Math. Commun. **29**(2024), 217–236

Fractal behavior of King's optimal eighth-order iterative method and its numerical application

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Received March 6, 2023; accepted April 14, 2024

Abstract. In this paper, the stability of an optimal eighth-order single-parameter King's method is analyzed via fractal behavior. Under the Möbius conjugate map on the Riemann sphere, we study the complex dynamic behavior of this iterative method. Firstly, supported by studying the strange fixed points, we draw the corresponding stability planes. Through defining a unified plane, we also obtain the global stability plane of the strange fixed points. Secondly, by generating the dynamical planes of the iterative method corresponding to the given parameters in the complex plane, we can get a stable parameter family. Finally, by selecting the parameter c in the stable parameter family, we apply the corresponding iterative methods to carry out numerical experiments, which illustrate the effectiveness and stability of these iterative methods.

AMS subject classifications: 37F10, 65H05, 65B99

Keywords: fractal behavior, complex dynamics, iterative methods, Möbius conjugacy, dynamical plane, nonlinear equations

1. Introduction

Fractals and the study of their dynamics is one of the emerging and interesting area for mathematicians. A fractal can be created by the iteration of a rational function on the complex plane. The Julia set [10] and the Mandelbrot set [14] generated by iterative methods for the complex function $f(z) = z^2 + c$ are examples of fractals. The Mandelbrot set is in the parameter space and the Julia set is in the dynamical space. The iteration of a rational function divided the complex plane into two parts, i.e., the Fatou set [8] and the Julia set. The basin of attraction of any fixed point belongs to the Fatou set; in addition, the boundaries of their basins of attraction belong to the Julia set. The parameter spaces and dynamical planes can show the complexity of the iterative method, which provides a new and important analytical tool to explore family members with especially stable behavior and suitable for solving nonlinear problems. More and more researchers are currently committed to studying the stability of the iterative method. Researchers such as Cordero et al. [7, 6, 5], Wang et al. [22, 23], Chicharro et al. [3, 2], and others [17, 20] have described the stability of some famous classes of methods, including Jarratt, King, Chebyshev-Halley, with the help of a parameter space and some dynamical planes.

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It is challenging to discover dynamical analysis of high-order (greater than 4) root seeker families in the literature. The dynamic behavior of a three-point optimal eighth-order single-parameter family will be analyzed in this paper. A variant of the optimal eighth-order method under the Kung-Traub conjecture [12] using the divided difference is proposed by King [11], and it is given by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = \phi_4(x_n, y_n), \\ x_{n+1} = x_n - \dfrac{f(x_n)}{mf(x_n)^2 - nf(x_n) + f'(x_n)}, \end{cases} \tag{1}$$

where the first two steps are the fourth-order iterative scheme, and

$$m = \frac{f'(x_n)[f(z_n)-f(y_n)]+f[x_n,z_n][f(y_n)-f(x_n)]+f[x_n,y_n][f(x_n)-f(z_n)]}{[f(y_n)-f(x_n)][f(y_n)-f(z_n)][f(x_n)-f(z_n)]}, \tag{2}$$

$$n = \frac{-m[f(y_n)-f(x_n)]^2 - f'(x_n) + f[x_n,y_n]}{f(y_n)-f(x_n)}, \tag{3}$$

where $f[\cdot,\cdot]$ is a forward divided difference of order one.

Petković proposed a widely used optimal two-point fourth-order iterative method in [16]. Its iterative scheme is as follows:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - p(t_n)\dfrac{f(y_n)}{f'(x_n)}, t_n = \dfrac{f(y_n)}{f(x_n)}. \end{cases} \tag{4}$$

Choosing $p(t) = \dfrac{t^2+(c-2)t-1}{ct-1} (c \in \mathbb{C})$ in (4), we get the iterative scheme:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f'(x_n)}\{1 + \dfrac{f(y_n)[f(y_n)-2f(x_n)]}{f(x_n)[cf(y_n)-f(x_n)]}\}. \end{cases} \tag{5}$$

Using (5) as the first two steps of method (1), we attain the following single-parametric family of eighth-order methods (OM):

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(y_n)}{f'(x_n)}\{1 + \dfrac{f(y_n)[f(y_n)-2f(x_n)]}{f(x_n)[cf(y_n)-f(x_n)]}\}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{mf(x_n)^2 - nf(x_n) + f'(x_n)}, \end{cases} \tag{6}$$

where m and n are defined earlier in expressions (2) and (3).

Our main interest in dynamical analysis of the proposed family of eighth-order methods (6) is to observe the effect of control parameter c on the dynamical planes describing the evolution of the correlated state space over time. This analysis enables us to avoid elements of bad behavior and adopt the most stable ones.

Now, we recall several primary dynamical concepts [1]. Provided a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the orbit of a point $z_0 \in \hat{\mathbb{C}}$ is defined as:

$$\{z_0, R(z_0), R^2(z_0), \cdots, R^n(z_0), \cdots\}.$$

Starting from the asymptotic behavior of their orbits, we classify the starting points to analyze the phase plane of the map R. $z_f \in \hat{\mathbb{C}}$ is called a fixed point if $R(z_f) = z_f$. Moreover, z_f is called an attractor if $|R'(z_f)| < 1$, a superattractor if $|R'(z_f)| = 0$, a repulsor if $|R'(z_f)| > 1$, and parabolic if $|R'(z_f)| = 1$. A critical point z^* is a point where the derivative of the rational function vanishes, $R'(z^*) = 0$. So, a superattracting fixed point is also a critical point. A periodic point z of period $p > 1$ is a point such that $R^p(z) = z$ and $R^k \neq z$, for $k < p$. A pre-periodic point is a point z that is not periodic but there exists a $k > 0$ such that $R^k(z)$ is periodic.

On the other hand, the basin of attraction of an attractor $\alpha \in \hat{\mathbb{C}}$ is defined as the set of starting points whose orbits tend to α:

$$A(\alpha) = \{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to \alpha, n \to \infty\}.$$

The Fatou set of the rational function R, $F(R)$ is the set of points $z \in \hat{\mathbb{C}}$ whose orbits tend to an attractor. Among them, attractors include fixed points, periodic orbits or infinity. Its complementary set in $\hat{\mathbb{C}}$ is the Julia set, $J(R)$, that is, the basin of attraction of any fixed point belongs to the Fatou set; in addition, the boundaries of their basins of attraction belong to the Julia set [15].

The remaining part of this paper is organized in five sections. Section 2 briefly describes the preliminary study of the dynamics on $\hat{\mathbb{C}}$. In Section 3, an application of the Möbius conjugacy map is investigated and also the strange fixed points of $R_f(z)$ and its stability are analyzed. In Section 4, we draw the dynamical planes of the family (6) elements matched up with free critical points. In Section 5, we perform numerical experiments on the proposed method and other methods. Finally, we summarize the whole paper.

2. Primary conclusions on the Riemann sphere

Definition 1. *Let $f(z)$ be a holomorphic function with no singularity except acnodes defined on the open interval $U \subset \mathbb{C}$. Such a function $f(z)$ is called a meromorphic function.*

Theorem 1. *Let $f(z)$ be a complex function. If $f(z)$ is an infinitely differentiable function on the Riemann sphere $\hat{\mathbb{C}}$ and $f'(z) = 0$, then $f(z)$ is a constant.*

Proof. Let $f(z) = u(x, y)+iv(x, y), (x, y) \in \hat{\mathbb{C}}$. Considering that $f(z)$ is an infinitely differentiable function on $\hat{\mathbb{C}}$, the following conclusions hold:

(1) $u(x, y), v(x, y)$ are differentiable on $\hat{\mathbb{C}}$ and $du = u_x dx+u_y dy, dv = v_x dx+v_y dy$.

(2) $u(x, y), v(x, y)$ satisfy the Cauchy-Riemann condition on $\hat{\mathbb{C}}$: $u_x = v_y, u_y = -v_x$.

(3) $f'(z) = u_x + iv_x = v_y - iu_y$.

According to $f'(z) = 0$, we have $u_x = v_y = 0, u_y = v_x = 0$; then $du = 0, dv = 0$.

Thus $u = c_1, v = c_2$ are established, where c_1, c_2 are constants.

So, $f(z) = u + iv = c_1 + ic_2$, that is, $f(z)$ is a constant. □

Theorem 2. *$f(z)$ is a meromorphic function on the Riemann sphere $\hat{\mathbb{C}}$ if and only if $f(z)$ is a rational function on the Riemann sphere $\hat{\mathbb{C}}$.*

Proof. ($\Leftarrow$) Apparently established.

($\Rightarrow$) Let $f(z)$ be a meromorphic function on the Riemann sphere $\hat{\mathbb{C}}$ and let $z_1, z_2, \cdots, z_k, \infty$ be acnodes of $f(z)$ on the Riemann sphere $\hat{\mathbb{C}}$. By using the Laurent expansion of $z_1, z_2, \cdots, z_k, \infty$, we have:

The singular parts of the Laurent expansion of $z_1, z_2, \cdots, z_k, \infty$ are

$$\phi_j(z) = \frac{c_{-1}^{(j)}}{z - z_j} + \cdots + \frac{c_{-m_j}^{(j)}}{(z - z_j)^{m_j}}, \quad (j = 1, 2, \cdots, k),$$

and $\varphi(z) = c_0 + c_1 z + \cdots + c_m z^m$.

The differentiable parts of the Laurent expansion of $z_1, z_2, \cdots, z_k, \infty$ are $\varphi_j(z)(j = 1, 2, \cdots, k)$ and $\phi(z)$.

Let $F(z) = f(z) - \varphi(z) - \sum_{j=1}^{k} \phi_j(z)$, and $F(z)$ is a meromorphic function on the Riemann sphere $\hat{\mathbb{C}}$. $F(z)$ is infinitely differentiable on the Riemann sphere $\hat{\mathbb{C}}$ without $z_1, z_2, \cdots, z_k, \infty$.

Since

$$\begin{aligned} &\lim_{x \to x_i} F(z) = \varphi_i(z_i) - \varphi(z_i) - \sum_{\substack{j=1 \\ j \neq i}}^{k} \phi_j(z_i)(i = 1, 2, \cdots, k), \\ &\lim_{z \to \infty} F(z) = 0. \end{aligned} \tag{7}$$

Therefore, $z_1, z_2, \cdots, z_k, \infty$ are removable singularities of $F(z)$. $F(z)$ is infinitely differentiable on the Riemann sphere $\hat{\mathbb{C}}$ after a supplementary definition.

Using Theorem 1 and (7), we obtain $F(z) \equiv 0$. So, $f(z) = \varphi(z) - \sum_{j=1}^{k} \phi_j(z)$ can be a rational function; then this proof is completed. $\square$

Lemma 1 (The scaling theorem)**.** *Let $f(z)$ be an analytic function on the Riemann sphere $\hat{\mathbb{C}}$, and let $\Gamma(z) = az + b, a \neq 0$, be an affine map. If $h(z) = f \circ \Gamma(z)$, then the fixed points operator R_f is analytically conjugated to R_h by Γ, that is, $\Gamma \circ R_h \circ \Gamma^{-1} = R_f(z)$.*

Proof. Theorem 1 in [21] has been proved. $\square$

It can be seen from the above lemma that we can conjugate the dynamic behavior of one operator with the related behavior of another operator by affine application.

Definition 2. *Let (X, f) and (Y, g) be two dynamic systems. It is reported that f is conjugate to g via h if there exists a homeomorphism mapping $h : Y \to X$ such that $h \circ f = g \circ h$. Like a mapping, h is called a conjugacy.*

Theorem 3. *Let f and g be defined by Definition 2 of class $\mathbf{C}^1$ let and there exist a conjugacy h between X and Y. Besides, let τ be a fixed point of g. Then, the following hold:*
(a) The fixed point property remains invariant under a topological conjugacy h, that is,

$$\tau = g(\tau),$$

if and only if

$$h(\tau) = f(h(\tau)).$$

(b) The Poincaré characteristic multiplier of τ by g, denoted by $m(g, \tau)$, is invariant under a diffeomorphic conjugacy h, that is,

$$m(h \circ g \circ h^{-1}, h(\tau)) = g'(\tau) = m(g, \tau).$$

Remark 1. *The above theorem states that a conjugacy h indeed preserves the dynamical behavior between the two dynamical systems X and Y, that is, two dynamical systems under a conjugacy h; the fixed point and its multiplier remain constant. For instance, let f and g be conjugate to each other via h; if τ is a fixed point of g, then $g(\tau)$ is a fixed point of f, and vice versa.*

Furthermore, we detect $f = h \circ g \circ h^{-1}$ and

$$f^n = (h \circ g \circ h^{-1}) \circ (h \circ g \circ h^{-1}) \circ \cdots \circ (h \circ g \circ h^{-1}) = h \circ g^n \circ h^{-1}.$$

If f and g are extra invertible, we can also discover $f^{-1} = h \circ g^{-1} \circ h^{-1}$ and $f^{-n} = h \circ g^{-n} \circ h^{-1}$. Then the topological conjugacy h maps an orbit:

$$\cdots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \cdots$$

of f onto an orbit:

$$\cdots, g^{-2}(y), g^{-1}(y), y, g(y), g^2(y), \cdots$$

of g, where $y = h(x)$. Thus, we discover the order of points is preserved.

Consequently, the orbits of the two maps behave similarly to homeomorphism h. It is undoubtedly invaluable to explore the dynamical properties of conjugate maps.

3. Fixed points and stability

Based on topological invariance, we convert R_f in (6) to Π by Möbius conjugacy map $M(z) = \dfrac{z-a}{z-b}$ $(a \neq b)$, satisfying the following:

$$\Pi(z; a, b, s, c) = \frac{\Phi(z; a, b, s, c)}{\Psi(z; a, b, s, c)}, \tag{8}$$

when applied to polynomial $f(z) = (z-a)(z-b)$, where Φ and Ψ are rational polynomials whose coefficients generally depend on parameters a, b, s, c. One of our purposes is to minimize both Φ and Ψ coefficients depending on parameters. Based on the above conjugate, we discover that all the coefficients of both Φ and Ψ are only dependent on the parameter c and independent of others. Accordingly, with the symbolic computation power of Mathematica, (8) can be written as

$$\Pi(z;c)=\frac{z^8\cdot\gamma(z;c)}{\omega(z;c)}, \tag{9}$$

where $\gamma(z;c)=12+106z+426z^2+1028z^3+1690z^4+2029z^5+1856z^6+1315z^7+714z^8+285z^9+78z^{10}+13z^{11}+z^{12}+c^2z(1+z)^4(2+z)^2(1+2z+z^2+z^3)-c(1+z)^2(4+40z+143z^2+268z^3+323z^4+274z^5+170z^6+72z^7+18z^8+2z^9)$, $\omega(z;c)=1+(13-2c)z+(78-22c+c^2)z^2+(285-110c+9c^2)z^3+(714-332c+36c^2)z^4+(1315-686c+87c^2)z^5+(1856-1041c+145c^2)z^6+(2029-1188c+175c^2)z^7+2(845-501c+75c^2)z^8+(1028-594c+85c^2)z^9+(426-227c+28c^2)z^{10}+2(53-24c+2c^2)z^{11}-4(-3+c)z^{12}$.

Lemma 2. *Relation* $\Pi(\frac{1}{z};c)=\frac{1}{\Pi(z;c)}$ *yields* $\forall\ c\in\mathbb{C}$ *and* $\forall\ z\in\hat{\mathbb{C}}$.

Proof. The proof is done using Theorem 3(a) via conjugacy $h(z)=\frac{1}{z}$, $\forall z\in\hat{\mathbb{C}}$. □

Considering that $M(z)$ is a fixed point of Π for a fixed point z of R_f with $M^{-1}(z)=\frac{zb-a}{z-1}$, z is a fixed point of R_f and $M(z)$ is a fixed point of Π under the Möbius conjugacy map $M(z)=\frac{z-a}{z-b}$.

We will explore the fixed points of $\Pi(z;c)$ and their stability. The fixed points of $\Pi(z;c)$ are given by the roots of:

$$\Pi(z;c)-z=\frac{z(z-1)(1+z+z^2)\cdot\theta(z;c)}{\omega(z;c)}, \tag{10}$$

where $\theta(z;c)=1+(13-2c)z+(78-22c+c^2)z^2+(286-110c+9c^2)z^3+(727-334c+36c^2)z^4+(1393-708c+88c^2)z^5+(2142-1151c+154c^2)z^6+(2744-1518c+211c^2)z^7+(2977-1662c+234c^2)z^8+(2744-1518c+211c^2)z^9+(2142-1151c+154c^2)z^{10}+(1393-708c+88c^2)z^{11}+(727-334c+36c^2)z^{12}+(286-110c+9c^2)z^{13}+(78-22c+c^2)z^{14}+(13-2c)z^{15}+z^{16}$.

Let the Möbius conjugacy map $M(z)=\frac{z-a}{z-b}$ be the conjugacy map, and the following properties yield:

$$(i)\ M(a)=0;\quad (ii)\ M(b)=\infty;\quad (iii)\ M(\infty)=1.$$

Evidently, $z=0$ or $z=\infty$ are two of their fixed points of $\Pi=M\circ R_f\circ M^{-1}$, regardless of the value of c. In spite of this, since $\Pi(0)=0,\Pi(\infty)=\infty$, that is, their orbits approach themselves, and such fixed points would have a slight affect on the dynamics, so we do not do too much research on them. The fixed points except $\{0,\infty\}$ are called strange fixed points, which differ from the roots of $f(z)$. To achieve additional strange fixed points, we have to respond to the equation $\Pi(z;c)-z=0$ in (10) for z for a given value of c. Through computation, we achieve the following lemma.

Lemma 3. *Relation* $\theta(\frac{1}{z}) = \frac{\theta(z)}{z^{16}}$ *is obtained,* $\forall\ c \in \hat{\mathbb{C}}$ *and* $\forall\ z \in \hat{\mathbb{C}}$.

Utilizing Theorem 3(a), the following corollary can be proved:

Corollary 1. *Let* Π *be defined as in (9). If* $\eta \in \hat{\mathbb{C}}$ *is any fixed point of* Π*, then so is* $\frac{1}{\eta}$.

We now investigate further strange fixed points and their stability in 3.2.

3.1. The fixed points and the strange fixed points

We first examine the existence of c-values for common factors (divisors) of $\theta(z;c)$ and $\omega(z;c)$. In addition, both $\theta(z;c)$ and $\omega(z;c)$ will be checked whether they have factors $(z-1)$ and $(1+z+z^2)$. The following theorem best describes the related properties of such existence.

Theorem 4.

(a) *If* $c = \frac{1}{2}$*, then* $\theta(z;c)$ *and* $\omega(z;c)$ *have a common factor* $(2+3z+2z^2)^2$.

(b) *If* $c_1 = \frac{41}{12}, c_2 = \frac{233}{60}$*, then* $\omega(z;c)$ *has a factor* $(z-1)$.

(c) *If* $c_3 = \frac{15}{4}, c_4 = \frac{169}{44}$*, then* $\theta(z;c)$ *has a factor* $(z-1)$.

(d) *If* $c_5 = \frac{1}{6}(9-i\sqrt{3}), c_6 = \frac{1}{6}(9+i\sqrt{3})$*, then* $\omega(z;c)$ *has a factor* $(1+z+z^2)$.

(e) *If* $c_7 = \frac{3}{2}, c_8 = 2$*, then* $\theta(z;c)$ *has a factor* $(1+z+z^2)$.

(f) *If* $z \neq 0$ *is a c-dependent strange fixed point of* $\Pi(z;c)$ *found from the roots of* $\theta(z;c) = 0$ *for* $c \notin \{\frac{1}{2}, \frac{41}{12}, \frac{233}{60}, \frac{15}{4}, \frac{169}{44}, \frac{3}{2}, 2, \frac{1}{6}(9-i\sqrt{3}), \frac{1}{6}(9+i\sqrt{3})\}$*, then so is* $\frac{1}{z}$*. The strange fixed points can be found from the 16 numerical roots of* $\theta(z;c) = 0$ *for a given c.*

Proof. (a) Suppose that $\theta(z;c) = 0$ and $\omega(z;c) = 0$ for some values of $z \in \mathbb{C}$. By eliminating c from the two polynomials, we obtain the relation $(2+3z+2z^2)^3 = 0$. Hence, $(2+3z+2z^2), (2+3z+2z^2)^2, (2+3z+2z^2)^3$ are candidates for common divisors of $\theta(z;c)$ and $\omega(z;c)$. First, dividing both $\theta(z)$ and $\omega(z)$ to get the same remainder $1-2c$, and when $c = \frac{1}{2}$, the remainder is 0. Certainly, $\theta(z)$ and $\omega(z)$ reduce to $\frac{1}{4}(2+3z+2z^2)^2(1+9z+36z^2+84z^3+136z^4+179z^5+202z^6+179z^7+136z^8+84z^9+36z^{10}+9z^{11}+z^{12})$ and $\frac{1}{4}(2+3z+2z^2)^2(1+9z+36z^2+83z^3+127z^4+143z^5+118z^6+53z^7+10z^8)$, respectively.

Hence, we find that $(2+3z+2z^2)^2$ is indeed a common divisor of $\theta(z)$ and $\omega(z)$. The remaining proof only needs to substitute the corresponding $\theta(z)$ and $\omega(z)$ into (10) and answer its roots to be the strange fixed points.

(b) We find $\omega(1) = 9553 - 5256c + 720c^2 = 0$ for $c_1 = \frac{41}{12}, c_2 = \frac{233}{60}$ to have a divisor $(z-1)$.

(c) We find $\theta(1) = 7(2535 - 1336c + 173c^2) = 0$ for $c_3 = \frac{15}{4}$, $c_4 = \frac{169}{44}$ to have a divisor $(z-1)$.

(d) We find $\omega(-(-1)^{\frac{1}{3}}) = \frac{1}{2}i(i+\sqrt{3})(7-9c+3c^2) = 0$ and $\omega((-1)^{\frac{2}{3}}) = -\frac{1}{2}i(-i+\sqrt{3})(7-9c+3c^2) = 0$ for $c_5 = \frac{1}{6}(9-i\sqrt{3})$, $c_6 = \frac{1}{6}(9+i\sqrt{3})$ to have a factor $(1+z+z^2)$.

(e) We find $\theta(-(-1)^{\frac{1}{3}}) = \frac{1}{2}i(i+\sqrt{3})(6-7c+2c^2) = 0$ and $\theta((-1)^{\frac{2}{3}}) = -\frac{1}{2}i(-i+\sqrt{3})(6-7c+2c^2) = 0$ for $c_7 = \frac{3}{2}$, $c_8 = 2$ to have a factor $(1+z+z^2)$.

(f) Considering Corollary 1, we find simply that $\frac{1}{z}$ is also a fixed point of $\Pi(z;c)$. We numerically acquire the strange fixed points z satisfying $\theta(z;c) = 0$ for given values of $c \notin \{\frac{1}{2}, \frac{41}{12}, \frac{233}{60}, \frac{15}{4}, \frac{169}{44}, \frac{3}{2}, 2, \frac{1}{6}(9-i\sqrt{3}), \frac{1}{6}(9+i\sqrt{3})\}$. By Corollary 1, if we know eight roots $\eta_1, \eta_2, \cdots, \eta_8$ of $\theta(z;c) = 0$, then we can factor $\theta(z;c) = \prod_{i=1}^{8}(z-\eta_i)(z-\frac{1}{\eta_i})$. A detailed analysis for the 16 roots of $\theta(z;c)$ will be shown later in Lemma 5. □

3.2. Stability of the fixed points and the strange fixed points

In order to determine the stability of the fixed points, the derivative of Π needs to be calculated from (9):

$$\Pi'(z;c) = \frac{z^7 \cdot \kappa(z;c)}{\omega^2(z;c)}, \tag{11}$$

where $\kappa(z;c) = 96+2046z+20900z^2+136126z^3+635276z^4+2265333z^5+6431636z^6+14962571z^7+29127592z^8+48190555z^9+68527336z^{10}+84396809z^{11}+90421212z^{12}+84396809z^{13}+68527336z^{14}+48190555z^{15}+29127592z^{16}+14962571z^{17}+6431636z^{18}+2265333z^{19}+635276z^{20}+136126z^{21}+20900z^{22}+2046z^{23}+96z^{24}+c^4z^3(1+z)^8(28+216z+737z^2+1548z^3+2339z^4+2684z^5+2339z^6+1548z^7+737z^8+216z^9+28z^{10})-c^3z^2(1+z)^6(88+1108z+6284z^2+21577z^3+51234z^4+91048z^5+126754z^6+141238z^7+126754z^8+91048z^9+51234z^{10}+21577z^{11}+6284z^{12}+1108z^{13}+88z^{14})+c^2z(1+z)^4(92+1696z+14019z^2+70226z^3+242301z^4+619986z^5+1235288z^6+1980658z^7+2609564z^8+2857420z^9+2609564z^{10}+1980658z^{11}+1235288z^{12}+619986z^{13}+242301z^{14}+70226z^{15}+14019z^{16}+1696z^{17}+92z^{18})-c(1+z)^2(32+900z+10582z^2+73529z^3+348188z^4+1215218z^5+3286530z^6+7132012z^7+12734214z^8+19043274z^9+24135662z^{10}+26100534z^{11}+24135662z^{12}+19043274z^{13}+12734214z^{14}+7132012z^{15}+3286530z^{16}+1215218z^{17}+348188z^{18}+73529z^{19}+10582z^{20}+900z^{21}+32z^{22})$.

Obviously, fixed points 0 and ∞, related to the roots of polynomial $f(z) = (z-a)(z-b)$, are superattracting points and also the critical points of $\Pi(z;c)$ owing to the right-hand side of (11). From the position of Corollary 2 (to be shown later), the multiplier $\Pi'(z;c) = 0$ at $z = \infty$ is found from the analyticity of the neighborhood of ∞ on the Riemann sphere. The rest of this subsection will describe in more detail the desired stability of the fixed points. By direct calculation, we get the following lemma.

Lemma 4. *Relation* $\kappa(\frac{1}{z}) = \frac{\kappa(z)}{z^{24}}$ *is obtained,* $\forall\ \lambda \in \mathbb{C}$ *and* $\forall\ z \in \hat{\mathbb{C}}$.

We first test the existence of c-values for common factors (divisors) of $\kappa(z)$ and $\omega(z)$. In addition, both $\kappa(z)$ and $\omega(z)$ will be examined whether they have factors of z^7.

The following theorem best describes the related properties of such existence as well as explicit strange fixed points.

Theorem 5.
(a) If $c = \frac{1}{2}$, *then*

$$\Pi'(z;c) = \frac{z^7 \cdot \kappa_1(z)}{(1+9z+36z^2+83z^3+127z^4+143z^5+118z^6+53z^7+10z^8)^2},$$

where $\kappa_1 = 80 + 1107z + 7156z^2 + 28637z^3 + 79852z^4 + 166487z^5 + 272732z^6 + 362873z^7 + 398632z^8 + 362873z^9 + 272732z^{10} + 166487z^{11} + 79852z^{12} + 28637z^{13} + 7156z^{14} + 1107z^{15} + 80z^{16}$.

(b) If $c = 3$, *then*

$$\Pi'(z;c) = \frac{-z^8 \cdot \kappa_2(z)}{(1+7z+21z^2+36z^3+42z^4+40z^5+38z^6+40z^7+34z^8+11z^9-3z^{10}-2z^{11})^2},$$

where $\kappa_2(z) = 18 + 142z + 362z^2 - 242z^3 - 4041z^4 - 13124z^5 - 26720z^6 - 42016z^7 - 57292z^8 + 72106z^9 - 84311z^{10} - 89280z^{12} - 72106z^{13} - 57292z^{14} - 42016z^{15} - 26720z^{16} - 13124z^{17} - 4041z^{18} - 242z^{19} + 362z^{20} + 142z^{21} + 18z^{22}$.

(c) If $c \neq \frac{1}{2}, 3$, *then, with the help of Theorem 7 (explained later in this section), the expected stability can be graphically realized by changing the parameter* c.

Proof. (a) Suppose that $\kappa(z) = 0$ and $\omega(z) = 0$ for some values of z. By eliminating c from the two polynomials, we obtain the relation: $(2+3z+2z^2)(1+16z+118z^2+541z^3+1737z^4+4108z^5+7196z^6+8835z^7+5772z^8-3302z^9-14643z^{10}-20772z^{11}-16749z^{12}-5137z^{13}+6150z^{14}+10877z^{15}+9002z^{16}+4648z^{17}+1444z^{18}+190z^{19}-24z^{20}-8z^{21}) = 0$. Hence $(2+3z+2z^2)$ is a candidate for common divisors of $\kappa(z)$ and $\omega(z)$. After checking constraints

$$\kappa(\frac{1}{4}(-3-i\sqrt{7})) = \omega(\frac{1}{4}(-3-i\sqrt{7})) = 0, \kappa(\frac{1}{4}(-3+i\sqrt{7})) = \omega(\frac{1}{4}(-3+i\sqrt{7})) = 0,$$

we find that $c = \frac{1}{2}$, yielding common divisors.

(b) For $\kappa(z)$ to have a factor z, we obtain $c = 3$ by solving $\kappa(0) = -32(-3+c) = 0$ for c.

(c) Graphical stability of z can be easily realized by Theorem 7. □

Corollary 2. *Let* $\eta \in \hat{\mathbb{C}}$ *be any fixed point of* Π *defined in (9). Then the following relation holds:*

$$\Pi'(\eta;c) = \Pi'(\frac{1}{\eta};c),$$

for any $c \in \mathbb{C}$.

The consequences of Theorem 5 are utilized to express the stability of the fixed points in Theorem 4, no matter what value c is. Table 1 shows the fixed points corresponding to the special parameter c in Theorem 4 and the stability of these fixed points, while c in Table 1 denotes different parameters, η denotes the strange fixed point, and g, f, e denote the repulsive, parabolic, and attractive fixed points, respectively.

c	η					No. of η
			$\lvert\Pi'(\eta;c)\rvert : m$			
$\frac{1}{2}$	1	−0.5 ± 0.866025i	−0.662565	1.05715	−3.32978 ± 0.737957i	19
	6.64828 : g	3.76923 : g	1.16483 : g	9.10288 : g	1505.36 : g	
	−2.67946 ± 1.89601i	−1.43902 ± 2.41158i	−0.490573 ± 0.356419i	−0.0456346 ± 2.24362i	0.188601 ± 0.577421i	
	1299.54 : g	740.797 : g	0.793037 : e	320.503 : g	0.826408 : e	
	0.936008 ± 1.43909i					
	103.467 : g					
$\frac{41}{12}$	1	−0.5 ± 0.866025i	−1.57805 ± 0.527117i	−0.570084 ± 0.190425i	−0.0101692 ± 0.999948i	19
	7.35119 : g	2.44547 : g	57.1192 : g	57.116 : g	3.8602 : g	
	0.86664 ± 0.498933i	−1.73287 ± 0.544017i	−0.525304 ± 0.164914i	−0.343419 ± 0.939182i	0.809927 ± 0.58653i	
	5.39743 : g	35.4899 : g	35.49 : g	0.398073 : e	12.1007 : g	
$\frac{233}{60}$	1	−0.5 ± 0.866025i	0.716729	1.39523	−1.57199 ± 0.485643i	19
	65.6845 : g	2.55757 : g	2.35323 : g	2.35337 : g	60.7642 : g	
	−0.580713 ± 0.179403i	0.0383904 ± 0.999263i	0.821502	1.21728	−1.72297 ± 0.490367i	
	60.7634 : g	2.64291 : g	3.31121 : g	3.31126 : g	36.6368 : g	
	−0.536903 ± 0.152805i	−0.317849 ± 0.948141i				
	36.6394 : g	0.507215 : e				
$\frac{15}{4}$	1($triple$)	−0.5 ± 0.866025i	1.57356 ± 0.496872i	−0.577884 ± 0.182475i	0.0264408 ± 0.99965i	17
	1 : m	2.53037 : g	70.7198 : g	59.7887 : g	3.76707 : g	
	−1.7255 ± 0.505059i	−0.533808 ± 0.156247i	−0.324346 ± 0.945938i	0.958656 ± 0.284569i		
	36.3568 : g	36.3582 : g	0.480419 : e	4.00659 : g		
$\frac{169}{44}$	1($triple$)	−0.5 ± 0.866025i	−1.72376 ± 0.494992i	−0.535935 ± 0.153899i	−0.319854 ± 0.947467i	17
	1 : m	2.54926 : g	36.5549 : g	36.5536 : g	0.499004 : e	
	0.759641	1.15969	−1.57248 ± 0.489167i	−0.579829 ± 0.180374i	0.0347331 ± 0.999397i	
	1.15974 : g	1.15969 : g	60.4538 : g	60.4599 : g	3.74768 : g	
$\frac{1}{6}(9-i\sqrt{3})$	1	−0.5 ± 0.866025i	−1.92143 + 0.818354i	−1.79644 − 0.91219i	−0.630021 − 0.864907i	19
	6.90218 : g	4.1659 : g	31.5087 : g	27.494 : g	4.0327 : g	
	−0.550242 + 0.755383i	−0.44255 + 0.224716i	−0.440533 − 0.18627i	−0.131936 + 0.93079i	0.149286 − 1.0532i	
	4.03275 : g	27.4944 : g	31.5098 : g	1.44337 : g	4.5422 : g	
	−1.72507 + 0.772758i	−1.59466 − 0.847944i	−0.559538 − 0.969453i	0.488867 + 0.259949i	−0.482804 − 0.216276i	
	41.0434 : g	33.6117 : g	3.84089 : g	0.233621 : e	41.0417 : g	
	−0.446587 + 0.773754i	0.385411 + 0.886933i	0.41212 − 0.948397i			
	3.84085 : g	4.41108 : g	4.41107 : g			
$\frac{1}{6}(9+i\sqrt{3})$	1	−0.5 ± 0.866025i	−1.72507 − 0.772758i	−1.59466 + 0.847944i	−0.559538 + 0.969453i	19
	6.90218 : g	99046.8 : g	41.0434 : g	33.6117 : g	3.84089 : g	
	−0.488867 − 0.259949i	−0.482804 + 0.216276i	−0.446487 − 0.773754i	0.385411 − 0.886933i	0.41212 + 0.948397i	
	33.6102 : g	41.0417 : g	3.84417 : g	4.41108 : g	4.41107 : g	
	−1.92143 − 0.818354i	−1.79644 + 0.91219i	−0.630021 + 0.864907i	−0.550242 − 0.755383i	−0.44255 − 0.224716i	
	31.5087 : g	27.494 : g	4.0327 : g	4.03275 : g	27.4944 : g	
	−0.440533 + 0.187627i	−0.131936 − 0.93079i	0.149286 + 1.0532i			
	31.5098 : g	1.44337 : g	4.5422 : g			
$\frac{3}{2}$	1	−0.5 ± 0.866025i	−1.66835 ± 0.813296i	−0.484304 ± 0.236092i	0.402652 ± 0.915353i	17
	6.91882 : g	0.999987 : e	37.8519 : g	37.8524 : g	4.44271 : g	
	−1.86617 ± 0.865842i	−0.59002 ± 0.807389i	−0.440939 ± 0.204582i	0.147123 ± 0.989118i		
	29.8889 : g	4.15236 : g	29.8884 : g	4.60883 : g		
2	1	−0.5 ± 0.866025i	−1.80299 ± 0.762859i	−0.47042 ± 0.199038i	0.273409 ± 0.961898i	17
	7.19833 : g	1.00001 : g	30.4498 : g	30.4498 : g	7.10865 : g	
	−1.62146 ± 0.712126i	−0.517006 ± 0.227063i	−0.331734 ± 0.943373i	0.470196 ± 0.882562i		
	42.4157 : g	42.4172 : g	3.92442 : g	4.9401 : g		

$|\Pi'(\eta;c)| : m$ denotes that η is attractive, parabolic and repulsive if $m = e(|\Pi'| < 1)$, $m = f(|\Pi'| = 1)$, $m = g(|\Pi'| > 1)$, respectively

Table 1: *Stability of strange fixed points η for special c-values*

Next, we will discuss the stability of the fixed point $z = 1$ when $c \notin \{\frac{1}{2}, \frac{41}{12}, \frac{233}{60}, \frac{15}{4}, \frac{169}{44}, \frac{3}{2}, 2, \frac{1}{6}(9-i\sqrt{3}), \frac{1}{6}(9+i\sqrt{3})\}$.

Theorem 6. *Let $c \notin \{\frac{1}{2}, \frac{41}{12}, \frac{233}{60}, \frac{15}{4}, \frac{169}{44}, \frac{3}{2}, 2, \frac{1}{6}(9-i\sqrt{3}), \frac{1}{6}(9+i\sqrt{3})\}$; then the related properties of the strange fixed point $z = 1$ are as follows:*

(*a*) *If $c = \frac{1}{276}(1041 \pm 14\sqrt{3})$, then $z = 1$ is a superattracting point;*

(*b*) *If $\frac{169}{44} < |c| < \frac{4821 + 14\sqrt{105}}{1284}$ or $\frac{4821 - 14\sqrt{105}}{1284} < |c| < \frac{15}{4}$, then $z = 1$ is* *an attractive point;*

(*c*) *If* $|c| = \dfrac{4821 + 14\sqrt{105}}{1284}$ *or* $|c| = \dfrac{4821 - 14\sqrt{105}}{1284}$, *then* $z = 1$ *is a parabolic fixed point;*

(*d*) *Finally, if* $|c| < \dfrac{169}{44}$ *or* $|c| > \dfrac{4821+14\sqrt{105}}{1284}$ *or* $|c| > \dfrac{15}{4}$ *or* $|c| < \dfrac{4821-14\sqrt{105}}{1284}$, *then* $z = 1$ *is a repulsive point.*

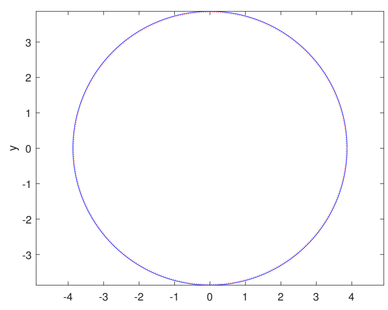
Proof. Substituting $z = 1$ into $\Pi'(z; c)$, we achieve

$$|\Pi'(1; c)| = |\frac{62788 - 33312c + 4416c^2}{9553 - 5256c + 720c^2}| = |\frac{4(15697 - 8328c + 1104c^2)}{(-41 + 12c)(-233 + 60c)}|.$$

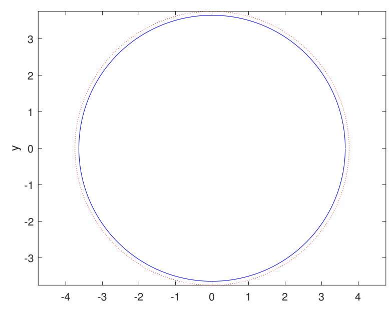
It is easy to confirm that $|\Pi'(1; \frac{1}{276}(1041 \pm 14\sqrt{3}))| = 0$.

Let $c = x + iy$ be an arbitrary complex number, and solve $|\Pi'(1; c)| \leq 1$. Then we can get $\dfrac{169}{44} < |c| \leq \dfrac{4821 + 14\sqrt{105}}{1284}$ or $\dfrac{4821 - 14\sqrt{105}}{1284} \leq |c| < \dfrac{15}{4}$. □

Figure 1 shows the stability plane for the strange fixed point $z = 1$.



(a) $(\frac{169}{44})^2 < x^2 + y^2 \leq (\frac{4821 + 14\sqrt{105}}{1284})^2$



(b) $(\frac{4821 - 14\sqrt{105}}{1284})^2 \leq x^2 + y^2 < (\frac{15}{4})^2$

Figure 1: *Stability planes for* $z = 1$

Further properties of $\theta(z)$ will be described in the following lemma to improve the desired graphic stability.

Lemma 5. *Let* $c \notin \{\frac{1}{2}, 3\}$ *and let* $\eta_1, \eta_2, \cdots, \eta_8 \in \mathbb{C}\backslash\{0\}$ *be eight roots of* $\theta(z)$. *Then,* $\theta(z)$ *can be factored out with eight second-degree polynomials in the form of* $\theta(z) = \prod_{i=1}^{8}(1 + d_i z + z^2) = \prod_{i=1}^{8}(z - \eta_i)(z - \frac{1}{\eta_i})$, *where* $d_i = -(\eta_i + \frac{1}{\eta_i})$ *or* $\eta_i = \frac{1}{2}(-d_i - \sqrt{-1 + d_i^2})$ *for* $1 \leq i \leq 8$ *in terms of* c.

Proof. Owing to Lemma 3 and Corollary 1, we find that if $\eta \neq 0$ is a strange fixed point of $\Pi(z; c)$ found from the roots of $\theta(z) = 0$ for $c \notin \{\frac{1}{2}, 3\}$, then so is $\frac{1}{\eta}$.

Consequently, the above factorization is valid. Then $\prod_{i=1}^{8}(z-\eta_i)(z-\frac{1}{\eta_i})$ is easily acquired under the sixteen-degree polynomials $\theta(z)$. $(1+d_i z+z^2)$ can be obtained by expanding $(z-\frac{1}{\eta_i})$ and it is obvious that $d_i=-(\eta_i+\frac{1}{\eta_i})$ holds.

With the aim of continuing to explore the stability of other fixed points from the root of $\theta(z)$, we need to find the root of $\theta(z)$ first by answering $\theta(z)=0$. Then, we obtain $\theta(z)=\theta_1(z)\cdot\theta_2(z)$, where $\theta_1(z)=1+(6-c)z+(15-4c)z^2+(20-6c)z^3+(21-6c)z^4+(20-6c)z^5+(15-4c)z^6+(6-c)z^7+z^8$, $\theta_2(z)=1+(7-c)z+(21-5c)z^2+(35-10c)z^3+(41-12c)z^4+(35-10c)z^5+(21-5c)z^6+(7-c)z^7+z^8$.

By comparing the expansion of $\prod_{i=1}^{4}(1+d_i z+z^2)$ with the coefficients of the same order term of $\theta_1(z)$ and $\theta_2(z)$, we find that the following two groups of four equations are related to $d_1, d_2, \cdots, d_8$,

$$\begin{cases} d_1+d_2+d_3+d_4=6-c, \\ 4+d_1d_2+d_1d_3+d_2d_3+d_1d_4+d_2d_4+d_3d_4=15-4c, \\ 3(d_1+d_2+d_3+d_4)+d_1d_2d_3+d_1d_2d_4+d_1d_3d_4+d_2d_3d_4=20-6c, \\ 6+2(d_1d_2+d_1d_3+d_1d_4+d_2d_3+d_2d_4+d_3d_4)+d_1d_2d_3d_4=21-6c. \end{cases} \tag{12}$$

and

$$\begin{cases} d_5+d_6+d_7+d_8=7-c, \\ 4+d_5d_6+d_5d_7+d_6d_7+d_5d_8+d_6d_8+d_7d_8=21-5c, \\ 3(d_5+d_6+d_7+d_8)+d_5d_6d_7+d_5d_6d_8+d_5d_7d_8+d_6d_7d_8=35-10c, \\ 6+2(d_5d_6+d_5d_7+d_5d_8+d_6d_7+d_6d_8+d_7d_8)+d_5d_6d_7d_8=41-12c. \end{cases}$$

We eliminate variables d_2, d_3, d_4 in (12) utilizing Mathematica to ultimately access a decic equation in d_1 below:

$$(2+3d_2-4d_1^2+d_1^3)c-2d_1+11d_1^2-6d_1^3+d_1^4-7=0$$

For convenience, we let $\vartheta(d_1;c)$ denote $(2+3d_2-4d_1^2+d_1^3)c-2d_1+11d_1^2-6d_1^3+d_1^4-7$. Eliminating d_1, d_3, d_4;$d_1, d_2, d_4$and d_1, d_2, d_3, respectively, we acquire a unique decic equation in $r\in\{d_1, d_2, d_3, d_4\}$ as follows: $\vartheta(r;c)=0$.

Next, we answer the roots of $\vartheta(r;c)$ and get:

$$\begin{aligned} d_1(c)&=\frac{6-c}{4}-\frac{1}{2}M(c)-\frac{1}{2}N(c), \\ d_2(c)&=\frac{6-c}{4}-\frac{1}{2}M(c)+\frac{1}{2}N(c), \\ d_3(c)&=\frac{6-c}{4}+\frac{1}{2}M(c)-\frac{1}{2}N(c), \\ d_4(c)&=\frac{6-c}{4}+\frac{1}{2}M(c)+\frac{1}{2}N(c), \end{aligned}$$

where:

$$M(c)=\sqrt{-11+\frac{1}{3}(11-4c)+\frac{1}{4}(-6+c)^2+4c+\frac{\iota(c)}{3o(c)}+\frac{o(c)}{\rho(c)}},$$

$$N(c)=\sqrt{-11+\frac{1}{2}(-6+c)^2+4c+\frac{1}{3}(-11+4c)-\frac{\iota(c)}{3o(c)}-\frac{o(c)}{\rho(c)}-\frac{\varrho(c)}{4M(c)}},$$

$\iota(c) = 2^{1/3}(1-4c+7c^2)$, $\varrho(c) = 4(11-4c)(-6+c)-(-6+c)^3-8(-2+3c)$, $o(c) = (322-204c+21c^2+34c^3-\sqrt{\nu(c)})^{1/3}$, $\rho(c) = 3\times 2^{1/3}$, $\nu(c) = 103680-131328c+54864c^2+14256c^3-15363c^4+3780c^5-216c^6$.

Solving the expression for d_5, d_6, d_7, d_8 is the same method.

Then, substituting $d_i(c)$ for $\eta_i = \dfrac{1}{2}(-d_i-\sqrt{-4+d_i^2})$, $i = 1,2,3,4,5,6,7,8$. The result is a c-dependent strange fixed point of Π. $\square$

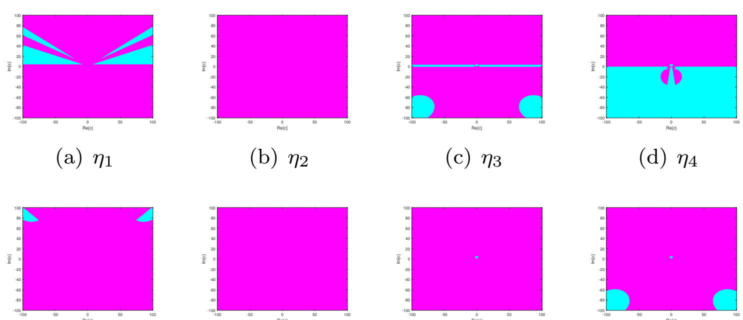
Definition 3 (Unified image(line or plane))**.** *Let $Q = \{(x,y)\in\mathbb{N}\times\mathbb{N} : 1 < x < Q_x, 1 < y < Q_y\}$, where Q_x and Q_y are the number of pixels of the image Y and $D = \{z\in\mathbb{Z} : 0\le z\le 255\}$. Let $Y_B, Y_G, Y_R : Q\to D$ be the intensity of blue, green and red of the pixels of an image Y, respectively.*

The binary image $T : Q\to\{0,1\}$ is defined as

$$T = \begin{cases} 0, if & Y_R = 0, Y_G = 1, Y_B = 1 \\ 1, if & Y_R = 1, Y_G = 0, Y_B = 1 \end{cases}$$

Let $X_F = \{x_{F,k}, i = 1,2,\ldots,k\}$ be the set of k strange fixed points, and let F_i be their associated binary images, $i = 1,2,\ldots,k$. The unified stability images $V : T\to\{0,1\}$ are defined as $V = \prod_{i=1}^{k} F_i$.

In order to demonstrate the stability for other c-dependent strange fixed points η contributed from the roots of $\theta(z) = 0$ in (10), we would better use the graphical method described in Algorithm 2.4 of [9]. For each root η_j of $\theta(\eta) = 0$, the stability planes are shown in Figure 2 for each root η_j of $\theta(\eta) = 0$.



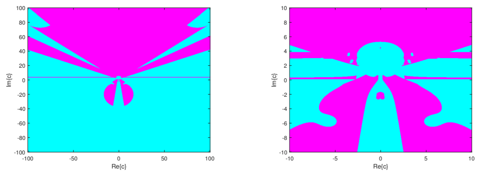
(a) η_1 (b) η_2 (c) η_3 (d) η_4

(e) η_5 (f) η_6 (g) η_7 (h) η_8

Figure 2: *Stability planes of the strange fixed points η_j, $1\le j\le 8$ from the roots of $\theta(\eta)$*

Figure 2 shows the stability planes of strange fixed points depending on parameter c.

In figures 2-3, we use a plane of 500×500 to represent the stability plane of the fixed point, and the maximum number of iterations is 25. According to Definition 3, Figure 2 can be uniformly represented by Figure 3. Among them, the cyan region indicates that the fixed point is attractive. We are mainly interested in the parameters in the pink region, because the iterative methods corresponding to the parameter values in the pink region do not converge to extra fixed points. Comparing



(a) Unified stability plane (b) Detail of (a)

Figure 3: *Unified stability plane*

Figure 3 (b) with Figure 2 in reference [13], it can be seen that the OM method is more stable than the method in [13] in the same range.

4. Dynamical planes

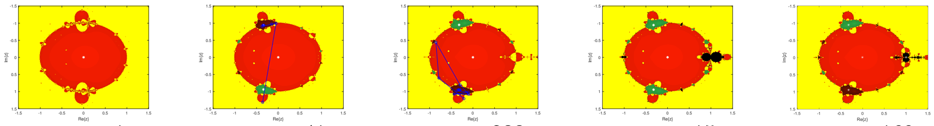
In the previous sections, by analyzing theorems 4-6, we obtain some special parameter c-values. Combined with Figure 3, we find that most of these special values are in the pink region. Therefore, we will use the dynamical planes to analyze the stability of the iterative method corresponding to these special parameter values.

This section gives the dynamical plane of the new family OM (6) with a given c-value. By using Matlab to run the program in [3], we obtain the dynamical planes of (6). We use a 500×500 points grid to generate dynamical planes, the maximum number of iterations is 50. Each color represents an attraction basin: convergence to 0 and ∞ is represented by red and yellow in sequence. If it converges to the fixed point $z = 1$, it shows green. If it does not converge to any root, it shows black. And we draw the track in blue. In addition, we use white $*$ to represent the attractive points.

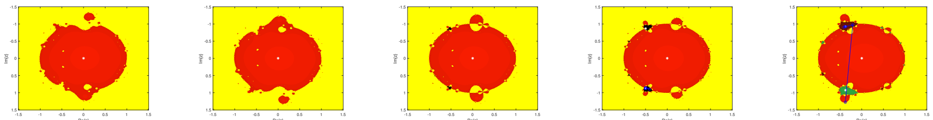
Firstly, we show the dynamical planes corresponding to the special c-values studied in theorems 4 and 5, that is, $c \in \{\frac{1}{2}, \frac{41}{12}, \frac{233}{60}, \frac{15}{4}, \frac{169}{44}, \frac{1}{6}(9-i\sqrt{3}), \frac{1}{6}(9+i\sqrt{3}), \frac{3}{2}, 2, 3\}$ in Figure 4, respectively. Figures 4(a), 4(f) and 4(g) have only red and yellow regions, that is, in 50 iterations, it only converges to 0 and ∞, indicating that the c-value has good convergence; other special values have black areas, but it can be seen from Figure 4(h) and Figure 4(i) that the black area of Figure 4(h) is smaller than that of Figure 4(i), indicating that $c = \frac{3}{2}$ is relatively stable.

In addition, when $c = -2i, 7 + 7i, 2.92 + 6.92i$, they correspond to the pink area in Figure 3, and when $c = 2.5i, -6i, 7.6 + 1.8i$, they correspond to the cyan area. By drawing the dynamical planes corresponding to these parameters, it can be seen that even if the stable parameter value is selected in the stable region, the corresponding dynamical planes also have black regions, that is, the region that does not converge to the roots, see Figure 5.

Besides, $c = \frac{1}{2}, \frac{9 \pm i\sqrt{3}}{6}$ correspond to the pink region in Figure 3, that is, the iterative method does not converge to the extra fixed points. By observing figures $4(a)$, $4(f)$, $4(g)$ and figures $5(a)$, $5(b)$, $5(c)$, it can be concluded that the stability of the iterative methods corresponding to the parameter values in the cyan region must not be good, and the c-values in the pink region are more stable.

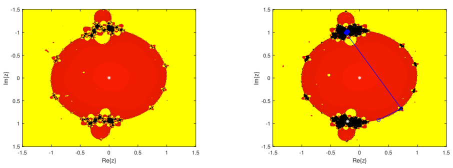


(a) $c = \dfrac{1}{2}$ (b) $c = \dfrac{41}{12}$ (c) $c = \dfrac{233}{60}$ (d) $c = \dfrac{15}{4}$ (e) $c = \dfrac{169}{44}$

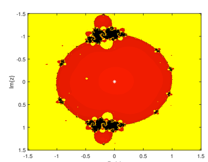


(f) $c = \dfrac{9 - i\sqrt{3}}{6}$ (g) $c = \dfrac{9 + i\sqrt{3}}{6}$ (h) $c = \dfrac{3}{2}$ (i) $c = 2$ (j) $c = 3$

Figure 4: *Dynamical planes of special c values*



(a) $c = -2i$ (b) $c = 7 + 7i$ (c) $c = 2.92 + 6.92i$



.

(d) $c = 2.5i$ (e) $c = -6i$ (f) $c = 7.6 + 1.8i$

Figure 5: *Dynamical planes of special c values*

In the above discussion, we can see that these methods corresponding to $c = \frac{1}{2}, \frac{9 \pm i\sqrt{3}}{6}$ are superior to the other members of the new family $OM(6)$.

5. Numerical experiments

In this section, based on the results obtained in the previous sections, we conduct several numerical experiments.

5.1. First experiment: Solving nonlinear equations

Table 2 gives four nonlinear equations and their zeros and the initial values of subsequent iterations. Tables 3-6 show the numerical performance of the iterative methods related to the c-values involved in theorems 4 and 5, which are used to compare the effects of different parameter selections on the convergence accuracy.

Tables 3-6 show the comparison results of the effective digits of each test function after 4 iterations. It can be seen from tables 3-6 that the accuracy is higher when $c = 3$. Regardless of $c = \frac{1}{6}(9 - i\sqrt{3})$ or $c = \frac{1}{6}(9 + i\sqrt{3})$), their operation results are the same.

i	$f_i(x)$	x_0	α
1	$10xe^{-x^2} - 1$	1.7	1.6796306104284499
2	$cos(\frac{\pi}{2}x) + x^2 - \pi$	2	2.0347248962791266
3	$11x^{11} - 1$	0.8	0.80413309750366432
4	$cos(x) - xe^x + x^2$	0.5	0.6391540963

Table 2: *The function $f_i(x)$ initial guesses x_0 and zeros α*

c	$\|x_1 - x_0\|$	$\|x_2 - x_1\|$	$\|x_3 - x_2\|$	$\|x_4 - x_3\|$
1/2	2.0369×10^{-2}	1.4167×10^{-13}	6.4532×10^{-103}	1.1962×10^{-817}
41/12	2.0369×10^{-2}	2.4701×10^{-14}	1.0196×10^{-109}	8.5893×10^{-873}
233/60	2.0369×10^{-2}	4.9603×10^{-14}	5.4598×10^{-107}	1.1762×10^{-850}
15/4	2.0369×10^{-2}	4.2534×10^{-14}	1.365×10^{-107}	1.5353×10^{-855}
169/44	2.0369×10^{-2}	4.7358×10^{-14}	3.5955×10^{-107}	3.9695×10^{-852}
$(1/6)(9 - i\sqrt{3})$	2.0369×10^{-2}	8.4034×10^{-14}	5.986×10^{-105}	3.968×10^{-834}
$(1/6)(9 + i\sqrt{3})$	2.0369×10^{-2}	8.4034×10^{-14}	5.986×10^{-105}	3.968×10^{-834}
3/2	2.0369×10^{-2}	8.2444×10^{-14}	5.04×10^{-105}	9.8315×10^{-835}
2	2.0369×10^{-2}	5.371×10^{-14}	1.0758×10^{-106}	2.787×10^{-848}
3	2.0369×10^{-2}	2.0867×10^{-15}	2.2401×10^{-119}	3.952×10^{-951}

Table 3: *Comparison for test function f_1 and special c-values*

c	$\|x_1 - x_0\|$	$\|x_2 - x_1\|$	$\|x_3 - x_2\|$	$\|x_4 - x_3\|$
1/2	3.4725×10^{-2}	2.8598×10^{-13}	4.9201×10^{-102}	3.7759×10^{-812}
41/12	3.4725×10^{-2}	4.2907×10^{-14}	2.0192×10^{-109}	4.8573×10^{-872}
233/60	3.4725×10^{-2}	9.2261×10^{-14}	1.9942×10^{-106}	9.4995×10^{-848}
15/4	3.4725×10^{-2}	7.8247×10^{-14}	4.5181×10^{-107}	5.5831×10^{-853}
169/44	3.4725×10^{-2}	8.781×10^{-14}	1.277×10^{-106}	2.555×10^{-849}
$(1/6)(9 - i\sqrt{3})$	3.4725×10^{-2}	1.721×10^{-13}	5.1895×10^{-104}	3.5466×10^{-828}
$(1/6)(9 + i\sqrt{3})$	3.4725×10^{-2}	1.721×10^{-13}	5.1895×10^{-104}	3.5466×10^{-828}
3/2	3.4725×10^{-2}	1.6907×10^{-13}	4.4225×10^{-104}	9.6922×10^{-829}
2	3.4725×10^{-2}	1.1228×10^{-13}	1.1211×10^{-104}	1.1071×10^{-841}
3	3.4725×10^{-2}	1.8859×10^{-15}	1.0277×10^{-121}	7.9933×10^{-972}

Table 4: *Comparison for test function f_2 and special c-values*

c	$\lvert x_1 - x_0\rvert$	$\lvert x_2 - x_1\rvert$	$\lvert x_3 - x_2\rvert$	$\lvert x_4 - x_3\rvert$
$1/2$	4.1331×10^{-3}	6.799×10^{-14}	3.1765×10^{-100}	7.2104×10^{-791}
$41/12$	4.1331×10^{-3}	1.9676×10^{-14}	5.0902×10^{-105}	1.0214×10^{-829}
$233/60$	4.1331×10^{-3}	3.2545×10^{-14}	4.7093×10^{-103}	9.0517×10^{-841}
$15/4$	4.1331×10^{-3}	2.8898×10^{-14}	1.6148×10^{-103}	1.5348×10^{-817}
$169/44$	4.1331×10^{-3}	3.1387×10^{-14}	3.3982×10^{-103}	6.4147×10^{-815}
$(1/6)(9 - i\sqrt{3})$	4.1331×10^{-3}	3.7441×10^{-14}	1.5072×10^{-102}	1.0395×10^{-809}
$(1/6)(9 + i\sqrt{3})$	4.1331×10^{-3}	3.7441×10^{-14}	1.5072×10^{-102}	1.0395×10^{-809}
$3/2$	4.1331×10^{-3}	3.6442×10^{-14}	1.1801×10^{-102}	1.4276×10^{-810}
2	4.1331×10^{-3}	2.1272×10^{-14}	9.2806×10^{-105}	1.2181×10^{-827}
3	4.1331×10^{-3}	7.9324×10^{-15}	1.487×10^{-108}	2.2678×10^{-858}

Table 5: *Comparison for test function f_3 and special c-values*

c	$\lvert x_1 - x_0\rvert$	$\lvert x_2 - x_1\rvert$	$\lvert x_3 - x_2\rvert$	$\lvert x_4 - x_3\rvert$
$1/2$	1.3915×10^{-1}	2.0403×10^{-8}	4.0858×10^{-63}	1.0567×10^{-500}
$41/12$	1.3915×10^{-1}	4.3804×10^{-9}	6.5978×10^{-69}	1.7479×10^{-547}
$233/60$	1.3915×10^{-1}	7.1755×10^{-9}	5.4981×10^{-67}	6.533×10^{-532}
$15/4$	1.3915×10^{-1}	6.4004×10^{-9}	1.9653×10^{-67}	1.5534×10^{-535}
$169/44$	1.3915×10^{-1}	6.9308×10^{-9}	4.0226×10^{-67}	5.1789×10^{-533}
$(1/6)(9 - i\sqrt{3})$	1.3915×10^{-1}	1.037×10^{-8}	1.0031×10^{-65}	7.6881×10^{-522}
$(1/6)(9 + i\sqrt{3})$	1.3915×10^{-1}	1.037×10^{-8}	1.0031×10^{-65}	7.6881×10^{-522}
$3/2$	1.3915×10^{-1}	1.0113×10^{-8}	7.9531×10^{-66}	1.164×10^{-522}
2	1.3915×10^{-1}	5.7728×10^{-9}	5.063×10^{-68}	1.7726×10^{-540}
3	1.3915×10^{-1}	1.6764×10^{-9}	1.3898×10^{-72}	3.1028×10^{-577}

Table 6: *Comparison for test function f_4 and special c-values*

5.2. Second experiment: The comparison of solving matrix sign function

The OM iterative method has advantages not only in terms of solving nonlinear equations, but also in terms of solving matrix sign functions. Firstly, we consider a nonlinear matrix function

$$X^2 - I = 0, \tag{13}$$

where I is an identity matrix. We can obtain the reciprocal form of the following iterative by applying (13) to (6), where $c = 3$:

$$\begin{aligned} X_{n+1} = {} & 4(X_n - 15X_n^3 + 196X_n^5 - 652X_n^7 + 2686X_n^9 + 23694X_n^{11} + 55284X_n^{13} + 39812X_n^{15} \\ & + 9417X_n^{17} + 649X_n^{19}) \times [I - 10X_n^2 + 117X_n^4 + 392X_n^6 - 2574X_n^8 + 42116X_n^{10} \\ & + 165570X_n^{12} + 218248X_n^{14} + 88013X_n^{16} + 12150X_n^{18} + 265X_n^{20}]^{-1} \end{aligned} \tag{14}$$

Besides, we will apply (14) to resolve the matrix sign for the famous Wilson matrix. It has the following form:

$$A = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix}.$$

In the calculation process, we use Matlab to record the number of iterations and computer running time to meet the stopping termination $R_n = \| X_n^2 - I \|_\infty \leq 10^{-100}$.

At the same time, we also carry out the same numerical experiments on the following eighth-order iterative methods, and get the experimental results, see Table 7. Table 8 shows the experimental results of four methods applied to solve random matrices of different orders. From tables 7 and 8, we can obtain that the OM method has advantages in both iterative count and CPU time.

Method 1

The method by Sharma et al. $M1$, see [18], is

$$\begin{cases} w_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = w_n - \dfrac{f(w_n)}{f'(x_n)} \dfrac{f(x_n) + af(w_n)}{f(x_n) + (a-2)f(w_n)}, \\ x_{n+1} = x_n - \dfrac{P+Q+R}{Pf[z_n, x_n] + Qf'(x_n) + Rf[w_n, x_n]} f(x_n), \end{cases}$$

where $P = (x_n - w_n)f(x_n)f(w_n)$, $Q = (w_n - z_n)f(w_n)f(z_n)$ and $R = (z_n - x_n)f(z_n)f(x_n)$. Here, we select $a = 1$.

Method 2

The method by Sharma and Sharma $M2$, see [19], is

$$\begin{cases} w_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = w_n - \dfrac{f(x_n)}{f(x_n) - 2f(w_n)} \dfrac{f(w_n)}{f'(x_n)}, \\ x_{n+1} = z_n - [1 + \dfrac{f(z_n)}{f(x_n)} + \alpha \dfrac{f(z_n)^2}{f(x_n)^2}] \dfrac{f[x_n, w_n]f(z_n)}{f[w_n, z_n]f[x_n, z_n]}, \end{cases}$$

where $\alpha \in \mathbb{R}$, and we select $\alpha = 1$, here.

Method 3

The method by Chun and Lee $M3$, see [4], is

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(y_n)}{f'(x_n)} \dfrac{1}{(1 - \frac{f(y_n)}{f(x_n)})^2}, \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f'(x_n)} \dfrac{1}{(1 - H(t_n) - J(s_n) - P(u_n))^2}, \end{cases}$$

where

$$H(t_n) = -\beta - \gamma + t_n + \frac{t_n^2}{2} - \frac{t_n^3}{2}, J(s_n) = \beta + \frac{s_n}{2}, P(u_n) = \gamma + \frac{u_n}{2},$$

and $t_n = \dfrac{f(y_n)}{f(x_n)}$, $s_n = \dfrac{f(z_n)}{f(x_n)}$, $u_n = \dfrac{f(z_n)}{f(y_n)}$, $\beta, \gamma \in \mathbb{R}$. We select $\beta = \gamma = 1$.

	OM	$M1$	$M2$	$M3$
Number of iterations	4	5	4	5
Time	$0.034760s$	$0.061319s$	$0.080200s$	$0.087021s$

Table 7: *Comparison results* (1)

	OM		$M1$		$M2$		$M3$	
Matrix No.	n[1]	Time	n	Time	n	Time	n	Time
$R_{4\times4}$	2	0.018940	2	0.025878	3	0.039122	5	0.103771
$R_{5\times5}$	3	0.015431	4	0.028522	6	0.012384	13	0.144543
$R_{8\times8}$	3	0.015703	4	0.031779	4	0.037287	45	0.191979
$R_{10\times10}$	4	0.065667	5	0.051149	6	0.080007	35	0.157630
$R_{15\times15}$	5	0.034656	6	0.041282	5	0.051692	148	0.360147
$R_{20\times20}$	4	0.061849	5	0.078071	7	0.130622	42	0.276220

[1] n represents the number of iterations

Table 8: *Comparison results* (2)

6. Conclusions

In this paper, the stability of an optimal eighth-order single-parameter OM iterative method for solving nonlinear equations is analyzed. Applying to the prototype polynomial $f(z) = (z-a)(z-b)$, iterative maps (9) under the Möbius conjugacy map are obtained, and the strange fixed points and their stability are analyzed. At the same time, we have investigated the complex dynamical behavior of OM at fixed and critical points. Through the analysis of the stability planes in figures 2-3 and dynamical planes in figures 4-5, we conclude that if the parameters are selected in the cyan region, the stability of the corresponding iterative method must not be good. However, parameters in the pink area are more stable. We verify the conclusion by drawing their dynamical planes. In particular, when $c = \frac{1}{2}, \frac{1}{6}(9 \pm i\sqrt{3})$, their corresponding iterative methods are the most stable. Numerical experiments show that the iterative method studied in this paper is superior to other eighth-order methods in some cases, whether in solving nonlinear equations or solving matrix sign function, especially $c = 3$.

Acknowledgement

This work was supported by the National Natural Science Foundation of China (No. 61976027), the Natural Science Foundation of Liaoning Province (No. 2022-MS-371), the Educational Commission Foundation of Liaoning Province of China (Nos. LJKMZ20221492, LJKMZ20221498) and the Key Project of Bohai University (No. 0522xn078).

The author would like to thank the referees for their helpful suggestions.

References

[1] P. Blanchard, *Complex analytic dynamics on the Riemann sphere*, B. Am. Math. Soc. **11**(1984), 85–141.

[2] F. Chicharro, A. Cordero, J. M. Gutiérrez, J. R. Torregrosa, *Complex dynamics of derivative-free methods for nonlinear euqations*, Appl. Math. Comput. **219**(2013), 7023–7035.

[3] F. Chicharro, A. Cordero, J. R. Torregrosa, *Drawing dynamical and parameters planes of iterative families and methods*, SCI. World J. **2013**(2013), 780153.

[4] C. Chun, M. Y. Lee, *A new optimal eighth-order family of iterative methods for the solution of nonlinear equations*, Appl. Math. Comput. **223**(2013), 506–519.

[5] A. Cordero, L. Guasp, J. R. Torregrosa, *Choosing the most stable members of Kou's family of iterative methods*, J. Comput. Appl. Math. **330**(2018), 759–769.

[6] A. Cordero, C. Jordán, E. Sanabria-Codesal, J. R. Torregrosa, *Design, Convergence and Stability of a Fourth-Order Class of Iterative Methods for Solving Nonlinear Vectorial Problems*, Fractal Fract. **5**(2021), 125–125.

[7] A. Cordero, M. Moscoso-Martínez, J. R. Torregrosa, *Chaos and Stability in a New Iterative Family for Solving Nonlinear Equations*, Algorithms **14**(2021), 101.

[8] P. Fatou, *Sur les équations fonctionnelles*, B. Soc. Math. Fr. **47**(1919), 161–271.

[9] Y. H. Geum, Y. I. Kim, Á. A. Magreñán, *A study of dynamics via Möbius conjugacy map on a family of sixth-order modified Newton-like multiple-zero finders with bivariate polynomial weight functions*, J. Comput. Appl. Math. **344**(2018), 608–623.

[10] G. Julia, *Mémoire sure l'itération des fonctions rationnelles*, J. Math. Pures Appl. **8**(1918), 47–245.

[11] Y. I. Kim, R. Behl,S. S. Motsa, *An Optimal Family of Eighth-Order Iterative Methods with an Inverse Interpolatory Rational Function Error Corrector for Nonlinear Equations*, Math. Model. Anal. **22**(2017), 321–336.

[12] H. T. Kung, J. F. Traub, *Optimal Order of One-Point and Multipoint Iteration*, J. Acm. **21**(1974), 643–651.

[13] F. I. C. López, A. Cordero, N. Garrido, J. R. Torregrosa, *On the choice of the best members of the Kim family and the improvement of its convergence*, Math. Method. Appl. Sci. **43**(2020), 8051–8066.

[14] B. B. Mandelbrot, J. A. Wheeler, *The Fractal Geometry of Nature*, Am. J. Phys. **51**(1983), 286–287.

[15] S. Mitsuhiro, *The Hausdorff Dimension of the Boundary of the Mandelbrot Set and Julia Sets*, Ann. Math. **147**(1998), 225–267.

[16] L. D. Petković, M. S. Petković, J. DžUnić, *A class of three-point root-solvers of optimal order of convergence*, Appl. Math. Comput. **216**(2010), 671–676.

[17] D. Sharma, I. K. Argyros, S. K. Parhi, S. K. Sunanda, *Local Convergence and Dynamical Analysis of a Third and Fourth Order Class of Equation Solvers*, Fractal Fract. **5**(2021), 27–27.

[18] J. R. Sharma, K. G. Rangan, P. Gupta, *Improved King's methods with optimal order of convergence based on rational approximations*, Appl. Math. Lett. **26**(2013), 473–480.

[19] J. R. Sharma, R. Sharma, *A new family of modified Ostrowski's methods with accelerated eighth order convergence*, Numer Algor. **54**(2010), 445–458.

[20] A. Tassaddiq, S. Qureshi, A. Soomro, E. Hincal, D. Baleanu, A. A. Shaikh, *A New Three-Step Root-Finding Numerical Method and Its Fractal Global Behavior*, Fractal Fract. **5**(2021), 204–204.

[21] X. Wang, X. Chen, *The Dynamical Analysis of a Biparametric Family of Six-Order Ostrowski-Type Method under the Möbius Conjugacy Map*, Fractal Fract. **6**(2022), 174.

[22] X. Wang, W. Li, *Choosing the Best Members of the Optimal Eighth-Order Petković's Family by Its Fractal Behavior*, Fractal Fract. **6**(2022), 749–749.

[23] X. Wang, W. Li, *Stability Analysis of Simple Root Seeker for Nonlinear Equation*, Axioms **12**(2023), 215.