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# On $k-$generalized Fibonacci Diophantine triples

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**Abstract.** For $n,\, k \geq 2,$ the $k$-generalized Fibonacci sequence $\left\{F_n^{(k)}\right\}$ is defined by each term being the sum of the $k$ preceding terms with the initial values $0, 0, \ldots, 0, 1$ ($k$ terms). In this paper, we prove that the system

$$ab + 1 = F_x^{(k)}$$
$$ac + 1 = F_y^{(k)}$$
$$bc + 1 = F_z^{(k)}$$

has no solution for $1 \leq a < b < c$ with $a \leq 10^3$ and some positive integers $x$, $y$ and $z$.
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## 1. Introduction

For $n,\, k \geq 2$ integers, the $k-$generalized Fibonacci sequence $\left\{F_n^{(k)}\right\}$ is given by the recurrence relation

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}$$

with initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \ldots = F_0^{(k)} = 0$ ($k-1$ terms) and $F_1^{(k)} = 1$. If $k = 2$, then we get the Fibonacci sequence denoted by $\{F_n\}$, while we obtain the Tribonacci sequence $\{T_n\}$ when $k = 3$. Now, we give the definition of Diophantine $m-$tuples.

**Definition 1.** *A Diophantine $m-$tuple is a set of $m$ distinct positive integers $\{a_1, a_2, \ldots, a_m\}$ such that $a_i a_j + 1$ is a square for all $1 \leq i < j \leq m$.*

When $m = 3$, it is called a Diophantine triple. For example, the set $\{1, 3, 8\}$ is a Diophantine triple. More generally, $\{F_{2n}, F_{2n+2}, F_{2n+4}\}$ is a family of Diophantine triples for all $n \in \mathbb{Z}_{\geq 1}$.

In number theory, finding Diophantine $m-$tuples, in particular linear recurrences, is a very popular research topic. Recently, some authors proposed variations of Diophantine triples in linear recurrences. Luca and Szalay [12, 13] gave a different

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perspective by replacing the square in Definition 1 with Fibonacci and Lucas numbers. Afterwards, Alp, Irmak and Szalay [1] showed that there are no balancing Diophantine triples. Then, Irmak and Szalay [11] generalized the previous result. Namely, they showed that there is no solution to the system $ab+1=u_x,\ ac+1=u_y$ and $bc+1=u_z$ with $1\le a<b<c$ such that $u_n=Au_{n-1}-u_{n-2},\ u_0=0,\ u_1=1$ for $A\ge 2$ and $n\ge 4$. Recently, the authors in [7] handled a Tribonacci version of this problem by proving that there are only finitely triples $(a,b,c)$ such that

$$\begin{aligned} ab+1&=T_x\\ ac+1&=T_y\\ bc+1&=T_z \end{aligned}$$

holds for $1<a<b<c$ and $k\ge 2$. This problem was generalized in [8] by replacing the Tribonacci numbers by $k-$generalized Fibonacci numbers.

In this paper, motivated by [8], we investigate the solutions to the system

$$\begin{aligned} ab+1&=F_x^{(k)}\\ ac+1&=F_y^{(k)}\\ bc+1&=F_z^{(k)} \end{aligned} \tag{1}$$

for $1\le a<b<c$ and $a\le 10^3$ for some positive integers $x$, $y$ and $z$.

The difference between our result and paper [8] is: Although the authors in [8] prove that system (1) has only *finitely* many solutions for $1<a<b<c$ integers, they did not give any information about what the solutions might be. In this paper, we put an upper bound on the smallest integer $a$. Thanks to this bound, we show that there is *no* solution of system (1). To do this, we use Baker's method on lower bounds for linear forms in logarithms of algebraic numbers. Our result is as follows:

**Theorem 1.** *Assume that $k\ge 2$. There is no sextuple of positive integers $(a,b,c;x,y,z)$ with $1\le a<b<c$ and $a\le 10^3$ that satisfies system (1).*

This theorem encourages us to offer the following conjecture.

**Conjecture 1.** *There is no solution of system (1) in positive integers $(a,b,c;x,y,z)$ with $1\le a<b<c$.*

Note that the condition $1\le a<b<c$ implies that $4\le x<y<z$. Now, we present several lemmas to prove Theorem 1.

## 2. Preliminary results

It is known that the characteristic polynomial of the $k$-generalized Fibonacci numbers is $\varphi(x)=x^k-x^{k-1}-\cdots-1$ and has just one root outside the unit circle. This root is called the dominant root and it is denoted by $\alpha$. We know that it is located between $2\left(1-2^{-k}\right)$ and 2 (see [15]). The following Binet-like formula for $F_n^{(k)}$ was given by Dresden and Du [6]:

$$F_n^{(k)}=\sum_{i=1}^{k}\frac{\alpha_i-1}{2+(k+1)(\alpha_i-2)}\alpha_i^{n-1},$$

where $\alpha = \alpha_1, \ldots, \alpha_k$ are the roots of $\varphi(x)$. They also proved the following result. Let $f_k(x) = \frac{x-1}{2+(k+1)(x-2)}$.

**Lemma 1.** *For $k \geq 2$ and $n \geq 2-k$, the inequality*

$$\left|F_n^{(k)} - f_k(\alpha)\alpha^{n-1}\right| < \frac{1}{2}$$

*holds.*

By using this lemma, the $n^{th}$ term of the $k-$generalized Fibonacci sequence can be written as

$$F_n^{(k)} = f_k(\alpha)\alpha^{n-1} + \xi_n, \tag{2}$$

where $|\xi_n| < \frac{1}{2}$.

Bravo and Luca (Lemma 1 in [3]) gave upper and lower bounds for the $k-$generalized Fibonacci numbers.

**Lemma 2.** *For $n \geq k+2$ and $k \geq 2$,*

$$\alpha^{n-2} < F_n^{(k)} < \alpha^{n-1}$$

*and*

$$\alpha^{n-2.5} < F_n^{(k)} - 1 < \alpha^{n-1}.$$

The following result is Lemma 4 in [3].

**Lemma 3.** *Let $k \geq 2$. If $r > 1$ is an integer satisfying $r - 1 < 2^{k/2}$, then*

$$f_k(\alpha)\alpha^{r-1} = 2^{r-2} + \frac{\delta_{r,k}}{2} + 2^{r-1}\eta_k + \eta_k\delta_{r,k},$$

*where $\delta_{r,k}$ and $\eta_k$ are real numbers such that*

$$|\delta_{r,k}| < 2^{r-k/2} \quad \textit{and} \quad |\eta_k| < 2^{1-k}k.$$

Together with (2), we note that the terms of the sequence $\left\{F_n^{(k)}\right\}$ can be written as

$$\begin{aligned} F_n^{(k)} &= f_k(\alpha)\alpha^{n-1} + \xi_n \\ &= 2^{n-2} + \frac{\delta_{n,k}}{2} + 2^{n-1}\eta_k + \eta_k\delta_{n,k} + \xi_n. \end{aligned} \tag{3}$$

The following result was given by Gómez Ruiz and Luca (see Lemma 3.2 in [9]).

**Lemma 4.** *For $k \geq 2$, $n \geq 1$, the inequality*

$$(F_{n+2}^{(k)} - 1)(F_n^{(k)} - 1) \leq (F_{n+1}^{(k)} - 1)^2$$

*holds.*

By this lemma, we deduce that

$$\frac{F_{n-s}^{(k)}-1}{F_{n+t-s}^{(k)}-1} \leq \frac{F_{n}^{(k)}-1}{F_{n+t}^{(k)}-1} \tag{4}$$

holds for all positive integers $s, t$ with $n \geq s+3$.

Cooper and Howard [5] gave the following formulas for $F_n^{(k)}$.

**Lemma 5.** *For $k \geq 2$ and $n \geq k+2$,*

$$F_n^{(k)} = 2^{n-2} + \sum_{j=1}^{\lfloor \frac{n+k}{k+1} \rfloor - 1} C_{n,j} 2^{n-(k+1)j-2},$$

*where*

$$C_{n,j} = (-1)^j \left[ \binom{n-jk}{j} - \binom{n-jk-2}{j-2} \right].$$

We note by Lemma 5 that

$$F_n^{(k)} = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ 2^{n-2}, & \text{if } 2 \leq n \leq k+1, \\ 2^{n-2} - (n-k)2^{n-k-3}, & \text{if } k+2 \leq n \leq 2k+2. \end{cases} \tag{5}$$

We deduce the following lemma by using Lemma 3 given in [10].

**Lemma 6.** *For $k+2 \leq n \leq 2^{k/2}$, the following estimates hold:*

$$(i) F_n^{(k)} = 2^{n-2}\left(1+\zeta^{'}(n,k)\right), \text{ where } \mid \zeta^{'}(n,k) \mid < \frac{2n}{2^k}.$$

$$(ii) F_n^{(k)} = 2^{n-2}\left(1-\frac{n-k}{2^{k+1}}\zeta^{''}(n,k)\right), \text{ where } \mid \zeta^{''}(n,k) \mid < \frac{4n^2}{2^{2k+2}}.$$

Let $\eta$ be an algebraic number and

$$f(X) = a_0 \prod_{i=1}^{d} \left(X - \eta^{(i)}\right) \in \mathbb{Z}[X]$$

the minimal primitive polynomial of $\eta$ over the integers having a positive leading coefficient $a_0$. The logarithmic height of $\eta$ is given by

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\left\{\left|\eta^{(i)}\right|, 1\right\}\right)\right).$$

Matveev [14] proved the following deep theorem.

**Theorem 2.** *Let $\mathbb{K}$ be a number field of degree $D$ over $\mathbb{Q}$, $\gamma_1, \gamma_2, \ldots, \gamma_t$ positive real numbers of $\mathbb{K}$, and $b_1, b_2, \ldots, b_t$ rational integers. Put*

$$B \geq \max\{|b_1|, |b_2|, \ldots, |b_t|\},$$

*and*

$$\Lambda = \gamma_1^{b_1} \ldots \gamma_t^{b_t} - 1.$$

*Let $A_1, \ldots, A_t$ be real numbers such that*

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \qquad i = 1, \ldots, t.$$

*Then, assuming that $\Lambda \neq 0$, we have*

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 \ldots A_t\right).$$

**Lemma 7.** *Assume that $1 \leq a < b < c$ are integers with $a \leq 10^3$. If $(x, y, z)$ is the solution to equation system (1), then the inequalities*

$$x + y - 32 \leq z \leq x + y$$

*hold.*

**Proof**. By using (1), we get

$$\left(F_x^{(k)} - 1\right)\left(F_y^{(k)} - 1\right) = a^2\left(F_z^{(k)} - 1\right).$$

Together with Lemma 2 and $1 \leq a \leq 10^3$, the inequalities

$$\alpha^{x+y-5} < \left(F_x^{(k)} - 1\right)\left(F_y^{(k)} - 1\right) < 10^6 \alpha^{z-1} < \alpha^{z+27.71}$$

and

$$\alpha^{z-2.5} < F_z^{(k)} - 1 \leq \left(F_x^{(k)} - 1\right)\left(F_y^{(k)} - 1\right) < \alpha^{x+y-2}$$

hold. If we compare the upper and lower bounds of the above inequalities, then we get the claimed result. □

## 3. The proof of Theorem 1

### 3.1. An upper bound for $x$ in terms $k$

Since $1 \leq a < b < c$, then $4 \leq x$. Firstly, assume that $4 \leq x < y < z < k + 2$. By using (5), equation (1) turns to

$$(2^{x-2} - 1)(2^{y-2} - 1) = a^2(2^{z-2} - 1). \tag{6}$$

If $z \geq 9$, then there is a primitive prime factor of $2^{z-2} - 1$ which can not divide $2^{x-2} - 1$ and $2^{y-2} - 1$. So, $z \leq 8$. For $4 \leq x < y < z \leq 8$ and $1 \leq a \leq 10^3$, we do not find any solution to equation (6).

Assume that $4 \le x < y < k+2 \le z$. Since $z \le x+y$, then $z \le 2k+1$. By using (5), we have

$$(2^{x-2}-1)(2^{y-2}-1) = a^2(2^{z-2}-(z-k)2^{z-k-2}). \tag{7}$$

Now, we use the 2−adic order of an integer $n$, denoted by $\nu_2(n)$, defined as the exponent of the highest power of 2 that divides $n$. After taking the 2−adic order of equation (7), we get

$$\begin{aligned}\nu_2((2^{x-2}-1)(2^{y-2}-1)) &= \nu_2(a^2(2^{z-2}-(z-k)2^{z-k-2}))\\ &= \nu_2(a^2 2^{z-k-2}(2^k-(z-k)))\\ &= \nu_2(a^2)+\nu_2(2^{z-k-2})+\nu_2(2^k-(z-k))\\ &= 2\nu_2(a)+(z-k-2)+\nu_2(2^k-(z-k)).\end{aligned}$$

Since $\nu_2((2^{x-2}-1)(2^{y-2}-1)) = 0$, then $\nu_2(a) = 0$, $z-k-2 = 0$ and $\nu_2(2^k-(z-k)) = 0$ must hold, thus $a$ must be an odd integer. Since $z-k-2=0$, then $\nu_2(2^k-(z-k)) = \nu_2(2^k-(k+2-k)) = \nu_2(2^k-2) = 1$, which is not possible.

Now, assume that $4 \le x < k+2 \le y < z$. By the fact that

$$x > \left(1-\frac{k}{k+1}\right)z-2, \tag{8}$$

(see [8] p. 1455) we get $z < (k+1)(k+4)$. For $k \le 142$ and $1 \le a \le 10^3$, we run Mathematica to find the solution to the equation

$$(2^{x-2}-1)(F_y^{(k)}-1) = a^2(F_z^{(k)}-1).$$

We do not find any solutions. Since the case $k > 142$ coindices with Section 3.3, we omit the details for now.

From now on, assume that $k \ge 2$ and $x \ge k+2$. The system of equations (1) gives that

$$\left(F_x^{(k)}-1\right)\left(F_y^{(k)}-1\right) = a^2\left(F_z^{(k)}-1\right). \tag{9}$$

If we use (2) for the terms $F_x^{(k)}$, $F_y^{(k)}$ and $F_z^{(k)}$, then we get

$$\begin{aligned}\left|f_k^2(\alpha)\,\alpha^{x+y-2}-a^2 f_k(\alpha)\,\alpha^{z-1}\right| = \big|&a^2(\xi_z-1)+(1-\xi_y)\,f_k(\alpha)\,\alpha^{x-1}\\ &+(1-\xi_x)\,f_k(\alpha)\,\alpha^{y-1}+(\xi_y-1)(1-\xi_x)\big|.\end{aligned}$$

After dividing both sides by $a^2 f_k(\alpha)\,\alpha^{z-1}$, we get

$$\begin{aligned}\left|1-\frac{f_k(\alpha)\,\alpha^{x+y-z-1}}{a^2}\right| &< \frac{1}{2f_k(\alpha)\,\alpha^{z-1}}+\frac{1}{2\alpha^{z-x}}+\frac{1}{2\alpha^{z-y}}+\frac{1}{4f_k(\alpha)\,\alpha^{z-1}}\\ &< \frac{1}{\alpha^{z-1}}+\frac{1}{2\alpha^{z-x}}+\frac{1}{2\alpha^{z-y}}+\frac{1}{2\alpha^{z-1}},\end{aligned}$$

where we used $1/2 < f_k(\alpha)$ and $|\xi_i-1| < 3/2$ for $i = x, y, z$. By Lemma 7, we deduce the following inequalities:

$$\left|1-\frac{f_k(\alpha)\,\alpha^{x+y-z-1}}{a^2}\right| < \frac{3}{\alpha^{x-32}} < \frac{1.3\cdot 10^{10}}{\alpha^x}. \tag{10}$$

In order to apply Theorem 2, we take $t=2$, $D=k$,

$$\gamma_1 = f_k(\alpha)/a^2, \ \gamma_2 = \alpha$$

with

$$b_1 = 1, \ b_2 = x+y-z-1.$$

Let

$$\Lambda := \gamma_1^{b_1}\gamma_2^{b_2} - 1.$$

To see $\Lambda \neq 0$, we assume on the contrary that $\Lambda = 0$. It gives that $a^2\alpha^{-(x+y-z-1)} = f_k(\alpha)$. We know that $a$, $\alpha$ and $\alpha^{-1}$ are algebraic integers. Therefore, $a^2$ and $\alpha^{-(x+y-z-1)}$ are also algebraic integers. Since the product of two algebraic integers is also an algebraic integer, then $a^2\alpha^{-(x+y-z-1)} = f_k(\alpha)$ is an algebraic integer. But this is a contradiction since $f_k(\alpha)$ is an algebraic number which is not an algebraic integer (see Lemma 2 (i) in [4]). So, we arrive at the claimed fact that $\Lambda \neq 0$.

Since $h(x/y) \leq h(x) + h(y)$, then

$$\begin{aligned} h(f_k(\alpha)/a^2) &\leq h(f_k(\alpha)) + h(a^2) \\ &\leq h(f_k(\alpha)) + 6\log 10 \\ &\leq 4\log k + 6\log 10 = \log(10^6 k^4). \end{aligned}$$

So, we can take $A_1 = k\log(10^6k^4)$ and $\log\alpha < 0.7 = A_2$ (see [2]). By Lemma 7, we take $B = 32$. Applying Theorem 2 to get a lower bound for $|\Lambda|$ and comparing this with inequality (10), we get

$$\exp\left(-C \cdot (1+\log 32)\left(k\log(10^6k^4)\right)0.7\right) < \frac{1.3\cdot 10^{10}}{\alpha^x},$$

where $C = 1.4\cdot 30^5 \cdot 2^{4.5}k^2(1+\log k)$. The above inequality yields

$$x < 3\cdot 10^{11}k^3(\log k)^2. \tag{11}$$

Thus, we found an upper bound for $x$ depending on $k$. From now on, we separate two cases depending on $k$.

## 3.2. The case of small $k$

Assume that $k \leq 142$. So, $x < 2.11\cdot 10^{19}$. Suppose first that $x \leq 39$. By (8), we deduce that $z \leq 5683$. For intervals $2 \leq k \leq 142$, $4 \leq x \leq 39$ and $4 \leq x < z \leq 5683$, we do not find any solution. Now, we assume $x \geq 40$ and $\Gamma := (x+y-z-1)\log\alpha - \log\left(a^2\right) + \log f_k(\alpha)$. Then, according to (10), we get

$$\left|1-e^{\Gamma}\right| < \frac{3}{\alpha^{x-32}} < \frac{1}{2}.$$

This gives

$$\left|(x+y-z-1) - \frac{\log(f_k(\alpha)/a^2)}{\log\alpha}\right| < \frac{6}{(\log\alpha)\alpha^{x-32}} < 0.27. \tag{12}$$

Thus, $x+y-z-1$ is the closest integer to $\log(f_k(\alpha)/a^2)/\log\alpha$. Every choice $(k,a)$ determines $x+y-z-1$ uniquely. We get

$$\|\log(f_k(\alpha)/a^2)/\log\alpha\| < \frac{6}{(\log\alpha)\alpha^{x-32}},$$

where $\|\delta\|$ is the distance to the nearest integer $\delta$. Then there follows

$$x < 32 + \frac{\log(\|(\log(f_k(\alpha)/a^2)/\log\alpha)\|^{-1}6(\log\alpha)^{-1})}{\log\alpha}.$$

For the range $1 \le a \le 10^3$ and $2 \le k \le 142$, we have $x \le 170$. This bound is significantly smaller than $2.11\cdot 10^{19}$. For $1 \le a \le 10^3$, $2 \le k \le 142$ and $4 \le x \le 170$, we choose $a$ for fixed $k$ and $x$ such that

$$F_x^{(k)} - 1 \equiv 0 \pmod{a}.$$

After doing this, we can determine $b$ easily since $F_x^{(k)} = ab+1$. As an example, we give all possible quadruples $(a,b,k,x)$ for $a=100$, $2 \le k \le 142$ and $4 \le x \le 170$ in Table 1. By the system of equations (1), we deduce

$$bF_y^{(k)} - aF_z^{(k)} = b-a.$$

Replacing $F_y^{(k)}$ and $F_z^{(k)}$ with $f_k(\alpha)\alpha^{y-1}$ and $f_k(\alpha)\alpha^{z-1}$ up to error terms we get

$$|\ f_k(\alpha)(b\alpha^{y-1} - a\alpha^{z-1})\ | \le b-a+b/2+a/2 < 3b/2,$$

so

$$\left|\alpha^{z-y} - \frac{b}{a}\right| \le \frac{3b}{2af_k(\alpha)\alpha^{y-1}} < \frac{3F_x^{(k)}}{a^2\alpha^{y-1}} < \frac{3}{a^2\alpha^{y-x}}.$$

For all possible quadruples $(a,b,x,k)$, we find

$$\left|\alpha^{z-y} - \frac{b}{a}\right| > 1.86\cdot 10^{-11}.$$

This gives that $y-x \le 53$. Since $x \le 170$, then $y \le 223$. We get $z \le 393$ as $z \le x+y$. When we run Mathematica again for these ranges, we do not find any solution.

| $a$ | $k$ | $b$ | $x$ |
|---|---|---|---|
| 100 | 2 | 5731478440138170841 | 101 |
| 100 | 3 | 1819976 | 33 |
| 100 | 3 | 193413225694157139589 | 86 |
| 100 | 3 | 7455279114146399174405820972944 | 126 |
| 100 | 3 | 104231474492951572664456508043769360152 | 153 |
| 100 | 4 | 357888291765073737616 | 81 |
| 100 | 5 | 273216628841081891273326618048 | 109 |
| 100 | 6 | 66417785663251578609454911997720785364315267 | 156 |
| 100 | 7 | 9554271045 | 42 |
| 100 | 7 | 1188962767680 | 49 |
| 100 | 7 | 2368371264315 | 50 |
| 100 | 7 | 8942494568691641122893 | 82 |
| 100 | 7 | 2216720361815965764027530 | 90 |
| 100 | 7 | 136211997423281766986544439137 | 106 |
| 100 | 7 | 16950638377816534043010778307968 | 113 |
| 100 | 7 | 33765064758209786319035012176799 | 114 |
| 100 | 7 | 31603029345657117543863449797162015920687142 | 154 |
| 100 | 8 | 647530578 | 38 |
| 100 | 9 | 11095521105671370588 | 72 |
| 100 | 9 | 6290944899536644442893900679592583312850690048 | 161 |
| 100 | 10 | 172843062354 | 46 |
| 100 | 10 | 6061474345574595264831610675 | 101 |
| 100 | 10 | 433001952102395119843136858451317459440888200888 | 167 |
| 100 | 13 | 51603626262316465903503484055716 | 114 |
| 100 | 15 | 3396856447630948497531417354494062756 | 130 |
| 100 | 15 | 14582263681780421297349091832547817402195217613 | 162 |
| 100 | 16 | 2950263200494827602 | 70 |
| 100 | 18 | 28537604459557341748579122993023499276910592 | 153 |
| 100 | 19 | 11528742258952110 | 62 |
| 100 | 19 | 13936330866528320614820309859432314605404 | 142 |
| 100 | 20 | 425331256834605571043870985817686016 | 127 |
| 100 | 21 | 41943 | 24 |
| 100 | 21 | 737861494634530734 | 68 |
| 100 | 24 | 7136211453650213545639831782996091154202624 | 151 |
| 100 | 25 | 13611272974669296714235155848528784261 | 132 |
| 100 | 26 | 435561069443292965498610827889971686277 | 137 |
| 100 | 27 | 25961475975265115286426167017472 | 113 |
| 100 | 37 | 57089907683938445946775318391524906128816210 | 154 |
| 100 | 41 | 43980465111 | 44 |
| 100 | 41 | 850705917285517911249446356268435374 | 128 |
| 100 | 42 | 773712524549316469021737 | 88 |
| 100 | 48 | 1784059615882129769546299841639811861530542 | 149 |
| 100 | 61 | 46116860184273879 | 64 |
| 100 | 63 | 1701411834604692311229447492834689024 | 129 |
| 100 | 70 | 27875931498163278926057815957686741630976 | 143 |
| 100 | 81 | 48357032784585166988247 | 84 |
| 100 | 101 | 50706024009129176059868128215 | 104 |
| 100 | 121 | 53169119831396634916152282411213783 | 124 |
| 100 | 141 | 55751862996326557853839295681620903764951 | 144 |

Table 1: *All quadruples $(a, k, b, x)$ for $a = 100$, $2 \leq k \leq 142$ and $4 \leq x \leq 170$ satisfying the equation $ab + 1 = F_n^{(k)}$.*

### 3.3. The case of large $k$

We assume that $k > 142$. If $4 \le x < k+2 \le y < z$. By (8), we get

$$z < (k+1)(k+4) < 2^{k/2}.$$

If $x \ge k+2$, then we know that $x < 3 \cdot 10^{11} k^3 (\log k)^2$ . By (8), we obtain

$$z < 3 \cdot 10^{11} k^3 (k+1)(\log k)^2 < 2^{k/2}.$$

In any case, the conditions of Lemma 3 are fullfilled. Since

$$a^2 = \frac{(F_x^{(k)} - 1)(F_y^{(k)} - 1)}{(F_z^{(k)} - 1)},$$

then we have the following after using equation (3):

$$\begin{aligned} \left|2^{x+y-4} - a^2 2^{z-2}\right| < & \left|2^{x-2}\left(\frac{\delta_{y,k}}{2} + 2^{x-1}\eta_k + \eta_k\delta_{y,k} + \frac{3}{2}\right)\right| \\ & + \left|2^{y-2}\left(\frac{\delta_{x,k}}{2} + 2^{y-1}\eta_k + \eta_k\delta_{x,k} + \frac{3}{2}\right)\right| \\ & + \left|\left(\frac{\delta_{y,k}}{2} + 2^{x-1}\eta_k + \eta_k\delta_{y,k} + \frac{3}{2}\right)\left(\frac{\delta_{x,k}}{2} + 2^{y-1}\eta_k + \eta_k\delta_{x,k} + \frac{3}{2}\right)\right| \\ & + \left|a^2\left(\frac{\delta_{z,k}}{2} + 2^{z-1}\eta_k + \eta_k\delta_{z,k} + \frac{3}{2}\right)\right|. \end{aligned}$$

Together with Lemma 3, we get

$$\begin{aligned} \left|2^{x+y-4} - a^2 2^{z-2}\right| & < 2^{x-2}\left(6\frac{2^y}{2^{k/2}} + \frac{3}{2}\right) + 2^{y-2}\left(6\frac{2^x}{2^{k/2}} + \frac{3}{2}\right) \\ & + \left(6\frac{2^x}{2^{k/2}} + \frac{3}{2}\right)\left(6\frac{2^y}{2^{k/2}} + \frac{3}{2}\right) + a^2\left(6\frac{2^z}{2^{k/2}} + \frac{3}{2}\right) \\ & < 12\frac{2^{x+y-2}}{2^{k/2}} + 3\frac{2^{x-2}}{2} + 3\frac{2^{y-2}}{2} \\ & + 36\frac{2^{x+y}}{2^k} + 9\frac{2^x}{2^{k/2}} + 9\frac{2^y}{2^{k/2}} + \frac{9}{4} + 6\cdot 10^6\frac{2^z}{2^{k/2}} + 3\frac{10^6}{2}. \end{aligned}$$

After dividing both sides by $2^{z-2}$, we obtain

$$\begin{aligned} \left|a^2 - 2^{x+y-z-2}\right| & < 12\frac{2^{x+y-2}}{2^{k/2}2^{z-2}} + 3\frac{2^{y-2}}{2^{z-2}} \\ & + 36\frac{2^{x+y}}{2^k 2^{z-2}} + 9\frac{2^x}{2^{k/2}2^{z-2}} + 9\frac{2^y}{2^{k/2}2^{z-2}} + \frac{9}{4\cdot 2^{z-2}} \\ & + 6\cdot 10^6\frac{2^z}{2^{k/2}2^{z-2}} + 3\frac{10^6}{2^{z-1}} \\ & \le \frac{12\cdot 2^{32}}{2^{k/2}} + 3\frac{2^{29}}{2^k} + \frac{9\cdot 2^{36}}{2^k} + \frac{9}{2^{k/2}} + \frac{9}{2^k} + \frac{24\cdot 10^6}{2^{k/2}} + 3\frac{10^6}{2^k} \\ & < \frac{5.16\cdot 10^{10}}{2^{k/2}} + \frac{6.21\cdot 10^{11}}{2^k} + \frac{1}{4}, \end{aligned}$$

where we used the inequalities $z-(x+y)>-32$, $y-32<z-x$ and $k+2<y$.

Assume that $a^2-2^{x+y-z-2}\neq 0$. Then we have

$$\frac{1}{2}\leq \min_{\substack{1\leq a\leq 10^3\\ -2\leq x+y-z-2\leq 30}} \left|a^2-2^{x+y-z-2}\right| < \frac{5.16\cdot 10^{10}}{2^{k/2}}+\frac{6.21\cdot 10^{11}}{2^k}.$$

This gives $k\leq 75$. But, this is impossible since $k>142$.

Assume that $a^2-2^{x+y-z-2}=0$. This yields that $a=2^u$ and $x+y-z-2=2u$ for $u\in\{0,1,2,\ldots,9\}$. Let $x\geq k+2$. With (1) and (4), we have

$$\begin{aligned}
2^{2u} &= \frac{(F_x^{(k)}-1)(F_y^{(k)}-1)}{F_{x+y-2u-2}^{(k)}-1}\\
&> \frac{(F_x^{(k)}-1)(F_{k+2}^{(k)}-1)}{F_{x+k-2u}^{(k)}-1} = \frac{(F_x^{(k)}-1)(2^k-2)}{F_{x+k-2u}^{(k)}-1}\\
&\geq \frac{(F_{k+2}^{(k)}-1)(2^k-2)}{F_{2k-2u+2}^{(k)}-1}.
\end{aligned}$$

Since $F_{2k-2u+2}^{(k)}-1=2^{2k-2u}-(k-2u+2)2^{k-2u-1}-1$, see (5), we deduce

$$(2^k-2)^2<2^{2u}(2^{2k-2u}-(k-2u+2)2^{k-2u-1}-1).$$

This yields $k\leq 23$ for $u\in\{0,1,2,\ldots,9\}$, which is not possible since $k>142$.

From now on, assume that $x<k+2$. Since $F_x^{(k)}-1=2^{x-2}-1$ and $a\mid F_x^{(k)}-1$, then $a=1$ and $x+y-2=z$. So, we get the equation

$$(2^{x-2}-1)(F_y^{(k)}-1)=F_{x+y-2}^{(k)}-1. \tag{13}$$

If $y=k+2$ or $y=k+3$, then the above equation turns to the following equations:

$$(2^{x-2}-1)(2^k-2)=2^{x+k-2}-x\cdot 2^{x-3}-1$$

and

$$(2^{x-2}-1)(2^{k+1}-4)=2^{x+k-1}-(x+1)2^{x-2}-1,$$

respectively. None of the above equations is possible since their left-hand sides are even and their right-hand sides are odd. If $y=k+4$, then we get

$$(2^{x-2}-1)(2^{k+2}-9)=2^{x+k}-(x+2)2^{x-1}-1,$$

which yields

$$(2x-5)2^{x-2}+10=2^{k+2}.$$

After taking the 2-adic order of both sides, we get

$$\nu_2((2x-5)2^{x-2}+10)=1 \quad\text{and}\quad \nu_2(2^{k+2})=k+2.$$

So $k+2=1$, which is a contradiction.

Now, we can assume, $y \geq k+5$. Firstly, assume that $x < k/2$. By using Lemma 6 (i), we can write

$$F_y = 2^{y-2}(1+\zeta_y^{'})$$

and

$$F_{x+y-2} = 2^{x+y-4}(1+\zeta^{'}),$$

where $\left|\zeta_y^{'}\right| < \frac{2y}{2^k}$ $\left|\zeta^{'}\right| < \frac{2(x+y-2)}{2^k}$. If we put them into equation (13) there follows:

$$(2^{x-2}-1)(2^{y-2}(1+\zeta_y^{'})-1) = 2^{x+y-4}(1+\zeta^{'})-1.$$

After several calculations, we get

$$\begin{aligned} \left|2^{y-2}-2\right| &= \left|2^{x+y-4}(\zeta_y-\zeta)-2^{x-2}-2^{y-2}\zeta_y\right| \\ &< 2^{x+y-4}(|\zeta_y|+|\zeta|)+2^{x-2}+2^{y-2}\left|\zeta_y\right| \\ &< \frac{2^{x+y-4}(x+2y-2)}{2^{k-1}}+2^{x-2}+\frac{2^{y-2}y}{2^{k-1}}. \end{aligned}$$

If we divide both sides by $2^{y-2}$, then

$$\left|1-\frac{1}{2^{y-3}}\right| < \frac{(x+2y-2)}{2^{k-x+1}}+\frac{1}{2^{y-x}}+\frac{y}{2^{k-1}}$$

holds. Since $x < k/2$, $k+5 \leq y < (k+1)(k+4) < 2^{k/2}$ and $k/2 < y-x$, we obtain

$$\frac{1}{2} < \left|1-\frac{1}{2^{y-3}}\right| < \frac{k/2+3(k+1)(k+4)-1}{2^{k/2-1}}.$$

This yields $k < 23$, which is not possible.

From now on, assume that $k/2 \leq x$. If we use the estimates given in Lemma 6 (ii) for the terms $F_y^{(k)}$ and $F_{x+y-2}^{(k)}$, then we get

$$(2^{x-2}-1)\left(2^{y-2}\left(1-\frac{y-k}{2^{k+1}}+\zeta_y^{''}\right)-1\right) = 2^{x+y-4}\left(1-\frac{x+y-2-k}{2^{k+1}}+\zeta^{''}\right)-1. \tag{14}$$

Then

$$\begin{aligned} \left|\frac{(x-2)2^{x+y-4}}{2^{k+1}}-2^{y-2}\right| &= \left|2^{x+y-4}(\zeta^{''}-\zeta_y^{''})-\frac{2^{y-2}(y-k)}{2^{k+1}}+2^{y-2}\zeta_y^{''}+2^{x-2}-2\right| \\ &< \frac{2^{x+y-2}((k+(k+1)(k+4))^2+((k+1)(k+4))^2)}{2^{2k+2}} \\ &+ \frac{2^{y-2}((k+1)(k+4)-k)}{2^{k+1}}+\frac{2^{y-2}((k+1)(k+4))^2}{2^{2k}}+2^{x-2}+2 \end{aligned}$$

holds, where we used $\left|\zeta^{''}\right| < \frac{4(x+y-2-k)^2}{2^{2k+2}} < \frac{4(k+(k+1)(k+4))^2}{2^{2k+2}}$, $\left|\zeta_y^{''}\right| < \frac{4y^2}{2^{2k+2}} <$

$\frac{4((k+1)(k+4))^2}{2^{2k+2}}$. After dividing both sides by $2^{x+y-4}$, we obtain

$$\begin{aligned}\frac{\left|x-2-2^{k-x+3}\right|}{2^{k+1}} &< \frac{4((k+(k+1)(k+4))^2+((k+1)(k+4))^2)}{2^{2k+2}}\\ &\quad+\frac{((k+1)(k+4)-k)}{2^{k+x-1}}+\frac{4((k+1)(k+4))^2}{2^{2k+x}}+\frac{1}{2^{y-2}}+\frac{1}{2^{x+y-5}}\\ &< \frac{4((k+(k+1)(k+4))^2+((k+1)(k+4))^2)}{2^{2k+2}}\\ &\quad+\frac{((k+1)(k+4)-k)}{2^{3k/2-1}}+\frac{4((k+1)(k+4))^2}{2^{5k/2}}+\frac{1}{2^{k+3}}+\frac{1}{2^{3k/2}},\end{aligned}$$

where we used $k/2 \le x$ and $k+5 \le y$. Since $x < k+2$ and $k-x+3 > 0$, then $x-2-2^{k-x+3} \in \mathbb{Z}$. If $\left|x-2-2^{k-x+3}\right| \ge 1$, we have:

$$\begin{aligned}\frac{1}{2^{k+1}} &< \frac{4((k+(k+1)(k+4))^2+((k+1)(k+4))^2)}{2^{2k+2}}+\frac{((k+1)(k+4)-k)}{2^{3k/2-1}}\\ &\quad+\frac{4((k+1)(k+4))^2}{2^{5k/2}}+\frac{1}{2^{k+3}}+\frac{1}{2^{3k/2}},\end{aligned}$$

which gives $k \le 23$, which is not possible.

Assume that $x-2-2^{k-x+3}=0$. Then equation (14) reduces to

$$2^{x+y-4}\zeta_y^{''}-2^{x-2}-2^{y-2}+\frac{2^{y-2}(y-k)}{2^{k+1}}-2^{y-2}\zeta_y^{''}+2=2^{x+y-4}\zeta^{''}.$$

Then we have

$$\begin{aligned}\left|2^{y-2}-2\right| &= \left|2^{x+y-4}\left(\zeta_y^{''}-\zeta^{''}\right)-2^{x-2}+\frac{2^{y-2}(y-k)}{2^{k+1}}-2^{y-2}\zeta_y^{''}\right|\\ &< \frac{2^{x+y-2}y^2}{2^{2k+2}}+\frac{2^{x+y-2}(x+y-2)^2}{2^{2k+2}}+2^{x-2}+\frac{2^{y-2}(y-k)}{2^{k+1}}+\frac{2^y y^2}{2^{2k+2}}.\end{aligned}$$

When we divide both sides by $2^{y-2}$, then we get

$$\begin{aligned}\frac{1}{2} < \left|1-\frac{1}{2^{y-3}}\right| &< \frac{2^x y^2}{2^{2k+2}}+\frac{2^x(x+y-2)^2}{2^{2k+2}}+\frac{1}{2^{y-x}}+\frac{(y-k)}{2^{k+1}}+\frac{4y^2}{2^{2k+2}}\\ &< \frac{((k+1)(k+4))^2+((k+1)(k+4)+k)^2}{2^k}+\frac{1}{4}+\frac{(k+1)(k+4)-k}{2^{k+1}}\\ &\quad+\frac{4((k+1)(k+4))^2}{2^{2k+2}},\end{aligned}$$

where we used $x < k+2$ and $2 \le y-x$. This gives $k \le 21$, which is a contradiction.

Finally, we complete the proof of Theorem 1.

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