty, then there exists a smooth path $\gamma \subset \Pi_6(\sigma)$ connecting a point of S with a point of $\Pi_6(\sigma, [0,2,0,1])$. One can choose the path avoiding the non-generic points of the set E_6 and intersecting this set transversally. Hence, there exists a point of γ belonging to the common boundary of S and $\Pi_6(\sigma, [0,2,0,1])$. After a linear change of the variable x, this point corresponds to a polynomial $Q = (x^2 - 1)R$, where for the roots of R, one has $a < f < g < b < 1$ and

$$q_5 := -a - b + f + g > 0 \quad \text{and } q_1 := abfg(1/a + 1/b - 1/f - 1/g) < 0, \text{ i.e.}$$
$$a + b < f + g \quad \text{and} \quad (a + b)/ab < (f + g)/fg.$$

This, however, is impossible. Indeed, for fixed f, g and $a+b$, the quantity $(a+b)/ab$ is the minimal possible, in which a and b are closest to one another. But for $b = g$, one should have $a < f$ and $1/a < 1/f$, which is contradictory; for $a = f$, this gives $b < g$, which is also a contradiction. Hence, $S = \emptyset$, i.e. the orders of moduli $[0, 2, 1, 0]$ and $[0, 3, 0, 0]$ are not realizable with $\Sigma_{2,1,2,2}$. □

Part (3). We prove that certain couples are realizable by concatenating couples corresponding to $d = 5$ with the ones corresponding to $d = 1$, see Subsection 2.5. It is shown in [15, Section 3] that for $d = 5$, the sign pattern $\Sigma_{2,2,2}$ is realizable with all compatible orders Ω (Ω is any string of 2 letters P and 3 letters N). Applying the involution i_m (see Definition 5) one sees that the sign pattern $i_m(\Sigma_{2,2,2}) = \Sigma_{1,2,2,1}$ is realizable with any order Ω' which is a string of 3 letters P and 2 letters N.

Denote by T (resp. U) a polynomial realizing the couple $(\Sigma_{2,2,2}, \Omega)$ (resp. $(\Sigma_{1,2,2,1}, \Omega')$). Hence, for $\varepsilon > 0$ small enough, the product $T(x)(x - \varepsilon)$ (resp. $U(x)(x + \varepsilon)$) realizes the order $P\Omega$ with the sign pattern $\Sigma_{2,2,2,1}$ (resp. the order $N\Omega'$ with the sign pattern $\Sigma_{1,2,2,2}$), see Subsection 2.5. Applying the involution i_r (see Definition 5) one understands that any order of the form $\Omega' N$ is realizable with the sign pattern $\Sigma_{2,2,2,1}$.

There are exactly 6 orders which are not of the form $P\Omega$ or $\Omega' N$. These are the 5 orders mentioned in part (3) of the theorem and the order $NPPNNP$. The latter is realizable with the sign pattern $\Sigma_{2,2,2,1}$:

$$(x + 4)(x - 5)(x - 6)(x + 8.74)(x + 9.41)(x - 9.59) = x^6 + 1.56x^5$$
$$- 165.7351x^4 - 145.848506x^3 + 7833.610842x^2 + 24.186884x - 94645.70472.$$

The order of moduli $NPNPNP$ is rigid (see Definition 4), so realizable only with the sign pattern $\Sigma_{1,2,2,2}$.

We prove non-realizability of the remaining 5 orders. Consider a degree 6 hyperbolic polynomial $Q := x^6 + \sum_{j=1}^{5} q_j x^j$ with distinct moduli of roots α_i and γ_j and defining the sign pattern $\Sigma_{2,2,2,1}$. Thus

$$Q = \prod_{i=1}^{3} (x - \alpha_i)(x + \gamma_i) \quad \text{and} \quad q_1 = \alpha_1\alpha_2\alpha_3\gamma_1\gamma_2\gamma_3 S_1,$$

where

$$S_1 := (1/\alpha_3 - 1/\gamma_3) + (1/\alpha_2 - 1/\gamma_2) + (1/\alpha_1 - 1/\gamma_1).$$

If the orders of moduli $[1, 2, 0, 0]$, $[2, 0, 1, 0]$, $[2, 1, 0, 0]$, and $[3, 0, 0, 0]$ are realizable, then the moduli of the roots satisfy the following inequalities:

$$\gamma_1 < \alpha_1 < \gamma_2 < \gamma_3 < \alpha_2 < \alpha_3, \qquad \gamma_1 < \gamma_2 < \alpha_1 < \alpha_2 < \gamma_3 < \alpha_3,$$
$$\gamma_1 < \gamma_2 < \alpha_1 < \gamma_3 < \alpha_2 < \alpha_3 \ \text{ and } \gamma_1 < \gamma_2 < \gamma_3 < \alpha_1 < \alpha_2 < \alpha_3, \quad \text{respectively.}$$

Thus $S_1 < 0$. Therefore, we have $q_1 < 0$, which leads to a contradiction.

□

Part (4). The order of moduli $PPNPNN$ is the canonical order for the sign pattern $\Sigma_{3,2,1,1}$, so the corresponding couple is realizable, see [11, Proposition 1]. The remaining 3 couples are realizable by perturbations of the following polynomials (see the beginning of the proof of part (1) of the theorem with the explanation about perturbations):

$$\begin{array}{ll} PPPNNN & (x-0.039)(x-0.4)(x-1)(x+1)(x+1.001)(x+4) \\ & = x^6+4.562x^5+0.824161x^4-6.2417404x^3 \\ & -1.7616986x^2+1.6797404x-0.0624624, \\ PPNNPN & (x-0.09)(x-0.19)(x+0.8)(x+1)(x-1)(x+13) \\ & = x^6+13.52x^5+5.5531x^4-16.19602x^3 \\ & -6.37526x^2+2.67602x-0.17784, \\ PNPPNN & (x-0.02)(x+1)(x-1)(x-3.1)(x+5)(x+20) \\ & = x^6+21.88x^5+21.062x^4-332.33x^3 \\ & -15.862x^2+310.45x-6.2. \end{array}$$

It has already been mentioned (see Proposition 2) that no orders of moduli with $u_4 = 0$ are realizable with the sign pattern $\Sigma_{3,2,1,1}$. Also, in the proof of part (1) we saw that the sign pattern $\Sigma_{3,2,1,1}$ is not realizable with any of the orders of moduli $[1,0,1,1]$ or $[1,1,0,1]$.

Non-realizability of the couple $(\Sigma_{3,2,1,1}, PNPNPN)$ was proved in the proof of part (1). The orders $[2,0,0,1]$ and $[0,2,0,1]$ are also not realizable with $\Sigma_{3,2,1,1}$, because their respective neighbours $[1,1,0,1]$, $[2,0,1,0]$ and $[1,1,0,1]$, $[0,1,1,1]$, $[0,2,1,0]$ are not, see Proposition 1.

It remains to prove non-realizability of the couple $(\Sigma_{3,2,1,1}, NPPPNN)$. It corresponds to the quadruple $[1,0,0,2]$ and has exactly two neighbours: $[0,1,0,2]$ and $[1,0,1,1]$, only the first of which is realizable. Proposition 1 and Remark 3 imply that if the couple is realizable, then there exists a polynomial $Q := (x^2-1)R$ such that for the roots a, b, $-f$ and $-g$ of R one has

$$0 < 1 < a < b < f < g.$$

Using the same notation as in the proof of part (1) we observe that

$$q_2 = (ab-1)fg + a(f-b) + ag + bf + bg > 0,$$

whereas one should have $q_2 < 0$. This contradiction implies non-realizability of the couple. □

References

[1] F. Cajori, *A history of the arithmetical methods of approximation to the roots of numerical equations of one unknown quantity*, Colorado College Publ. Sci. Ser. 12-7, Colorado College, Colorado Springs, 1910.

[2] D. R. Curtiss, *Recent extensions of Descartes' rule of signs*, Ann. of Math. **19**(1918), 251–278.

[3] J.-P. Gua de Malves, *Démonstrations de la Règle de Descartes, Pour connaître le nombre des racines positives & négatives dans les équations qui n'ont point de racines imaginaires*, Mémoires de Mathématique et de Physique tirés des registres de l'Académie Royale des Sciences, 1741.

[4] *The Geometry of René Descartes with a facsimile of the first edition*, translated by D. E. Smith and M. L. Latham, New York, Dover Publications, 1954.

[5] J. Forsgård, V. P. Kostov, B. Shapiro, *Could René Descartes have known this?* Exp. Math. 24 bf 4(2015), 438–448.

[6] J. Fourier, *Sur l'usage du théorème de Descartes dans la recherche des limites des racines*, Bulletin des sciences par la Société philomatique de Paris, 1820, 156–165, 181–187; œuvres 2, 291–309, Gauthier-Villars, 1890.

[7] Y. Gati, V. P. Kostov and M. C. Tarchi, *Sign patterns and rigid moduli orders*, Grad. J. Math. **6**(2021), 60–72.

[8] C. F. Gauss, *Beweis eines algebraischen Lehrsatzes*, J. Reine Angew. Math. **3**(1828), 1–4; Werke **3**, 67–70, Göttingen, 1866.

[9] J. L. W. Jensen, *Recherches sur la théorie des équations*, Acta Math. **36**(1913), 181–195.

[10] V. P. Kostov, *Descartes' rule of signs and moduli of roots*, Publ. Math. Debrecen **96**(2020), 161–184.

[11] V. P. Kostov, *Hyperbolic polynomials and canonical sign patterns*, Serdica Math. J. **46**(2020), 135–150.

[12] V. P. Kostov, *Hyperbolic polynomials and rigid moduli orders*, Publ. Math. Debrecen **100**(2022), 119–128.

[13] V. P. Kostov, *Univariate polynomials and the contractibility of certain sets*, God. Sofiî. Univ. "Sv. Kliment Ohridski." Fac. Mat. Inform. **107**(2020), 11–35.

[14] V. P. Kostov, *Which Sign Patterns are Canonical?*, Results Math. **77**(2022), No 6, paper 235, `https://doi.org/10.1007/s00025-022-01769-3`.

[15] V. P. Kostov, *Beyond Descartes' rule of signs*, Constr. Math. Anal. **6**(2023), 128–141.

[16] V. P. Kostov, *Moduli of roots of hyperbolic polynomials and Descartes' rule of signs*, in: *Constructive Theory of Functions*, (B. Draganov, K. Ivanov, G. Nikolov and R. Uluchev, Eds.), Sozopol 2019, Prof. Marin Drinov Academic Publishing House, Sofia, 2020, 131–146.

[17] E. Laguerre, *Sur la théorie des équations numériques*, Journal de Mathématiques pures et appliquées, **9**(1883), 99–146; œuvres 1, Paris, 1898, Chelsea, New-York, 1972, pp. 3–47.

[18] B. E. Meserve, *Fundamental Concepts of Algebra*, New York, Dover Publications, 1982.