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Lower bounds for the number of local nearrings on groups of order p^{3*}

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Abstract. Lower bounds for the number of local nearrings on groups of order p^3 are obtained. On each non-metacyclic non-abelian or metacyclic abelian groups of order p^3 there exist at least $p+1$ non-isomorphic local nearrings.

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1. Introduction

A study of local nearrings was first initiated in [11] and it was found that the additive group of a finite zero-symmetric local nearring is a p-group. In [12], it is shown that, up to isomorphism, there exist $p-1$ local zero-symmetric nearrings with elementary abelian additive groups of order p^2 in which the subgroups of non-invertible elements have order p, that is, those nearrings which are not nearfields. Together with the fundamental paper [22] and [5], a complete description of all zero-symmetric local nearrings of order p^2 is obtained. For instance, every nearring with identity on a cyclic group is a commutative ring.

Note that there is no nearring with identity whose additive group is isomorphic to the quaternion group Q_8 [4]. The dihedral group D_4 of order 8 cannot be the additive group of local nearrings [14]. The existence of local nearrings on finite abelian p-groups is proved in [13], i.e. every non-cyclic abelian p-group of order $p^n > 4$ is the additive group of a zero-symmetric local nearring which is not a ring. Also, it is established in [18] that an arbitrary non-metacyclic Miller–Moreno p-group of order $p^n > 8$ is the additive group of some local nearring, and the multiplicative group of such nearring has order $p^{n-1}(p-1)$. All nearrings with identity up to the order of 31 are contained in the package SONATA [1] of the computer algebra system GAP [21].

In [17], it is proved that, up to an isomorphism, there exist at least p local nearrings on elementary abelian additive groups of order p^3 which are not nearfields. Lower bounds for the number of local nearrings on groups of order p^3 are obtained.

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It is established that on each non-metacyclic non-abelian or metacyclic abelian groups of order p^3 there exist at least $p+1$ non-isomorphic local nearrings.

2. Preliminaries

We will give the basic definitions.

Definition 1. *A non-empty set R with two binary operations "$+$" and "$\cdot$" is a nearring if:*

1) $(R,+)$ *is a group with neutral element* 0*;*

2) $(R,\cdot)$ *is a semigroup;*

3) $x\cdot(y+z)=x\cdot y+x\cdot z$ *for all* x*,* y*,* $z\in R$*.*

Such a nearring is called a left nearring. If axiom 3) is replaced by an axiom $(x+y)\cdot z=x\cdot z+y\cdot z$ *for all* x*,* y*,* $z\in R$*, then we get a right nearring.*

The group $(R,+)$ of a nearring R is denoted by R^+ and called the *additive group* of R. It is easy to see that for each subgroup M of R^+ and for each element $x\in R$ the set $xM=\{x\cdot y|y\in M\}$ is a subgroup of R^+ and in particular $x\cdot 0=0$. If, in addition, $0\cdot x=0$ for all $x\in R$, then the nearring R is called *zero-symmetric*. Furthermore, R is a *nearring with identity* i if the semigroup $(R,\cdot)$ is a monoid with identity element i. In the latter case, the group of all invertible elements of the monoid $(R,\cdot)$ is denoted by R^* and called the *multiplicative group* of R. A subgroup M of R^+ is called R^**-invariant* if $rM\le M$ for each $r\in R^*$, and (R,R)*-subgroup* if $xMy\subseteq M$ for arbitrary x, $y\in R$.

The following assertion is well-known (see, for instance, [5], Theorem 3).

Lemma 1. *The exponent of the additive group of a finite nearring R with identity i is equal to the additive order of i which coincides with the additive order of every invertible element of R.*

Definition 2. *A nearring R with identity is called* ***local*** *if the set L of all non-invertible elements of R forms a subgroup of the additive group R^+.*

Throughout this paper, L will denote the subgroup of non-invertible elements of R.

The following lemma characterizes the main properties of finite local nearrings (see [2], Lemma 3.2).

Lemma 2. *Let R be a local nearring with identity i. Then the following statements hold:*

1) L *is an* (R,R)*-subgroup of* R^+*;*

2) *each proper* R^**-invariant subgroup of* R^+ *is contained in* L*;*

3) *the set* $i+L$ *forms a subgroup of the multiplicative group* R^**.*

Finite local nearrings with a cyclic subgroup of non-invertible elements are described in [16, Theorem 1].

Theorem 1. *Let R be a local nearring of order p^n with $n > 1$, whose subgroup L is cyclic and non-trivial. Then the additive group R^+ is either cyclic or an elementary abelian group of order p^2. In the first case, R is a commutative local ring, which is isomorphic to the residual ring $\mathbb{Z}/p^n\mathbb{Z}$ with $n \geq 2$; in the other case, there exist p non-isomorphic such nearrings R with $|L| = p$, from which $p-1$ are zero-symmetric nearrings and their multiplicative groups R^* are isomorphic to a semidirect product of two cyclic subgroups of orders p and $p-1$.*

As a direct consequence of Theorem 1, we have the following result.

Corollary 1. *Let R be a local nearring of order p^3 which is not isomorphic to $\mathbb{Z}/p^3\mathbb{Z}$ or is not a nearfield. Then the subgroup of non-invertible elements L is an elementary abelian group of order p^2.*

The following statement contains a classification of groups of order p^3 (see [6]).

Proposition 1. *Let G be a group of order p^3. The defining relations of such non-isomorphic groups are given:*

Abelian groups:

1) $a^{p^3} = 1$.

2) $a^{p^2} = 1$, $b^p = 1$, $ab = ba$.

3) $a^p = b^p = c^p = 1$, $ab = ba$, $ac = ca$, $cb = bc$.

Non-abelian groups of order $2^3 = 8$:

4) *a dihedral group,* $a^4 = 1$, $b^2 = 1$, $a^{-1}b = ba$.

5) *a quaternion group,* $a^4 = 1$, $b^2 = a^2$, $a^{-1}b = ba$.

Non-abelian groups of order p^3, p is odd:

6) $a^{p^2} = 1$, $b^p = 1$, $b^{-1}ab = a^{1+p}$.

7) $a^p = 1$, $b^p = 1$, $c^p = 1$, $ab = bac$, $ac = ca$, $bc = cb$.

Next, we denote by G_1 a group with relations 7), by G_2 a group with relations 6), and by G_3 a group with relations 2) of Proposition 1.

We define group G_1 to be the additively written group generated by a, b, c subject to the relations $ap = bp = cp = 0$, $a + b = b + a + c$, $a + c = c + a$, $b + c = c + b$.

The following two lemmas are given in [18].

Lemma 3. *Let $k,\ l \in \mathbb{N}$. Then in G_1, the equalities $-ak - bl + ak + bl = c(kl)$ and $bl + ak = -c(kl) + ak + bl$ hold.*

Proof. Since $-b + a + b = a + c$, we have $-bl + a + bl = a + cl$. Then

$$-bl + ak + bl = (a + cl)k = ak + ckl.$$

Therefore, $-ak - bl + ak + bl = ckl$. □

Lemma 4. *Let $k,\ l, r \in \mathbb{N}$. Then in G_1, we have $(ak+bl)r = akr + blr - ckl\binom{r}{2}$.*

Proof. The proof will be carried out by induction on r. For $r = 1$, the equality is valid. Let for r the equality hold, i.e.,

$$(ak+bl)r = akr + blr - ckl\binom{r}{2}.$$

Let us prove the equality for $r+1$:

$$\begin{aligned}(ak+bl)(r+1) &= akr + blr + ak + bl - ckl\binom{r}{2}\\ &= ak(r+1) + bl(r+1) - cklr - ckl\binom{r}{2}\\ &= ak(r+1) + bl(r+1) - ckl(r + \binom{r}{2})\\ &= ak(r+1) + bl(r+1) - ckl\binom{r+1}{2}.\end{aligned}$$

Therefore, the equality is valid for any r. $\square$

Let additively written groups of type H have a finite representation in the form

$$\langle a, b | ap^k, bp, -b + a + b - a(1 + p^{k-1})\rangle,$$

where $k \geq 2$ and p is prime (see [8]).

The number of non-isomorphic nearrings with identity on groups of type H is given in [8]. It is obvious that for $k = 2$ and $p > 2$ a group of type H will be isomorphic to the group G_2.

As noted above, there exist local nearrings on all abelian groups. Also, according to [18] and [8], there exist local nearrings on G_1 and G_2, respectively. So, we have the following result.

Proposition 2. *On each group of order p^3 with $p > 2$ there exists a local nearring.*

Denote by $n(G)$ the number of all non-isomorphic local nearrings on the group G.

3. Nearrings with identity whose additive groups are isomorphic to G_1

Let R be a nearring with identity whose additive group of R^+ is isomorphic to G_1. Then $R^+ = \langle a\rangle + \langle b\rangle + \langle c\rangle$ for some elements a, b and c of R satisfying the relations $ap = 0$, $bp = 0$, $cp = 0$, $a + b = b + a + c$, $a + c = c + a$ and $b + c = c + b$. In particular, each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2 + cx_3$ with coefficients $0 \leq x_1 < p$, $0 \leq x_2 < p$ and $0 \leq x_3 < p$.

We will show that there does not exist a nearring in which the identity is in the center of the additive group G_1. Note that the subgroup $\langle c\rangle$ is the center of G_1.

Remark 1. *Let c be an identity of R, i. e. $xc = cx = x$ for each $x \in R$. Furthermore, for each $x \in R$ there exist coefficients $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\lambda(x)$, $\mu(x)$ and $\nu(x)$ such that $xb = a\alpha(x) + b\beta(x) + c\gamma(x)$ and $xa = a\lambda(x) + b\mu(x) + c\nu(x)$. It is clear that they are uniquely defined modulo p, so that some mappings $\alpha : R \to \mathbb{Z}_p$, $\beta : R \to \mathbb{Z}_p$, $\gamma : R \to \mathbb{Z}_p$, $\lambda : R \to \mathbb{Z}_p$, $\mu : R \to \mathbb{Z}_p$ and $\nu : R \to \mathbb{Z}_p$ are determined.*

Further, using Lemma 3, we derive

$$\begin{aligned} xc =& - xa - xb + xa + xb = -c\nu(x) - b\mu(x) - a\lambda(x) - c\gamma(x) - b\beta(x) - a\alpha(x) \\ &+ a\lambda(x) + b\mu(x) + c\nu(x) + a\alpha(x) + b\beta(x) + c\gamma(x) \\ =& - b\mu(x) - a\lambda(x) - b\beta(x) - a\alpha(x) + a\lambda(x) + b\mu(x) + a\alpha(x) + b\beta(x) \\ =& - b\mu(x) + c\lambda(x)\beta(x) - b\beta(x) - a\lambda(x) - a(\alpha(x) - \lambda(x)) \\ &+ b\mu(x) + a\alpha(x) + b\beta(x) = c\lambda(x)\beta(x) - b(\mu(x) + \beta(x)) - a\alpha(x) \\ &+ b\mu(x) + a\alpha(x) + b\beta(x) = c\lambda(x)\beta(x) - b(\mu(x) + \beta(x)) \\ &- a\alpha(x) - c\mu(x)\alpha(x) + a\alpha(x) + b\mu(x) + b\beta(x) \\ =& c(\lambda(x)\beta(x) - \mu(x)\alpha(x)) - b(\mu(x) + \beta(x)) + b\mu(x) + b\beta(x) \\ =& c(\lambda(x)\beta(x) - \mu(x)\alpha(x)) \neq x. \end{aligned}$$

Therefore, there does not exist a nearring in which the identity is in the center of the additive group G_1.

Since the order of the element a is equal to the exponent of group G, then by Lemma 1 we can assume that a is an identity of R, i.e. $ax = xa = x$ for each $x \in R$. Furthermore, for each $x \in R$ there exist coefficients $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ such that $xb = a\alpha(x) + b\beta(x) + c\gamma(x)$. It is clear that they are uniquely defined modulo p, so that some mappings $\alpha\colon R \to \mathbb{Z}_p$, $\beta : R \to \mathbb{Z}_p$ and $\gamma : R \to \mathbb{Z}_p$ are determined.

Nearrings with identity and local nearrings on non-metacyclic Miller–Moreno groups were studied in [18] and [15]. Lemmas 5, 6, 9 are based on the results of these papers.

Lemma 5. *Let R be a nearring whose additive group is G_1 with identity a. If $x = ax_1 + bx_2 + cx_3$, $y = ay_1 + by_2 + cy_3 \in R$, $xb = a\alpha(x) + b\beta(x) + c\gamma(x)$, then*

$$\begin{aligned} xy =& a(x_1y_1 + \alpha(x)y_2) + b(x_2y_1 + \beta(x)y_2) + c(-x_1x_2\binom{y_1}{2} \\ &- \alpha(x)\beta(x)\binom{y_2}{2} - x_2\alpha(x)y_1y_2 + x_3y_1 + \gamma(x)y_2 + x_1\beta(x)y_3 - x_2\alpha(x)y_3). \end{aligned}$$

Moreover, for the mappings $\alpha\colon R \to \mathbb{Z}_p$, $\beta\colon R \to \mathbb{Z}_p$ and $\gamma\colon R \to \mathbb{Z}_p$ the following statements hold:

(0) $\alpha(0) \equiv 0 \pmod p$, $\beta(0) \equiv 0 \pmod p$ *and* $\gamma(0) \equiv 0 \pmod p$ *if and only if the nearring R is zero-symmetric;*

(1) $\alpha(xy) \equiv x_1\alpha(y) + \alpha(x)\beta(y) \pmod p$,

(2) $\beta(xy) \equiv x_2\alpha(y) + \beta(x)\beta(y) \pmod p$,

$$\text{(3)}\ \ \gamma(xy) \equiv -x_1x_2\tbinom{\alpha(y)}{2} - \alpha(x)\beta(x)\tbinom{\beta(y)}{2} - x_2\alpha(x)\alpha(y)\beta(y)$$
$$+x_3\alpha(y) + \gamma(x)\beta(y) + x_1\beta(x)\gamma(y) - x_2\alpha(x)\gamma(y) \pmod{p}.$$

Proof. Since $0 \cdot a = a \cdot 0 = 0$, it follows that R is a zero-symmetric nearring if and only if
$$0 = 0 \cdot b = a\alpha(0) + b\beta(0) + c\gamma(0),$$
or equivalently $\alpha(0) \equiv 0 \pmod{p}$, $\beta(0) \equiv 0 \pmod{p}$ and $\gamma(0) \equiv 0 \pmod{p}$. Moreover, since $c = -a - b + a + b$ and due to the left distributive law we have $0 \cdot c = -0 \cdot a - 0 \cdot b + 0 \cdot a + 0 \cdot b = 0$, whence

$$0 \cdot x = 0 \cdot (ax_1 + bx_2 + cx_3) = (0 \cdot a)x_1 + (0 \cdot b)x_2 + (0 \cdot c)x_3 = 0,$$

so that statement (0) holds.

Further, using Lemma 3, we derive

$$\begin{aligned}
xc =& - xa - xb + xa + xb = -cx_3 - bx_2 - ax_1 - c\gamma(x) - b\beta(x) - a\alpha(x)\\
&+ ax_1 + bx_2 + cx_3 + a\alpha(x) + b\beta(x) + c\gamma(x)\\
=& - bx_2 - ax_1 - b\beta(x) - a\alpha(x) + ax_1 + bx_2 + a\alpha(x) + b\beta(x)\\
=& - bx_2 + cx_1\beta(x) - b\beta(x) - ax_1 - a(\alpha(x) - x_1) + bx_2 + a\alpha(x) + b\beta(x)\\
=& cx_1\beta(x) - b(x_2 + \beta(x)) - a\alpha(x) + bx_2 + a\alpha(x) + b\beta(x)\\
=& cx_1\beta(x) - b(x_2 + \beta(x)) - a\alpha(x) - cx_2\alpha(x) + a\alpha(x)\\
&+ bx_2 + b\beta(x) = c(x_1\beta(x) - x_2\alpha(x)) - b(x_2 + \beta(x))\\
&+ bx_2 + b\beta(x) = c(x_1\beta(x) - x_2\alpha(x)).
\end{aligned}$$

Further, using the left distributive law, we obtain

$$\begin{aligned}
xy =&(ax_1 + bx_2 + cx_3)y_1 + (a\alpha(x) + b\beta(x) + c\gamma(x))y_2\\
&+ (cx_1\beta(x) - x_2\alpha(x))y_3.
\end{aligned}$$

By Lemma 4, we get

$$(ax_1 + bx_2)y_1 = ax_1y_1 + bx_2y_1 - cx_1x_2\binom{y_1}{2},$$
$$(a\alpha(x) + b\beta(x))y_2 = a\alpha(x)y_2 + b\beta(x)y_2 - c\alpha(x)\beta(x)\binom{y_2}{2}$$

and
$$bx_2y_1 + a\alpha(x)y_2 = a\alpha(x)y_2 + bx_2y_1 - cx_2\alpha(x)y_1y_2.$$

Hence and using the left distributive law, we have

$$\begin{aligned}
xy =& a(x_1y_1 + \alpha(x)y_2) + b(x_2y_1 + \beta(x)y_2) + c(-x_1x_2\binom{y_1}{2}\\
&- \alpha(x)\beta(x)\binom{y_2}{2} - x_2\alpha(x)y_1y_2 + x_3y_1 + \gamma(x)y_2 + x_1\beta(x)y_3 - x_2\alpha(x)y_3).
\end{aligned}$$

The associativity of multiplication in R implies that for all $x,\, y \in R$

1) $$(xy)b = x(yb).$$

According to $xb = a\alpha(x) + b\beta(x) + c\gamma(x)$, we obtain

2) $$(xy)b = a\alpha(xy) + b\beta(xy) + c\gamma(xy)$$

and $yb = a\alpha(y) + b\beta(y) + c\gamma(y)$. Substituting the last equation into the right part of equality 1), we also have

$$\begin{aligned} x(yb) = {} & a(x_1\alpha(y) + \alpha(x)\beta(y)) + b(x_2\alpha(y) + \beta(x)\beta(y)) \\ & + c(-x_1x_2\binom{\alpha(y)}{2} - \alpha(x)\beta(x)\binom{\beta(y)}{2} - x_2\alpha(x)\alpha(y)\beta(y) \\ & + x_3\alpha(y) + \gamma(x)\beta(y) + x_1\beta(x)\gamma(y) - x_2\alpha(x)\gamma(y)). \end{aligned}$$

Since equality 1) implies the congruence of the corresponding coefficients in formulas 2) and 3), we obtain statements (1)–(3). $\square$

4. Local nearrings whose additive groups are isomorphic to G_1

Let R be a local nearring whose additive group of R^+ is isomorphic to G_1. Then $R^+ = \langle a\rangle + \langle b\rangle + \langle c\rangle$ for some elements a, b and c of R satisfying the relations $ap = 0$, $bp = 0$, $cp = 0$, $a + b = b + a + c$, $a + c = c + a$ and $b + c = c + b$. In particular, each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2 + cx_3$ with coefficients $0 \le x_1 < p$, $0 \le x_2 < p$ and $0 \le x_3 < p$.

Since order of the element a is equal to the exponent of group G, then by Lemma 1 we can assume that a is an identity of R, i.e., $ax = xa = x$ for each $x \in R$. Furthermore, for each $x \in R$ there exist coefficients $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ such that $xb = a\alpha(x) + b\beta(x) + c\gamma(x)$. It is clear that they are uniquely defined modulo p, so that some mappings $\alpha : R \to \mathbb{Z}_p$, $\beta : R \to \mathbb{Z}_p$ and $\gamma : R \to \mathbb{Z}_p$ are determined.

By Corollary 1, L is the normal subgroup of order p^2 in R. Since L contains the derived subgroup of R^+, it follows that $L = \langle b\rangle + \langle c\rangle$ and subgroup $\langle c\rangle$ is the center of R^+. Since $R^* = R \setminus L$, it follows that $R^* = \{ax_1 + bx_2 + cx_3 \mid x_1 \not\equiv 0 \pmod{p}\}$ and $x = ax_1 + bx_2 + cx_3$ is invertible if and only if $x_1 \not\equiv 0 \pmod{p}$.

Lemma 6. *Let R be a local nearring whose additive group is G_1 with identity a. If $x = ax_1 + bx_2 + cx_3$, $y = ay_1 + by_2 + cy_3 \in R$, $xb = a\alpha(x) + b\beta(x) + c\gamma(x)$, then*

$$x \cdot y = ax_1y_1 + b(x_2y_1 + \beta(x)y_2) + c(-x_1x_2\binom{y_1}{2} + x_3y_1 + \gamma(x)y_2 + x_1\beta(x)y_3). \quad (*)$$

Moreover, for the mappings $\beta\colon R \to \mathbb{Z}_p$ and $\gamma\colon R \to \mathbb{Z}_p$ the following statements hold:

(0) $\alpha(0) \equiv 0 \pmod{p}$, $\beta(0) \equiv 0 \pmod{p}$ *and* $\gamma(0) \equiv 0 \pmod{p}$ *if and only if the nearring R is zero-symmetric;*

(1) $\alpha(x) \equiv 0 \pmod{p}$*;*

(2) *if* $\beta(x) \equiv 0 \pmod p$*, then* $x_1 \equiv 0 \pmod p$*;*

(3) $\beta(xy) \equiv \beta(x)\beta(y) \pmod p$*;*

(4) $\gamma(xy) \equiv \gamma(x)\beta(y) + x_1\beta(x)\gamma(y) \pmod p$*.*

Proof. Since $L = \langle b\rangle + \langle c\rangle$ and L is the (R,R)-subgroup in R^+, by statement 1) of Lemma 2 it follows that $xb \in L$, hence $a\alpha(x) \in L$ for each $x \in R$. Thus $\alpha(x) \equiv 0 \pmod p$ and we get statement (1). Substituting the obtained value of $\alpha(x) \equiv 0 \pmod p$ into the formulas from Lemma 5, we obtain statements (3) and (4) of the lemma and the formula for the product xy. Putting $y = c$, we get $xc = c(x_1\beta(x))$. Hence, if $\beta(x) \equiv 0 \pmod p$, then $xc = 0$, and so $x \in L$. Therefore, $x_1 \equiv 0 \pmod p$, as claimed in statement (2). Indeed, statement (0) repeats statement (0) of Lemma 5. □

It is known that for such groups the commutator $D(R^+)$ coincides with the center $Z(R^+)$ and has the order p.

Lemma 7. *The commutator* $D(R^+)$ *is an ideal in the local nearring* R*.*

Proof. Let $x = ax_1 + bx_2 + cx_3$, $y = ay_1 + by_2 + cy_3 \in R$, $z = cz_3 \in D(R^+)$. Let us check whether $D(R^+)$ is an ideal in R, i.e., $(z + x)y - xy \in D(R^+)$. To do this, we use formula $(*)$ for multiplying elements in R. We obtain

$$
\begin{aligned}
&(z+x)y - xy \\
&\quad =(cz_3 + ax_1 + bx_2 + cx_3)(ay_1 + by_2 + cy_3) - (ax_1 + bx_2 + cx_3)(ay_1 + by_2 + cy_3) \\
&\quad =(ax_1 + bx_2 + c(x_3 + z_3))(ay_1 + by_2 + cy_3) - (ax_1 + bx_2 + cx_3)(ay_1 + by_2 + cy_3) \\
&\quad = ax_1y_1 + b(x_2y_1 + \beta(x)y_2) + c(-x_1x_2\binom{y_1}{2} + x_3y_1 + z_3y_1 + \gamma(z+x)y_2 + x_1\beta(x)y_3) \\
&\qquad - ax_1y_1 - b(x_2y_1 + \beta(x)y_2) - c(-x_1x_2\binom{y_1}{2} + x_3y_1 + \gamma(x)y_2 + x_1\beta(x)y_3) \\
&\quad = ax_1y_1 + b(x_2y_1 + \beta(x)y_2) - ax_1y_1 - b(x_2y_1 + \beta(x)y_2) + c(z_3y_1 + (\gamma(z + x) \\
&\qquad - \gamma(x))y_2) = ax_1y_1 - ax_1y_1 + b(x_2y_1 + \beta(x)y_2) - c(x_1y_1(x_2y_1 + \beta(x)y_2)) \\
&\qquad - b(x_2y_1 + \beta(x)y_2) + c(z_3y_1 + (\gamma(z + x) - \gamma(x))y_2) \\
&\quad = c(z_3y_1 + (\gamma(z + x) - \gamma(x))y_2 - x_1x_2y_1^2 - x_1y_1y_2\beta(x)) \in D(R^+).
\end{aligned}
$$

Therefore, $D(R^+)$ is an ideal of R. □

Lemma 8. *Let* R *be a local nearring whose additive group of* R^+ *is isomorphic to* G_1*. If* $x = ax_1 + bx_2 + cx_3$*,* $y = ay_1 + by_2 + cy_3 \in R$*, then the mappings* $\beta\colon R \to \mathbb{Z}_{p^2}$ *and* $\gamma\colon R \to \mathbb{Z}_p$ *from* $(*)$ *can be one of the following:*

1) $\beta(x) = {x_1}^i$ *and* $\gamma(x) = 0$ $(0 < i < p)$*;*

2) $\beta(x) = 1$ *and* $\gamma(x) = 0$*;*

3) $\beta(x) = x_1^2$ *and* $\gamma(x) = x_1x_2$*.*

Proof. 1) Since zero-symmetric local nearrings of order p^2 are classified in [12], it follows that all non-isomorphic factor-nearrings with derived subgroup $N = R/D(R^+)$ are described. That is, you can apply the multiplication formula from the specified work, pre-adapting it for the left local nearrings. It is clear that $N^+ = \langle a\rangle + \langle b\rangle$. Namely, let $x = ax_1 + bx_2$ and $y = ay_1 + by_2$ be elements of N; then

$$xy = ax_1y_1 + b(x_2y_1 + \rho(x_1)y_2).$$

Moreover, in [12, Theorem 1.6] and [12, Corollary 1.11] it was shown that there are, up to isomorphism, $p-1$ local nearrings (which are not nearfields) with the additive group of order p^2 and exponent p. These local nearrings are completely determined by maps $\rho\colon \mathbb{Z}_p \to \mathbb{Z}_p$ such that (i) $\rho(x) = 0$ if and only if $x = 0$ and (ii) ρ is a group endomorphism of $(\mathbb{Z}_p - \{0\}, \cdot)$ [10]. So ρ takes one of $p-1$ values for zero-symmetric nearrings.

On the other hand, by the formula $(*)$ we have:

$$xy = ax_1y_1 + b(x_2y_1 + \beta(x)y_2).$$

Equating the coefficients for generators, we obtain: $\beta(x)y_2 = \rho(x_1)y_2$. Hence $\beta(x) = \rho(x_1)$. So there exist $p-1$ different zero-symmetric nearrings.

2) It is obvious that the multiplication $(*)$ with the functions $\beta(x) = 1$ and $\gamma(x) = 0$ is the constant nearring multiplication.

3) We will show further that for $\beta(x) = x_1^2$ and $\gamma(x) = x_1x_2$ the multiplication $(*)$, i.e.

$$x * y = ax_1y_1 + b(x_2y_1 + x_1^2y_2) + c(-x_1x_2\binom{y_1}{2} + x_3y_1 + x_1x_2y_2 + x_1^3y_3),$$

is a nearring multiplication. By Lemma 6 (3) and (4), the functions $\beta(x)$ and $\gamma(x)$ need to satisfy the following conditions: $\beta(xy) = \beta(x)\beta(y)$ and $\gamma(xy) = \beta(y)\gamma(x) + x_1\beta(x)\gamma(y)$. We check that $\beta(xy) = (x_1y_1)^2 = x_1^2y_1^2 = \beta(x)\beta(y)$ and $\gamma(xy) = x_1y_1(x_2y_1 + x_1y_2) = x_1x_2y_1^2 + x_1^2y_1y_2 = \beta(y)\gamma(x) + x_1\beta(x)\gamma(y)$. Therefore, $\beta(x)$ and $\gamma(x)$ satisfy conditions (3) and (4) of Lemma 6.

It is easy to see that the element a is a multiplicative identity for $(R,*)$ and $x * b = bx_1^2 + cx_1x_2$ for each $x \in R$. We show that with respect to the operations "$+$" and "$*$" the system $(R,+,*)$ is a nearring with identity element a. Clearly, it suffices to check that if $z = az_1 + bz_2 + cz_3$ is an arbitrary element of R, then $x * (y + z) = x * y + x * z$ and $(x * y) * b = x * (y * b)$.

Indeed, we have

$$x * z = ax_1z_1 + b(x_2z_1 + x_1^2z_2) + c(-x_1x_2\binom{z_1}{2} + x_3z_1 + x_1x_2z_2 + x_1^3z_3)$$

and

$$b(x_2y_1 + x_1^2y_2) + a(x_1z_1) = a(x_1z_1) + b(x_2y_1 + x_1^2y_2) - c(x_1z_1(x_2y_1 + x_1^2y_2))$$

by Lemma 3. We get

$$\begin{aligned}
x*y+x*z=&a(x_1y_1+x_1z_1)+b(x_2y_1+x_1^2y_2+x_2z_1+x_1^2z_2)\\
&+c(-x_1x_2\binom{y_1}{2}+x_3y_1+x_1x_2y_2+x_1^3y_3\\
&-x_1x_2\binom{z_1}{2}+x_3z_1+x_1x_2z_2+x_1^3z_3-x_1z_1(x_2y_1+x_1^2y_2))\\
=&ax_1(y_1+z_1)+b(x_2(y_1+z_1)+x_1^2(y_2+z_2))\\
&+c(-x_1x_2\binom{(y_1+z_1)}{2}+x_3(y_1+z_1)+x_1x_2(y_2+z_2)+x_1^3(y_3+z_3-z_1y_2)).
\end{aligned}$$

On the other hand, $y+z=(ay_1+by_2+cy_3)+(az_1+bz_2+cz_3)=a(y_1+z_1)+b(y_2+z_2)+c(y_3+z_3-z_1y_2)$ because $by_2+az_1=az_1+by_2-cz_1y_2$ by Lemma 3. Therefore,

$$\begin{aligned}
x*(y+z)=&ax_1(y_1+z_1)+b(x_2(y_1+z_1)+x_1^2(y_2+z_2))\\
&+c(-x_1x_2\binom{(y_1+z_1)}{2}+x_3(y_1+z_1)+x_1x_2(y_2+z_2)+x_1^3(y_3+z_3-z_1y_2)).
\end{aligned}$$

Therefore, $x*(y+z)=x*y+x*z$, as desired.

Next, $y*b=by_1^2+cy_1y_2$ and so $x*(y*b)=b(x_1^2y_1^2)+c(x_1x_2y_1^2+x_1^3y_1y_2)=b(x_1y_1)^2+c(x_1x_2(x_2y_1+x_1^2y_2)$.

On the other hand, $(x*y)*b=(ax_1y_1+b(x_2y_1+x_1^2y_2)+c(-x_1x_2\binom{y_1}{2}+x_3y_1+x_1x_2y_2+x_1^3y_3)*b=b(x_1y_1)^2+c(x_1x_2(x_2y_1+x_1^2y_2)=x*(y*b)$. Thus, the system $(R,+,*)$ is a nearring with identity element a, and so multiplication "$*$" is a nearring multiplication, as desired.

□

So, we have examples of $p+1$ nearring multiplications and we formulate the following result.

Theorem 2. *There exist at least $p+1$ non-isomorphic local nearrings on each non-metacyclic non-abelian groups of order p^3.*

Proof. The nearrings $R=(R,+,\cdot)$ with functions $\beta(x)={x_1}^i$ and $\gamma(x)=0$ ($0<i<p$) are non-isomorphic zero-symmetric local nearrings according to the above and paper [12]. It is easy to check that the local nearrings $R=(R,+,\cdot)$ with functions $\beta(x)=1$ and $\gamma(x)=0$ are non-zero-symmetric. It is obvious that $R=(R,+,\cdot)$ with functions $\beta(x)=x_1^2$ and $\gamma(x)=x_1x_2$ are zero-symmetric and non-isomorphic to the nearrings considered above. Therefore, there exist at least $p+1$ non-isomorphic local nearrings on each non-metacyclic non-abelian groups of order p^3. □

Example 1. *Let $G\cong(C_5\times C_5)\rtimes C_5$. If $x=ax_1+bx_2+cx_3$ and $y=ay_1+by_2+cy_3\in G$ and $(G,+,\cdot)$ is a local nearring, then by Lemma 8 "$\cdot$" is one of the following multiplications:*

(1) $x\cdot y=ax_1y_1+b(x_2y_1+y_2)+c(-x_1x_2\binom{y_1}{2}+x_3y_1+x_1y_3)$;

(2) $x \cdot y = ax_1y_1 + b(x_2y_1 + x_1^3(x)y_2) + c(-x_1x_2\binom{y_1}{2} + x_3y_1 + x_1x_2y_2 + x_1^4y_3)$;

(3) $x \cdot y = ax_1y_1 + b(x_2y_1 + x_1y_2) + c(-x_1x_2\binom{y_1}{2} + x_3y_1 + x_1^2y_3)$;

(4) $x \cdot y = ax_1y_1 + b(x_2y_1 + x_1^2y_2) + c(-x_1x_2\binom{y_1}{2} + x_3y_1 + x_1^3y_3)$;

(5) $x \cdot y = ax_1y_1 + b(x_2y_1 + x_1^3(x)y_2) + c(-x_1x_2\binom{y_1}{2} + x_3y_1 + x_1^4y_3)$;

(6) $x \cdot y = ax_1y_1 + b(x_2y_1 + x_1^4y_2) + c(-x_1x_2\binom{y_1}{2} + x_3y_1 + x_1^5y_3)$.

A computer program verified that for $p = 5$, the nearring obtained in Lemma 8 (3) is indeed a local nearring (see Example 1 (2)) deposited on GitHub:

`https://github.com/raemarina/Examples/blob/main/LNR_125-3.txt`

From the packages SONATA and LocalNR [19] we have the following number of non-isomorphic local nearrings:

$StructureDescription(R^+)$	$n(R^+)$
$(C_3 \times C_3) \rtimes C_3$	4
$(C_5 \times C_5) \rtimes C_5$	6
$(C_7 \times C_7) \rtimes C_7$	8
$(C_{11} \times C_{11}) \rtimes C_{11}$	12

.

5. Local nearrings whose additive groups are isomorphic to G_2

Let R be a local nearring whose additive group of R^+ is isomorphic to G_2. Then $R^+ = \langle a\rangle + \langle b\rangle$ for some elements a and b of R satisfying the relations $ap^2 = 0$, $bp = 0$ and $-b + a + b = a(1 + p)$. In particular, each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2$ with coefficients $0 \le x_1 < p^2$ and $0 \le x_2 < p$.

By [8, Theorem 7.1], for $p = 3$ there exist three zero-symmetric nearrings with identity on G_2, and for $p > 3$ one. At the same time, by [8, Theorem 4.2] there exists one non-zero-symmetric nearring with identity on G_2. On the other hand, in [14], it was shown that for each group G_2 there exists a zero-symmetric local nearring. With the SONATA nearring library, it is easy to make sure that all nearrings with identity on G_2 of order 27 are local. The formula for multiplying elements of local nearrings on Miller–Moreno metacyclic groups is defined in [20]. Since G_2 is a Miller–Moreno metacyclic group, using [20, Corollary 2], for arbitrary elements $x = ax_1 + bx_2$ and $y = ay_1 + by_2$ of G_2 and putting $\alpha(x) = 0$ and $\beta(x) = 1$, we obtain the following multiplication formula:

$$x \cdot y = a(x_1y_1 - x_1x_2\binom{y_1}{2}p) + b(x_2y_1 + y_2). \ (**)$$

It is easy to see that $R = (G_2, +, \cdot)$ is a non-zero-symmetric local nearring.

The multiplication formula for arbitrary elements of a zero-symmetric local nearring on G_2 is given when proving [20, Theorem 2], namely:

$$x \cdot y = a(x_1y_1 - x_1x_2\binom{y_1}{2}p) + b(x_2y_1 + \beta(x)y_2), \ (***)$$

where $\beta(x) = \begin{cases} 1, \; if \; x_1 \not\equiv 0 \pmod p; \\ 0, \; if \; x_1 \equiv 0 \pmod p). \end{cases}$

So, from the results presented in this section, we have:

1) $n(G_2) = 4$ for $p = 3$;
2) $n(G_2) = 2$ for $p > 3$. Moreover, the multiplication formulas in such nearrings are determined by formulas $(**)$ or $(***)$.

6. Local nearrings whose additive groups are isomorphic to G_3

Let R be a local nearring, which is not a nearfield, whose additive group of R^+ is isomorphic to G_3. Then $R^+ = \langle a\rangle + \langle b\rangle$ for some elements a and b of R satisfying the relations $ap^2 = 0$, $bp = 0$ and $b + a = a + b$. In particular, each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2$ with coefficients $0 \leq x_1 < p^2$ and $0 \leq x_2 < p$.

We can assume, without loss of generality, that a is an identity of R, i.e., $ax = xa = x$ for each $x \in R$. Furthermore, for each $x \in R$ there exist coefficients $\alpha(x)$ and $\beta(x)$ such that $xb = a\alpha(x) + b\beta(x)$. It is clear that they are uniquely defined modulo p^2 and p, respectively, so that some mappings $\alpha : R \to \mathbb{Z}_{p^2}$ and $\beta : R \to \mathbb{Z}_p$ are determined.

Lemma 9. *Let R be a local nearring, which is not a nearfield, whose additive group of R^+ is isomorphic to G_3. If a coincides with identity element of R, $x = ax_1 + bx_2$, $y = ay_1 + by_2 \in R$, $xb = a\alpha(x) + b\beta(x)$, then*

$$xy = a(x_1y_1 + \alpha(x)y_2) + b(x_2y_1 + \beta(x)y_2). \; (****)$$

Moreover, for the mappings $\alpha\colon R \to \mathbb{Z}_{p^2}$ and $\beta\colon R \to \mathbb{Z}_p$ the following statements hold:

(0) $\alpha(0) = \beta(0) = 0$ *if and only if R is zero-symmetric;*

(1) $\alpha(a) = 0$ *and* $\beta(a) = 1$*;*

(2) $\alpha(x) \equiv 0 \pmod p$*;*

(3) $\alpha(xy) \equiv x_1\alpha(y) + \alpha(x)\beta(y) \pmod p$*;*

(4) $\beta(xy) \equiv x_2\alpha(y) + \beta(x)\beta(y) \pmod p$.

Proof. Since $0 \cdot a = a \cdot 0 = 0$, it follows that R is a zero-symmetric nearring if and only if $0 = 0 \cdot b = a\alpha(0) + b\beta(0)$ or equivalently, $\alpha(0) = \beta(0) = 0$. Moreover, since $b = ab = a\alpha(a) + b\beta(a)$, we have $\alpha(a) = 0$ and $\beta(a) = 1$, so that statements (0) and (1) hold.

Further, using the left distributive law, we derive

$$xy = (xa)y_1 + (xb)y_2 = (ax_1 + bx_2)y_1 + (a\alpha(x) + b\beta(x))y_2.$$

We also have $(ax_1 + bx_2)y_1 = ax_1y_1 + bx_2y_1$ and $(a\alpha(x) + b\beta(x))y_2 = a\alpha(x)y_2 + b\beta(x)y_2$. Thus $xy = a(x_1y_1 + \alpha(x)y_2) + b(x_2y_1 + \beta(x)y_2)$ and so statement (2) holds.

By Corollary 1, $L = \langle ap \rangle + \langle b \rangle$. Since $xL \subseteq L$ for each $x \in R$ by Lemma 2, we have $xb = a\alpha(x) + b\beta(x) \in L$, whence $\alpha(x) \equiv 0 \pmod{p}$ for each $x \in R$.

Finally, the associativity of multiplication in R implies that

$$(xy)b = x(yb) = a\alpha(xy) + b\beta(xy).$$

Furthermore, substituting $yb = a\alpha(y) + b\beta(y)$ instead of y into formula $(****)$, we also have

$$x(yb) = a(x_1\alpha(y) + \alpha(x)\beta(y)) + b(x_2\alpha(y) + \beta(x)\beta(y)).$$

Comparing the coefficients under a and b in two expressions obtained for $x(yb)$, we derive statements (3) and (4) of the lemma. □

Lemma 10. *$\langle ap \rangle$ is an ideal of R.*

Proof. Let $x = ax_1 + bx_2$, $y = ay_1 + by_2 \in R$, $z = apz_1 \in \langle ap \rangle$. Check whether $\langle ap \rangle$ is an ideal of R, i.e. $(z + x)y - xy \in \langle ap \rangle$. Using the formula $(****)$, we have

$$\begin{aligned}(z+x)y - xy =&(apz_1 + ax_1 + bx_2)(ay_1 + by_2) - (ax_1 + bx_2)(ay_1 + by_2)\\ =&(a(pz_1 + x_1) + bx_2)(ay_1 + by_2) - (ax_1 + bx_2)(ay_1 + by_2)\\ =&\, a((pz_1 + x_1)y_1 + \alpha(z + x)y_2) + b(x_2y_1 + \beta(z + x)y_2)\\ &- (a(z_1y_1 + \alpha(x)y_2) + b(x_2y_1 + \beta(x)y_2))\\ =&\, a(pz_1y_1 + (\alpha(z + x) - \alpha(x))y_2) + b(\beta(z + x) - \beta(x))y_2\\ =&\, a(pz_1y_1 + (\alpha(z + x) - \alpha(x))y_2) \in \langle ap \rangle,\end{aligned}$$

since $\beta(z + x) - \beta(x) = 0$.

Therefore, $\langle ap \rangle$ is an ideal of R. □

Lemma 11. *Let R be a local nearring, which is not a nearfield, whose additive group of R^+ is isomorphic to G_3. If $x = ax_1 + bx_2$, $y = ay_1 + by_2 \in R$, then the mappings $\alpha\colon R \to \mathbb{Z}_{p^2}$ and $\beta\colon R \to \mathbb{Z}_p$ from $(****)$ can be one of the following:*

(1) *$\alpha(x) = 0$ and $\beta(x) = {x_1}^i \pmod{p}$ $(0 < i < p)$;*

(2) *$\alpha(x) = 0$ and $\beta(x) = 1$;*

(3) *$\alpha(x) = px_2$ and $\beta(x) \equiv x_1 \pmod{p}$.*

Proof. 1) Since zero-symmetric local nearrings of order p^2 are classified in [12], it follows that all non-isomorphic factor-nearrings with a derived subgroup $N = R/\langle ap \rangle$ are described. That is, you can apply the multiplication formula from the specified work, pre-adapting it for the left local nearrings. It is clear that $N^+ = \langle \overline{a} \rangle + \langle \overline{b} \rangle$. Namely, let $\overline{x} = \overline{a}x_1 + \overline{b}x_2$ and $\overline{y} = \overline{a}y_1 + \overline{b}y_2$ be elements of N; then

$$\overline{xy} = \overline{a}x_1y_1 + \overline{b}(x_2y_1 + \rho(x_1)y_2).$$

Moreover, by [12, Theorem 1.6], ρ takes one of $p - 1$ values for zero-symmetric nearrings.

On the other hand, by the formula $(****)$ we have $\overline{xy} = \overline{a}x_1y_1 + \overline{b}(x_2y_1 + \beta(x)y_2)$.

Equating the coefficients for generators, we obtain $\beta(x)y_2 = \rho(x_1)y_2$. Hence $\beta(x) = \rho(x_1)$. So there exist $p-1$ different zero-symmetric nearrings.

2) It is obvious that the multiplication $(***\,*)$ with the functions $\alpha(x) = 0$ and $\beta(x) = 1$ is the constant nearring multiplication.

3) By direct checking of conditions (3) and (4) of Lemma 9 for $\alpha(x) = px_2$ and $\beta(x) \equiv x_1 \pmod p$ and by using the multiplication $(***\,*)$, we prove the lemma. □

So, we have examples of $p+1$ nearring multiplications and we formulate the following result.

Theorem 3. *There exist at least $p+1$ non-isomorphic local nearrings on each metacyclic abelian groups of order p^3.*

Proof. Nearrings $R = (R, +, \cdot)$ with functions $\alpha(x) = 0$ and $\beta(x) = {x_1}^i \pmod p$ $(0 < i < p)$ are non-isomorphic zero-symmetric local nearrings according to the above and paper [12]. It is easy to check that the local nearring $R = (R, +, \cdot)$ with functions $\alpha(x) = 0$ and $\beta(x) = 1$ is non-zero-symmetric.

Consider $R = (R, +, \cdot)$ with functions $\alpha(x) = px_2$ and $\beta(x) \equiv x_1 \pmod p$. In this case, we have $|\{xb = apx_2 + bx_1 | x \in R\}| = p^2$ and R is zero-symmetric. For example, in the above case with $\alpha(x) = 0$ and $\beta(x) = x_1 \pmod p$ we get $|\{xb = bx_1 | x \in R\}| = p$, and so R is non-isomorphic to the nearrings considered above. Therefore, there exist at least $p+1$ non-isomorphic local nearrings on each metacyclic abelian group of order p^3. □

From the packages SONATA and LocalNR we have the following number of non-isomorphic local nearrings:

$StructureDescription(R^+)$	$n(R^+)$
$C_9 \times C_3$	13
$C_{25} \times C_5$	31
$C_{49} \times C_7$	31
$C_{121} \times C_{11}$	47

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