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On some properties of k-generalized Fibonacci numbers

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Abstract. Some identities between positive integer powers of matrices X satisfying the equation

$$X^{k+1}-(r+1)X^k+rI+\sum_{i=1}^{k-1}(r-1)X^i=\mathbf{0}$$

and k-generalized Fibonacci numbers are first established, where k is an integer with $k \geq 2$. Then, two main formulas are given, one of which is related to the k-generalized Fibonacci sequence.

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1. Introduction and preliminaries

The k-generalized Fibonacci sequence $\{G_{(n)}^{(k)}\}$ is defined by the recurrence relation

$$G_{(n)}^{(k)}=G_{(n-1)}^{(k)}+G_{(n-2)}^{(k)}+\cdots+G_{(n-k)}^{(k)}$$

for all positive integers n and k with $n \geq k \geq 2$ and the initial conditions

$$G_{(0)}^{(k)}=G_{(1)}^{(k)}=\cdots=G_{(k-2)}^{(k)}=0,\ G_{(k-1)}^{(k)}=1.$$

Meanwhile, it is clear that the Fibonacci sequence $\{F_n\}$ is a special case of $\{G_{(n)}^{(k)}\}$ for $k=2$ [2].

The definition of the k-generalized Fibonacci sequence can also be adopted by having negative indices, that is, from the recurrence relation, we get

$$G_{(-n)}^{(k)}=-G_{(-n+1)}^{(k)}-G_{(-n+2)}^{(k)}-\cdots-G_{(-n+k-1)}^{(k)}+G_{(-n+k)}^{(k)},$$

replacing here n by $-n$. So, we can use the recurrence relation of the sequence $\{G_{(n)}^{(k)}\}$ for all integers n [7].

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The relationships between some sequences and matrices have been investigated in different studies. A relation between k-generalized Fibonacci numbers and matrices was shown in [2] with the generalized Q_k-matrix for $k \geq 2$ integers:

$$Q_k^n = \begin{pmatrix} G_{(n+k-1)}^{(k)} & G_{(n+k-2)}^{(k)} & \cdots & G_{(n)}^{(k)} \\ \sum_{i=0}^{k-2} G_{(n+k-2-i)}^{(k)} & \sum_{i=0}^{k-2} G_{(n+k-3-i)}^{(k)} & \cdots & \sum_{i=0}^{k-2} G_{(n-1-i)}^{(k)} \\ \vdots & \vdots & \cdots & \vdots \\ G_{(n+k-2)}^{(k)} & G_{(n+k-3)}^{(k)} & \cdots & G_{(n-1)}^{(k)} \end{pmatrix}$$

for all $n \in \mathbb{N}$. The characteristic polynomial of the Q_k-matrix is

$$h_k(x) = x^k - x^{k-1} - \cdots - x - 1,$$

which is also the auxiliary polynomial for $\{G_{(n)}^{(k)}\}$ [2]. The roots of this polynomial were investigated in different studies, and some properties were obtained using the roots. One of the important properties of these roots is that they are distinct [4, 6, 8]. The relation between these roots and the sequence $\{G_{(n)}^{(k)}\}$ was given in [5]. If λ is a root of the polynomial $h_k(x)$,

$$\begin{aligned} \lambda^n = & G_{(n)}^{(k)} \lambda^{k-1} + (G_{(n-1)}^{(k)} + G_{(n-2)}^{(k)} + \cdots + G_{(n-k+1)}^{(k)}) \lambda^{k-2} \\ & + (G_{(n-1)}^{(k)} + G_{(n-2)}^{(k)} + \cdots + G_{(n-k+2)}^{(k)}) \lambda^{k-3} + \cdots + (G_{(n-1)}^{(k)} + G_{(n-2)}^{(k)}) \lambda + G_{(n-1)}^{(k)} \end{aligned}$$

is valid for all positive integers $n \geq k$.

There are lots of well-known sequences in the literature. In many studies, authors obtained equalities for these sequences. Some of the equalities are explicit formulas used to obtain the terms of sequences. For example, a well-known explicit formula for the Fibonacci sequence is that

$$F_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i}$$

is true for all $n \in \mathbb{N}$. Again, such formulas have been obtained for other well-known sequences (for example, see [3]). Relations have also been established for the sequence $\{G_{(n)}^{(k)}\}$. In [1], the authors showed lots of important results for the sequence $\{G_{(n)}^{(k)}\}$. Some of these results are about finding the terms of the sequence $\{G_{(n)}^{(k)}\}$ for special cases of n. First, they showed that $G_{(n)}^{(k)} = 2^{n-k}$ is true for all integers n with $k \leq n \leq 2k-1$. Then, they showed that

$$G_{(n)}^{(k)} = 2^{n-k} + \sum_{i=1}^{\lfloor \frac{n+k}{k+1} \rfloor - 1} (-1)^i (\binom{n-ki-k+2}{i} - \binom{n-ki-k}{i-2}) 2^{n-(k+1)i-k}$$

is true for all integers n with $n \geq 2k$ (Theorem 2.4 in [1]). Thus, thanks to these two equalities, they enabled obtaining the terms of the sequence $\{G_{(n)}^{(k)}\}$ for all integers n with $n \geq k$.

It is important to obtain new results. In addition, as can be seen in the results in the literature, different methods were used to achieve these results. It is known that not only the results obtained but also the proof methods used to obtain them are important.

The main purpose of this paper is to obtain a new explicit formula for the sequence $\{G^{(k)}_{(n)}\}$. Meanwhile, the main result obtained in this paper is similar to Theorem 2.4 in [1]. However, there are two notable differences between our result and the result mentioned in [1]. The first of these is that while the result mentioned above is valid for all integers n with $n \geq 2k$, the result we obtained is valid for all integers n with $n \geq k$. Another and more important thing is that matrix algebra is primarily used in addition to the properties of the sequence in obtaining the main result and other results.

In this paper, we first show the relationship between the matrices satisfying the equation

$$x^{k+1} - (r+1)x^k + r + \sum_{i=1}^{k-1}(r-1)x^i = 0$$

and the sequence $\{G^{(k)}_{(n)}\}$, where r is a nonzero real number with $h_k(r) \neq 0$. Then, we introduce a new sequence $\{F^{(k,r)}_{(n)}\}$ and get some identities related to the sequences $\{F^{(k,r)}_{(n)}\}$ and $\{G^{(k)}_{(n)}\}$. Moreover, we establish two main formulas, one of which is related to the k-generalized Fibonacci sequence.

Before giving the main results, it will be beneficial to introduce some terminologies which will be used in the subsequent lines. $p(x) = \det(xI - A)$ is the characteristic polynomial of the square matrix A. The eigenvalues of A are the roots of the characteristic equation $p(x) = 0$ or, equivalently, the zeros of the characteristic polynomial $p(x)$. The well-known Cayley-Hamilton theorem states that $p(A) = \mathbf{0}$. The minimal polynomial of A is the monic polynomial $q(x)$ of least degree such that $q(A) = \mathbf{0}$. If $m(x)$ is a polynomial such that $m(A) = \mathbf{0}$, then the minimal polynomial $q(x)$ divides $m(x)$. Moreover, $q(x)$ divides $p(x)$ and, $q(x)$ and $p(x)$ have the same zeros.

We want to note that r will also be a nonzero real number with $h_k(r) \neq 0$, and k will be an integer with $k \geq 2$ throughout the paper to avoid uncertainty. Furthermore, the symbol $\lfloor a \rfloor$ denotes the greatest integer that is less than or equal to the real number a.

2. Results

Now, we give two lemmas that will be used throughout the study. Their proofs can be done easily using the recurrence relation and the assumptions which will be given, and are therefore omitted here.

Lemma 1. *It is true that* $G^{(k)}_{(-1)} = 1$ *and* $G^{(k)}_{(-2)} = -1$. *Also,* $G^{(k)}_{(p)} = 0$ *if* p *is an integer with* $-k \leq p < -2$.

Lemma 2. $f(n+1) - rf(n) = G^{(k)}_{(n)}$ *is true for all* $n \in \mathbb{N}$, *where* $f(n) = \sum_{i=0}^{n-1} r^{n-1-i} G^{(k)}_{(i)}$.

Note that throughout the study, $f(n)$ will be as in Lemma 2.

Theorem 1. *If X is a square matrix satisfying*

$$X^{k+1} - (r+1)X^k + rI + \sum_{i=1}^{k-1}(r-1)X^i = \boldsymbol{0}$$

for any k, then

$$X^n = f(n)X^k - (\sum_{i=1}^{k-1}(f(n) - \sum_{j=1}^{i+1} G_{(n-j)}^{(k)})X^i) - (f(n) - G_{(n-1)}^{(k)})I$$

is true for all $n \in \mathbb{N}$.

Proof. We will use induction on n. If necessary calculations are made using Lemma 1, it is easily seen that the equality is true for $n = 1$. Now, let us assume that the assertion is true for some $n \in \mathbb{N}$. First, if necessary arrangements are made in the equality $X^{n+1} = X^n X$,

$$X^{n+1} = (rf(n) + G_{(n)}^{(k)})X^k - rf(n)I - \sum_{i=1}^{k-1}(rf(n) - \sum_{j=1}^{i} G_{(n-j)}^{(k)})X^i \qquad (1)$$

is obtained. If Lemma 2 is used in (1),

$$X^{n+1} = f(n+1)X^k - (f(n+1) - G_{(n)}^{(k)})I - \sum_{i=1}^{k-1}(f(n+1) - G_{(n)}^{(k)} - \sum_{j=1}^{i} G_{(n-j)}^{(k)})X^i$$

or, equivalently,

$$X^{n+1} = f(n+1)X^k - (f(n+1) - G_{(n)}^{(k)})I - \sum_{i=1}^{k-1}(f(n+1) - \sum_{j=1}^{i+1} G_{(n+1-j)}^{(k)})X^i$$

is obtained, and the proof is completed. □

Corollary 1. *The following identities are valid for all $n \in \mathbb{N}$:*

(*a*) *If X is a square matrix satisfying $h_k(X) = \boldsymbol{0}$ for any k, then*

$$X^n = (\sum_{i=1}^{k-1}\sum_{j=1}^{i+1} G_{(n-j)}^{(k)}X^i) + G_{(n-1)}^{(k)}I,$$

(*b*) ((3.3) *in [5]*)

$$\lambda^n = (\sum_{i=1}^{k-1}\sum_{j=1}^{i+1} G_{(n-j)}^{(k)}\lambda^i) + G_{(n-1)}^{(k)},$$

where λ is a root of the equation $h_k(x) = 0$.

Proof. Since $h_k(X) = \mathbf{0}$, then $h_k(X)(X - rI) = \mathbf{0}$ or, equivalently, $X^{k+1} - (r + 1)X^k + rI + \sum_{i=1}^{k-1}(r-1)X^i = \mathbf{0}$ is obtained. So, we can use Theorem 1, and we get

$$X^n = f(n)X^k - (\sum_{i=1}^{k-1}(f(n) - \sum_{j=1}^{i+1} G^{(k)}_{(n-j)})X^i) - (f(n) - G^{(k)}_{(n-1)})I.$$

Here, by considering $h_k(X) = \mathbf{0}$ and performing necessary calculations, we get

$$X^n = (\sum_{i=1}^{k-1}\sum_{j=1}^{i+1} G^{(k)}_{(n-j)}X^i) + G^{(k)}_{(n-1)}I.$$

This is the proof of item (a).

Now, let D be a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_k$, where λ_i's are the roots of the equation $h_k(x) = 0$. It is obvious that the characteristic equation of the matrix D is $h_k(x) = 0$, and from the Cayley-Hamilton theorem, $h_k(D) = \mathbf{0}$. So, if item (a) is used, the proof of item (b) is easily obtained. □

Now, let us introduce a new sequence $\{F^{(k,r)}_{(n)}\}$ by the recurrence relation

$$F^{(k,r)}_{(n)} = (r+1)F^{(k,r)}_{(n-1)} - (\sum_{i=1}^{k-1}(r-1)F^{(k,r)}_{(n-1-i)}) - rF^{(k,r)}_{(n-k-1)},$$

for all integers $n > k$ and the initial conditions $F^{(k,r)}_{(0)} = F^{(k,r)}_{(1)} = \cdots = F^{(k,r)}_{(k-1)} = 0$, $F^{(k,r)}_{(k)} = 1$.

Theorem 2. *If X is a square matrix satisfying*

$$X^{k+1} - (r+1)X^k + rI + \sum_{i=1}^{k-1}(r-1)X^i = \boldsymbol{0}$$

for any k, then

$$X^n = F^{(k,r)}_{(n)}X^k - (\sum_{i=1}^{k-1}(rF^{(k,r)}_{(n-1-i)} + (r-1)\sum_{j=1}^{i} F^{(k,r)}_{(n-j)})X^i) - rF^{(k,r)}_{(n-1)}I$$

is true for all integers $n > k$.

Proof. The proof can be done in two steps. Let us first show that it is true for $k = 2$. We need to show that

$$X^n = F^{(2,r)}_{(n)}X^2 - (rF^{(2,r)}_{(n-2)} + (r-1)F^{(2,r)}_{(n-1)})X - rF^{(2,r)}_{(n-1)}I$$

is true for matrices satisfying the condition $X^3 - (r+1)X^2 + (r-1)X + rI = \mathbf{0}$, where $n > 2$ integer. The proof is easily obtained by induction when the necessary calculations are performed. Now let us show that the assertion is true for $k > 2$ by induction on n again. First, we must show that the equality is true for $n = k + 1$. The equality is seen by considering the equalities

$$F^{(k,r)}_{(k+1)} = r + 1,\ F^{(k,r)}_{(k)} = 1 \text{ and } F^{(k,r)}_{(0)} = F^{(k,r)}_{(1)} = F^{(k,r)}_{(2)} = \cdots = F^{(k,r)}_{(k-1)} = 0,$$

and necessary calculations are performed. Finally, we assume that the equality is true for some positive integer $n > k$. We must show that the equality is true for $n + 1$. The desired result is obtained from the matrix equality $X^{n+1} = X^n X$, by considering the equality

$$F^{(k,r)}_{(n)} = (r+1)F^{(k,r)}_{(n-1)} - (\sum_{i=1}^{k-1}(r-1)F^{(k,r)}_{(n-1-i)}) - rF^{(k,r)}_{(n-k-1)}$$

and performing the necessary calculations. □

Corollary 2. *$f(n) = F^{(k,r)}_{(n)}$ is true for all integers $n > k$.*

Proof. Let $A = \begin{pmatrix} r & \mathbf{0} \\ \mathbf{0} & D \end{pmatrix}$, where D is the matrix as in Corollary 1. By using Theorem 1 and Theorem 2 we get

$$A^n = f(n)A^k - (\sum_{i=1}^{k-1}(f(n) - \sum_{j=1}^{i+1} G^{(k)}_{(n-j)})A^i) - (f(n) - G^{(k)}_{(n-1)})I \tag{2}$$

and

$$A^n = F^{(k,r)}_{(n)}A^k - (\sum_{i=1}^{k-1}(rF^{(k,r)}_{(n-1-i)} + (r-1)\sum_{j=1}^{i} F^{(k,r)}_{(n-j)})A^i) - rF^{(k,r)}_{(n-1)}I, \tag{3}$$

respectively. Hence from (2) and (3), we get

$$\begin{aligned}(f(n) - F^{(k,r)}_{(n)})A^k &- (\sum_{i=1}^{k-1}(f(n) - rF^{(k,r)}_{(n-1-i)} - (r-1)\sum_{j=1}^{i} F^{(k,r)}_{(n-j)} - \sum_{j=1}^{i+1} G^{(k)}_{(n-j)})A^i) \\ &- (f(n) - rF^{(k,r)}_{(n-1)} - G^{(k)}_{(n-1)})I = \mathbf{0}.\end{aligned} \tag{4}$$

Now, let

$$\begin{aligned} m(x) = &(f(n) - F^{(k,r)}_{(n)})x^k \\ &- (\sum_{i=1}^{k-1}(f(n) - rF^{(k,r)}_{(n-1-i)} - (r-1)\sum_{j=1}^{i} F^{(k,r)}_{(n-j)} - \sum_{j=1}^{i+1} G^{(k)}_{(n-j)})x^i) \\ &- (f(n) - rF^{(k,r)}_{(n-1)} - G^{(k)}_{(n-1)}).\end{aligned}$$

It is easily seen that $m(A) = \mathbf{0}$ by using (4). Also, it is easily seen that the characteristic polynomial of the matrix A is $p(x) = h_k(x)(x-r)$. Now let $q(x)$ be the minimal polynomial of the matrix A, i.e., the monic (that is, with the leading coefficient 1) polynomial of minimum degree such that $q(A) = \mathbf{0}$. We know that the eigenvalues of the matrix A are distinct. So, $q(x)=p(x)$. Moreover, $q(x) \mid m(x)$. Hence $m(x)$ must be the zero polynomial. Thus $f(n) - F_{(n)}^{(k,r)} = 0$ is obtained, and the proof is completed. □

Using Corollary 2, we will establish two main results. To achieve these, we will first give two auxiliary results.

Corollary 3. *$F_{(n)}^{(k,1)} = 2^{n-k}(1 - \frac{1}{2}\sum_{i=0}^{n-k-1} \frac{F_{(i)}^{(k,1)}}{2^i})$ is true for all integers $n > k$.*

Proof. We get immediately from the definition of the sequence $\{F_{(n)}^{(k,r)}\}$ that

$$\begin{aligned} F_{(n)}^{(k,1)} &= 2F_{(n-1)}^{(k,1)} - F_{(n-k-1)}^{(k,1)} \\ 2F_{(n-1)}^{(k,1)} &= 4F_{(n-2)}^{(k,1)} - 2F_{(n-k-2)}^{(k,1)} \\ 4F_{(n-2)}^{(k,1)} &= 8F_{(n-3)}^{(k,1)} - 4F_{(n-k-3)}^{(k,1)} \\ &\vdots \\ 2^{n-(k+1)}F_{(k+1)}^{(k,1)} &= 2^{n-k}F_{(k)}^{(k,1)} - 2^{n-(k+1)}F_{(0)}^{(k,1)}. \end{aligned}$$

If these equations are summed side by side, then

$$F_{(n)}^{(k,1)} = 2^{n-k}(1 - \frac{1}{2}\sum_{i=0}^{n-k-1} \frac{F_{(i)}^{(k,1)}}{2^i})$$

is obtained, and the proof is completed. □

Corollary 4. *If n is an integer with $k \le n \le 2k$, then $F_{(n)}^{(k,1)} = 2^{n-k}$.*

Proof. We know that $F_{(k)}^{(k,1)} = 1$, that is, the equality is true for $n = k$. Now, if $k+1 \le n \le 2k$, then $0 \le n-k-1 \le k-1$. So, we get

$$\sum_{i=0}^{n-k-1} \frac{F_{(i)}^{(k,1)}}{2^i} = 0$$

from the definition of $\{F_{(n)}^{(k,r)}\}$. Hence the desired result is obtained by using Corollary 3. □

By using Corollary 3, the elements of the sequence $\{F_{(n)}^{(k,1)}\}$ can be obtained in appropriate intervals, as in Corollary 4. For example, if $2k+1 \le n \le 3k+1$, then $k \le n-k-1 \le 2k$. Hence it is easily obtained that $F_{(n)}^{(k,1)} = 2^{n-k}(1 - \frac{n-2k}{2^{k+1}})$ for $2k+1 \le n \le 3k+1$ by using Corollary 3 and Corollary 4. So, of course, it is possible to continue this way.

Now let us give a more general result that gives the elements of the sequence $\{F_{(n)}^{(k,1)}\}$.

Theorem 3. $F_{(n)}^{(k,1)} = 2^{n+1} \sum_{i=1}^{\lfloor \frac{n+1}{k+1} \rfloor} \frac{(-1)^{i+1}}{2^{ik+i}} \binom{n-ik}{i-1}$ *is true for all integers* $n \geq k$.

Proof. First, let us show that the equality is true for all integers n such that $k \leq n \leq 2k$. In this case, we have $F_{(n)}^{(k,1)} = 2^{n-k}$ from Corollary 4. Also, it is clear that $\lfloor \frac{n+1}{k+1} \rfloor = 1$. By combining these facts, the desired result is easily obtained.

Now let us show that the equality is true for $n \geq 2k+1$. We use strong induction on n. First of all, it is directly seen that

$$F_{(n+1)}^{(k,1)} = 2F_{(n)}^{(k,1)} - F_{(n-k)}^{(k,1)}$$

from the definition of the sequence $\{F_{(n)}^{(k,r)}\}$. Hence

$$F_{(2k+1)}^{(k,1)} = 2^{k+1} - 1$$

is obtained. On the other hand, it is clear that $\lfloor \frac{2k+2}{k+1} \rfloor = 2$. By combining these facts, it is seen that the assertion is true for $n = 2k+1$. Now assume that the equality holds for all positive integers less than or equal to n, where n is an arbitrary integer with $n \geq 2k+1$. We must show that the equality is true for $n+1$. Due to this assumption, we have the equality

$$F_{(n)}^{(k,1)} = 2^{n+1} \sum_{i=1}^{\lfloor \frac{n+1}{k+1} \rfloor} \frac{(-1)^{i+1}}{2^{ik+i}} \binom{n-ik}{i-1}. \tag{5}$$

It is also clear that $n-k \geq k+1$ since $n \geq 2k+1$. Now either $k+1 \leq n-k \leq 2k$ or $n-k \geq 2k+1$. Hence, we have the equality

$$F_{(n-k)}^{(k,1)} = 2^{n-k+1} \sum_{i=1}^{\lfloor \frac{n-k+1}{k+1} \rfloor} \frac{(-1)^{i+1}}{2^{ik+i}} \binom{n-k-ik}{i-1}, \tag{6}$$

because we showed that (6) is true for $k+1 \leq n-k \leq 2k$. Again, if $n-k \geq 2k+1$, we can also write (6) due to the assumption.

Let $\lfloor \frac{n+1}{k+1} \rfloor = b$. Hence we have

$$n+1 = b(k+1) + u, \tag{7}$$

where u is an integer with $0 \leq u < k+1$. Also, since $n \geq 2k+1$, then $b > 1$. Now we consider the cases $u = k$ and $0 \leq u < k$ separately.

First, let $u = k$. In this case, $\lfloor \frac{n-k+1}{k+1} \rfloor = b$ is obtained. Hence if (5) and (6) are used in the equality $F_{(n+1)}^{(k,1)} = 2F_{(n)}^{(k,1)} - F_{(n-k)}^{(k,1)}$, then

$$F_{(n+1)}^{(k,1)} = 2^{n+2} \sum_{i=1}^{b} \frac{(-1)^{i+1}}{2^{ik+i}} \binom{n-ik}{i-1} - 2^{n-k+1} \sum_{i=1}^{b} \frac{(-1)^{i+1}}{2^{ik+i}} \binom{n-k-ik}{i-1} \tag{8}$$

is obtained. Furthermore, we have $n = b(k+1)+k-1$ from (7). Therefore, if the necessary calculations are made taking the equality $n = b(k+1)+k-1$ into consideration,

$$F^{(k,1)}_{(n+1)} = (-1)^b + 2^{n-k+1} + 2^{n+2}\sum_{i=2}^{b}\frac{(-1)^{i+1}}{2^{ik+i}}\left(\binom{n-ik}{i-1}+\binom{n-ik}{i-2}\right)$$

is obtained from (8). Again, if the necessary calculations are made for the last equality, we get

$$F^{(k,1)}_{(n+1)} = 2^{n+2}\sum_{i=1}^{b+1}\frac{(-1)^{i+1}}{2^{ik+i}}\binom{n+1-ik}{i-1}.$$

Additionally, it is directly seen that $\lfloor\frac{n+2}{k+1}\rfloor = b+1$. So,

$$F^{(k,1)}_{(n+1)} = 2^{n+2}\sum_{i=1}^{\lfloor\frac{n+2}{k+1}\rfloor}\frac{(-1)^{i+1}}{2^{ik+i}}\binom{n+1-ik}{i-1}$$

and the desired result is obtained for $u = k$.

Now, let $0 \le u < k$. In this case, we get $\lfloor\frac{n-k+1}{k+1}\rfloor = b-1$. Again, if (5) and (6) are used in the equality $F^{(k,1)}_{(n+1)} = 2F^{(k,1)}_{(n)} - F^{(k,1)}_{(n-k)}$,

$$F^{(k,1)}_{(n+1)} = 2^{n+2}\sum_{i=1}^{b}\frac{(-1)^{i+1}}{2^{ik+i}}\binom{n-ik}{i-1} - 2^{n-k+1}\sum_{i=1}^{b-1}\frac{(-1)^{i+1}}{2^{ik+i}}\binom{n-k-ik}{i-1}$$

is obtained. Again, if the necessary calculations are made for this equality, it is easily seen that

$$F^{(k,1)}_{(n+1)} = 2^{n-k+1} + 2^{n+2}\sum_{i=2}^{b}\frac{(-1)^{i+1}}{2^{ik+i}}\left(\binom{n-ik}{i-1}+\binom{n-ik}{i-2}\right). \tag{9}$$

It is also clear that $\lfloor\frac{n+2}{k+1}\rfloor = b$ for $0 \le u < k$. Therefore, if the necessary calculations are made in (9),

$$F^{(k,1)}_{(n+1)} = 2^{n+2}\sum_{i=1}^{\lfloor\frac{n+2}{k+1}\rfloor}\frac{(-1)^{i+1}}{2^{ik+i}}\binom{n+1-ik}{i-1},$$

and the desired result is obtained for $0 \le u < k$. So, the proof is completed. □

Now, let us finally give the result that can be used to obtain the elements of the sequence $\{G^{(k)}_{(n)}\}$.

Theorem 4. $G^{(k)}_{(n)} = 2^{n+1}\left(\sum_{i=1}^{\lfloor\frac{n+2}{k+1}\rfloor}\frac{(-1)^{i+1}}{2^{ik+i-1}}\binom{n+1-ik}{i-1} + \sum_{i=1}^{\lfloor\frac{n+1}{k+1}\rfloor}\frac{(-1)^{i}}{2^{ik+i}}\binom{n-ik}{i-1}\right)$ *is true for all integers* $n \ge k$.

Proof. Considering that $G^{(k)}_{(k)} = 1$, it is easily seen that the equality is true for $n = k$. Then, using Lemma 2 and Corollary 2, we easily get

$$G^{(k)}_{(n)} = F^{(k,1)}_{(n+1)} - F^{(k,1)}_{(n)} \tag{10}$$

for all integers $n > k$. Now, if Theorem 3 is used in (10), then the desired result is obtained. □

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