MATHEMATICAL COMMUNICATIONS
Math. Commun. **29**(2024), 237–251

A note on the asymptotic analysis of heat transfer in flow through a dilated pipe

Andrijana Ćurković[1,*] and Eduard Marušić-Paloka[2]

[1] *Faculty of Science, University of Split, Ruđera Boškovića 33, HR-21 000 Split, Croatia*
[2] *Department of Mathematics, Faculty of Science and Mathematics, University of Zagreb, BijeniÄŊka cesta 30, HR-10 000 Zagreb, Croatia*

Received May 31, 2023; accepted November 7, 2023

Abstract. The flow of a fluid through a pipe subjected to heating is studied. The fluid is governed by the pressure drop, and the heat exchange between the fluid and the environment is described by Newton's cooling law. The longitudinal heat expansion of the pipe is taken into account. Error estimates for the asymptotic approximation in the extended domain are derived.

AMS subject classifications: 80M35, 35Q35

Keywords: asymptotic analysis, heat transfer problem, dilated pipe, linear heat expansion law

1. Intoduction

The study of fluid flows subjected to heating has been initiated by situations occurring in heat exchangers, underground piping structures, geothermal systems, etc. (see [1, 5, 6, 7, 8]). In such systems, pipe walls are subjected to a large temperature gradient during heat transfer, and as materials expand when heated and contract when cooled, pipes also expand and contract at different temperatures. We assume that longitudinal expansion is described by the law of linear heat expansion: The length changes by an amount proportional to the original length and the temperature change. The proportionality coefficient is the heat expansion, a small parameter of size 10^{-5}, which allows us to search for an approximate solution by asymptotic analysis. The asymptotic behavior of the solutions of partial differential equations with respect to a small parameter, which is a physical parameter [9] or a feature of the domain shape, is intensively studied in applied mathematics. Domains are usually considered to be rods, plates [3], tubes [1, 2, 11], etc., as well as some unions of these elements [12]. The problem is usually recognized for its engineering importance and studied in mechanics, without rigorous estimates for the difference between the exact solution and the approximate solution. They were later studied mathematically. The mathematical justification implies the estimation of the difference between the exact solution and its asymptotic approximation or gives some convergence theorems.

*Corresponding author. *Email addresses:* `andrijana@pmfst.hr` (A. Ćurković), `emarusic@math.hr` (E. Marušić-Paloka)

https://www.mathos.unios.hr/mc

In the present paper, we study the asymptotic behavior of the fluid temperature in a heat conduction problem in a dilated pipe. The temperature is the solution of the convection-diffusion equation with a steady state Poiseuille velocity. Due to the pipe dilation, the fluid domain is no longer fixed and changes depending on the unknown temperature. By introducing a suitable change of variables, the domain becomes fixed, but the PDE becomes nonlinear. The existence and uniqueness of the solution was considered [10], as well as the asymptotic behavior on a fixed domain [9]. The main goal of this paper is to give an analysis of the approximation on the original extended domain. In Section 2, we give a problem statement and observation on the difference between the pipe length for the exact fluid temperature and its approximation, as well as a note on variable change functions for exact and approximated solutions.

The study of thin structures is motivated by the engineering of industrial equipment, so in Section 3 we assume that the pipe has a circular cross-section and denote by the small parameter δ the ratio between the thickness and the length of the pipe. For the chosen parameters, we give a graphical representation of the variable change function, the first order approximation and corrector. This visualization illustrates the importance of considering the approximation on the original, unfixed domain. In Section 4, error estimates are derived mathematically.

2. Flow through a dilated pipe

We assume that the cylindrical domain is filled with a fluid and that the fluid inside the pipe is heated (or cooled) by the surrounding medium and its temperature is described by a stationary convection-diffusion equation. Due to the temperature change, the pipe changes its length in the longitudinal direction, which is equal to 1 at the reference temperature, while the cross-section remains constant and is assumed to be circular in this paper. The length change in the longitudinal direction is proportional to the temperature change. Since the fluid temperature equation is considered in a dilated domain, the average temperature change of the pipe is also considered in a dilated domain. Thus, the length change is described by the heat expansion law

$$l_{\theta^\epsilon} = 1 + \epsilon \int_0^{l_{\theta^\epsilon}} \langle \theta^\epsilon \rangle(\xi)\, d\xi, \tag{1}$$

where l_{θ^ϵ} stands for the pipe's length, $\epsilon \ll 1$ is a small parameter denoting the heat expansion coefficient and θ^ϵ is the increment in the temperature, i.e. the difference between the temperature of the pipe or fluid and a reference temperature. The mean temperature over the pipe's cross-section is given by $\langle \theta^\epsilon \rangle$. This value is approximated by the mean value of the fluid temperature per cross-section ω. Namely,

$$\langle \theta^\epsilon \rangle(\cdot) = \frac{1}{|\omega|} \int_\omega \theta^\epsilon(\cdot, y, z)\, dydz$$

is used instead of $1/|\partial\omega| \int_{\partial\omega} \theta^\epsilon(\cdot, y, z)\, dS_{y,z}$.

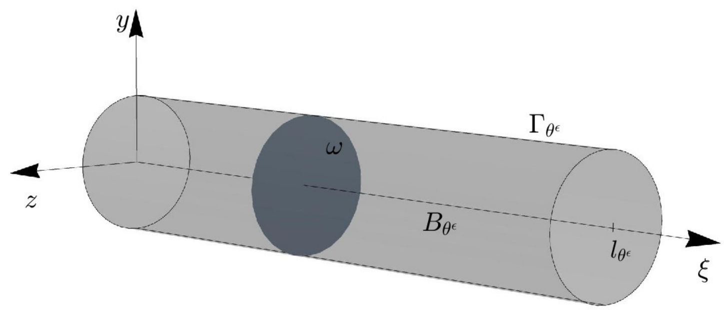

Figure 1: *The flow domain* B_{θ^ϵ}

The fluid-pipe system is heated by the surrounding medium of the exterior temperature $f = f(\xi, y, z)$ of class C^1 and we consider the stationary convection-diffusion equation for the fluid temperature in the domain $B_{\theta^\epsilon} = (0, l_{\theta^\epsilon}) \times \omega$ supplemented with the mixed Dirichlet-Robin boundary conditions:

$$-\kappa\left(\frac{\partial^2\theta^\epsilon}{\partial\xi^2}+\frac{\partial^2\theta^\epsilon}{\partial y^2}+\frac{\partial^2\theta^\epsilon}{\partial z^2}\right)+u\frac{\partial\theta^\epsilon}{\partial\xi}=0 \qquad \text{in } B_{\theta^\epsilon} \tag{2}$$

$$\kappa\frac{\partial\theta^\epsilon}{\partial\boldsymbol{n}}=\sigma(f-\theta^\epsilon) \qquad \text{on } \Gamma_{\theta^\epsilon}=(0,l_{\theta^\epsilon})\times\partial\omega \tag{3}$$

$$\theta^\epsilon(0,\cdot,\cdot)=\theta_l,\ \theta^\epsilon(l_{\theta^\epsilon},\cdot,\cdot)=\theta_r \qquad \text{on } \omega. \tag{4}$$

The positive constants κ and σ are independent of ϵ and stand for thermal conductivity of the fluid and the heat transfer coefficient, respectively, $\boldsymbol{n}$ denotes the exterior unit normal on the lateral boundary Γ_{θ^ϵ}, while θ_l and θ_r are prescribed constant temperatures on the ends of the pipe. We study the laminar regime of the flow (governed by a prescribed pressure drop) and therefore assume that the fluid velocity u has Poiseuille form:

$$u = (p_l - p_r)v,$$

where v is the parabolic profile described by

$$\begin{aligned} -\mu\triangle v &= 1 \quad \text{in } \omega \\ v &= 0 \quad \text{on } \partial\omega. \end{aligned}$$

Here μ stands for the viscosity of the fluid, while p_l and p_r represent the prescribed (constant) pressures at the pipe's left and right end, respectively.

The main difficulty arises from the nonlinearity of the coupled system due the fact that the domain B_{θ^ϵ} depends on the solution θ^ϵ. The existence and uniqueness of the weak solution $\theta^\epsilon \in H^1(B_{\theta^\epsilon}) \cap L^\infty(B_{\theta^\epsilon})$ under the assumption that

ϵ is small enough have been proven in [10]. We emphasize that the assumption $\epsilon \cdot \max\{|\theta_l|, |\theta_r|, \|g\|_{L^\infty}\} < 1/4$, by a maximum principle (see [10, Proposition 1.]), yields

$$\epsilon |\langle \theta^\epsilon \rangle| < \frac{1}{4}. \tag{5}$$

It is obvious that one cannot hope to derive the exact solution of such nonlinear problem. Therefore we study the appropriate asymptotic model defined on the fixed fluid domain obtained by introducing the variable change

$$x = \xi - \epsilon \int_0^\xi \langle \theta^\epsilon \rangle (s)\, ds. \tag{6}$$

Due to the inequality (5), this mapping is strictly increasing bijection from B_{θ^ϵ} in $B_1 = (0,1) \times \omega$. Considering the heat law (1), one can imagine that the variable ξ of the dilated pipe is connected to the corresponding coordinate $x \in (0,1)$ of the unexpanded pipe. In view of that, we introduce a new unknown temperature

$$T^\epsilon(x, y, z) = \theta^\epsilon(\xi, y, z)$$

and the exterior temperature

$$g(x, y, z) = f(\xi, y, z).$$

Taking into account the introduced substitutions and

$$\begin{aligned}
\frac{dx}{d\xi} &= 1 - \epsilon \langle \theta^\epsilon \rangle (\xi) = 1 - \epsilon \langle T^\epsilon \rangle (x), \qquad (7)\\
\frac{\partial \theta^\epsilon}{\partial \xi} &= \big(1 - \epsilon \langle T^\epsilon \rangle\big) \frac{\partial T^\epsilon}{\partial x}\\
\frac{\partial^2 \theta^\epsilon}{\partial \xi^2} &= \left(-\epsilon \left\langle \frac{\partial T^\epsilon}{\partial x} \right\rangle \frac{\partial T^\epsilon}{\partial x} + \big(1 - \epsilon \langle T^\epsilon \rangle\big) \frac{\partial^2 T^\epsilon}{\partial x^2}\right) \big(1 - \epsilon \langle T^\epsilon \rangle\big),
\end{aligned}$$

problem (2)–(4) reads:

$$\begin{aligned}
&- \kappa \left(\triangle T^\epsilon + \epsilon \left(-2 \langle T^\epsilon \rangle \frac{\partial^2 T^\epsilon}{\partial x^2} - \left\langle \frac{\partial T^\epsilon}{\partial x} \right\rangle \frac{\partial T^\epsilon}{\partial x} \right. \right.\\
&\left. \left. + \epsilon^2 \langle T^\epsilon \rangle \frac{\partial}{\partial x} \left(\langle T^\epsilon \rangle \frac{\partial T^\epsilon}{\partial x} \right) \right) \right) + u \left(1 - \epsilon \langle T^\epsilon \rangle\right) \frac{\partial T^\epsilon}{\partial x} = 0 \quad \text{in } B_1 \qquad (8)\\
&\kappa \frac{\partial T^\epsilon}{\partial \boldsymbol{n}} = \sigma (g - T^\epsilon) \quad \text{on } \Gamma_1 = (0,1) \times \partial\omega \qquad (9)\\
&T^\epsilon(0, \cdot, \cdot) = \theta_l,\ T^\epsilon(1, \cdot, \cdot) = \theta_r \quad \text{on } \omega. \qquad (10)
\end{aligned}$$

Due to the nonlinearity of the problem, we do not expect to find an exact solution, but rather look for an approximation. Considering the small parameter ϵ, we expand the temperature T^ϵ in an asymptotic series by powers of ϵ as follows:

$$T^\epsilon = T_0 + \epsilon T_1 + \cdots + \epsilon^k T_k + \cdots$$

The substitution of the introduced expansion in the heat equation and the collection of the members with the same powers of ϵ leads to a system of linear elliptic PDEs

$$\begin{aligned} -\kappa\triangle T_0 + u\frac{\partial T_0}{\partial x} &= 0 \\ -\kappa\triangle T_k + u\frac{\partial T_k}{\partial x} &= \beta_k^\epsilon, \quad k > 1, \end{aligned} \tag{11}$$

where β_k^ϵ depends on T_m, $0 \le m < k$, and their derivatives with respect to x. The expressions for β_k^ϵ are long and tedious and we give it only for $k = 1$:

$$\beta_1^\epsilon = -2\kappa\langle T_0\rangle\frac{\partial^2 T_0}{\partial x^2} - \kappa\left\langle\frac{\partial T_0}{\partial x}\right\rangle\frac{\partial T_0}{\partial x} + u\langle T_0\rangle\frac{\partial T_0}{\partial x}$$

and omit others. By using the same approach to impose the boundary conditions we get

$$\begin{aligned} \kappa\frac{\partial T_0}{\partial \boldsymbol{n}} &= \sigma(g - T_0), \quad T_0(0,\cdot,\cdot) = \theta_l,\ T_0(1,\cdot,\cdot) = \theta_r \\ \kappa\frac{\partial T_k}{\partial \boldsymbol{n}} &= -\sigma T_k, \quad T_k(0,\cdot,\cdot) = 0,\ T_0(1,\cdot,\cdot) = 0. \end{aligned} \tag{12}$$

The obtained system is recursive and in each step the right-hand side of the equation is known from the previous equation. Therefore, the existence, uniqueness and regularity of the solutions $T_k \in H^2(B_1)$ are obvious (e. g. see [4]).

Moreover, paper [9] considers the evaluation of the difference between the exact solution of (8)–(10) and the derived asymptotic solution, and the following error estimate holds:

$$\|T^\epsilon - (T_0 + \epsilon T_1 + \cdots + \epsilon^k T_k)\|_{H^1(B_1)} \le C\epsilon^{k+1}, \tag{13}$$

where C is a constant independent of ϵ. In what follows, the positive constant C denotes a generic constant independent of ϵ (and δ in the next section).

Note that the problem was formulated at the beginning for θ^ϵ and that the domain of θ^ϵ, i.e. dilation of the pipe, is determined by the corresponding temperature T^ϵ. By using the asymptotic approximation, the domain itself, as well as the length of the pipe, is no longer determined exactly. Let us state the result on the error estimation for the pipe dilation.

Observation 1. *Let l_{θ^ϵ} be the length of the pipe corresponding to the solution θ^ϵ of the original problem (2)–(4), and let l_k^ϵ be the approximation of the length of the pipe corresponding to the solution $T_k^\epsilon = T_0 + \epsilon T_1 + \cdots + \epsilon^k T_k$ of the asymptotic problem. Then the following estimate holds:*

$$|l_{\theta^\epsilon} - l_k^\epsilon| \le C\epsilon^{k+2}.$$

Proof. By changing variable (6) we deduce

$$dx = d\xi - \epsilon\langle\theta^\epsilon\rangle(\xi)\,d\xi = (1 - \epsilon\langle T^\epsilon\rangle(x))\,d\xi$$

$$l_{\theta^\epsilon} = 1 + \epsilon \int_0^{l_{\theta^\epsilon}} \langle \theta_\epsilon \rangle (\xi)\, d\xi = 1 + \epsilon \int_0^1 \frac{\langle T^\epsilon \rangle (x)}{1 - \epsilon \langle T^\epsilon \rangle (x)}\, dx.$$

From inequality (5) it follows that the expressions $1 - \epsilon \langle T^\epsilon \rangle (x)$ and $1 - \epsilon \langle T_k^\epsilon \rangle (x)$ are bounded from below by a positive real number (e.g. $3/4$ for a sufficiently small ϵ). Therefore, we have

$$\begin{aligned} |l_{\theta^\epsilon} - l_k^\epsilon| &= \left| \epsilon \int_0^1 \left(\frac{\langle T^\epsilon \rangle (x)}{1 - \epsilon \langle T^\epsilon \rangle (x)} - \frac{\langle T_k^\epsilon \rangle (x)}{1 - \epsilon \langle T_k^\epsilon \rangle (x)} \right) dx \right| \\ &= \left| \epsilon \int_0^1 \frac{\langle T^\epsilon \rangle (x) - \langle T_k^\epsilon \rangle (x)}{(1 - \epsilon \langle T^\epsilon \rangle (x))(1 - \epsilon \langle T_k^\epsilon \rangle (x))}\, dx \right| \\ &\leq C\epsilon \int_0^1 \int_\omega |T^\epsilon(x,y,z) - T_k^\epsilon(x,y,z)|\; dxdydz \\ &\leq C\epsilon \|T^\epsilon - T_k^\epsilon\|_{L^2(B_1)} \leq C\epsilon^{k+2}. \end{aligned} \tag{14}$$

In the above conclusion we have used estimate (13). □

It should be emphasized that the regressions from the variable x to the initial variable ξ via the exact solution and the approximation are not the same. An error of size $C\epsilon^{k+2}$ occurs along the entire length of the pipe. More precisely, let $\xi^\epsilon(x)$ be the corresponding variable value in the pipe for the exact solution and $\xi_k^\epsilon(x)$ for the approximation of the solution. Then we have

$$|\xi^\epsilon(x) - \xi_k^\epsilon(x)| = \left| \epsilon \int_0^x \left(\frac{\langle T^\epsilon \rangle (t)}{1 - \epsilon \langle T^\epsilon \rangle (t)} - \frac{\langle T_k^\epsilon \rangle (t)}{1 - \epsilon \langle T_k^\epsilon \rangle (t)} \right) dt \right| \leq C\epsilon^{k+2}. \tag{15}$$

Furthermore,

$$\begin{aligned} \left| \frac{d}{dx}\xi^\epsilon(x) - \frac{d}{dx}\xi_k^\epsilon(x) \right| &= \epsilon \left| \frac{\langle T^\epsilon \rangle (x)}{1 - \epsilon \langle T^\epsilon \rangle (x)} - \frac{\langle T_k^\epsilon \rangle (x)}{1 - \epsilon \langle T_k^\epsilon \rangle (x)} \right| \\ &\leq C\epsilon \left| \langle T^\epsilon \rangle (x) - \langle T_k^\epsilon \rangle (x) \right| \leq C\epsilon \| \langle T^\epsilon \rangle - \langle T_k^\epsilon \rangle \|_{H^1(0,1)} \\ &\leq C\epsilon \|T^\epsilon - T_k^\epsilon\|_{H^1(B_1)} \leq C\epsilon^{k+2}, \end{aligned}$$

leading to the conclusion that the changes in the variable ξ^ϵ can be approximated by the change in ξ_k^ϵ with respect to x. Since the pipe does not necessarily expand linearly, but a part may be subject to a greater expansion, we conclude that the change in the rate of expansion described by the variable ξ^ϵ can be approximated by the expansion determined by the approximation ξ_k^ϵ.

3. Thin (or long) pipe

In this section, we study the flow in a thin (or long) pipe with a small circular cross-section of radius $\delta > 0$. For this purpose, we introduce another small parameter δ. Our goal is to find an asymptotic approximation for the temperature θ^ϵ. However, we start with the approximation for T^ϵ introduced in the previous section since this solution is defined on a fixed length domain. Due to the small thickness of the pipe, we expect the heat flux to be mainly in the x direction. We will introduce fast variables and once again use an asymptotic approach, but this time in terms of δ and for the zero-order approximation T_0 and the corrector T_1.

We study the problem for T_0 in domain $B_1^\delta = (0,1) \times \omega_\delta = (0,1) \times \{(y,z) \in \mathbb{R}^2 : y^2 + z^2 < \delta^2\}$:

$$\begin{aligned} &-\kappa \triangle T_0 + u \frac{\partial T_0}{\partial x} = 0 \quad \text{in } B_1^\delta \\ &\kappa \frac{\partial T_0}{\partial \boldsymbol{n}} = \sigma(g - T_0) \quad \text{on } \Gamma_1^\delta = (0,1) \times \partial\omega_\delta \\ &T_0(0,\cdot,\cdot) = \theta_l,\ T_0(1,\cdot,\cdot) = \theta_r. \end{aligned}$$

Introducing the fast variables $\rho = \frac{y}{\delta}$ and $\tau = \frac{z}{\delta}$ and the following notation for the partial differential operator $\triangle_{\rho,\tau}\Phi = \frac{\partial^2\Phi}{\partial\rho^2} + \frac{\partial^2\Phi}{\partial\tau^2}$, we get a problem in δ independent domain B_1^1 which reads:

$$\begin{aligned} &-\kappa \left(\frac{1}{\delta^2} \triangle_{\rho,\tau} T_0 + \frac{\partial^2 T_0}{\partial x^2} \right) + u \frac{\partial T_0}{\partial x} = 0 \quad \text{in } B_1^1 \\ &\kappa \frac{\partial T_0}{\partial \boldsymbol{n}} = \delta\sigma(g - T_0) \quad \text{on } \Gamma_1^1 \\ &T_0(0,\cdot,\cdot) = \theta_l,\ T_0(1,\cdot,\cdot) = \theta_r. \end{aligned}$$

Recall that the velocity u is given in Poiseuille form

$$u = \frac{p_l - p_r}{4\mu}(\delta^2 - y^2 - z^2) = \delta^2 \frac{p_l - p_r}{4\mu}(1 - \rho^2 - \tau^2) = \mathcal{O}(\delta^2).$$

We expand the unknown temperature

$$T_0 = T_0^0 + \delta T_0^1 + \cdots + \delta^n T_0^n + \cdots . \tag{16}$$

The equations for the first-order approximation were derived in [9]. A standard application of asymptotic analysis for non-isothermal flows in a thin pipe was used (e.g. see [1]). We will briefly repeat the derivation here to give the readers a better overview.

Let us first consider the meaning of the heat transfer coefficient. Assuming $\sigma \gg \mathcal{O}(\delta)$, the leading approximation in the temperature expansion is equal to the external temperature g. In this case, the temperature exchange through the lateral boundary dominates the process, and the effect of the temperature of the fluid entering the pipe is negligible. If it is $\sigma \ll \mathcal{O}(\delta)$, then the effect of temperature exchange at the lateral boundary is negligible. The most interesting critical case $\sigma = \mathcal{O}(\delta)$ is between the above cases, where all physically relevant effects are preserved in the proposed problem. Therefore, we limit further consideration to the case

$$\sigma = \alpha\delta, \quad \alpha = \mathcal{O}(1).$$

Substituting expansion (16) into heat equation (11) and taking into account lateral boundary condition (12), by collecting leading order terms we get

$$\begin{aligned} -\triangle_{\rho,\tau} T_0^0 &= 0 \quad \text{in } B_1^1 \\ \kappa \frac{\partial T_0^0}{\partial \boldsymbol{n}} &= 0 \quad \text{on } \Gamma_1^1, \end{aligned}$$

from which we conclude $T_0^0 = T_0^0(x)$. Similarly, by collecting the terms in the heat equation with δ^{-1} and in the lateral boundary condition with δ, we obtain that T_0^1 satisfies the same system, which also leads to a conclusion $T_0^1 = T_0^1(x)$. To determine T_0^0, we proceed by defining the problem for T_0^2:

$$-\triangle_{\rho,\tau} T_0^2 - \frac{\partial^2 T_0^0}{\partial x^2} = 0 \quad \text{in } B_1^1 \tag{17}$$

$$\kappa \frac{\partial T_0^2}{\partial \boldsymbol{n}} = \alpha(g - T_0^0) \quad \text{on } \Gamma_1^1. \tag{18}$$

The necessary condition for the existence of T_0^2 results in a linear second order ODE for T_0^0:

$$\frac{d^2 T_0^0}{dx^2} - \frac{2\alpha}{\kappa} T_0^0 = -\frac{\alpha}{\kappa\pi} \int_{\partial B(0,1)} g(x,y,z)\, dydz = -\frac{2\alpha}{\kappa} g(x) \quad \text{in } (0,1) \tag{19}$$

leading to

$$T_0^0(x) = A_1 \cosh \frac{\sqrt{2\alpha}x}{\sqrt{\kappa}} + A_2 \sinh \frac{\sqrt{2\alpha}x}{\sqrt{\kappa}} + \sqrt{\frac{2\alpha}{\kappa}} \int_0^x \sinh \frac{\sqrt{2\alpha}(t-x)}{\sqrt{\kappa}} \cdot g(t)\, dt.$$

For simplicity, we have assumed $g = g(x)$. Otherwise, in the rest of the paper, in all calculations, g should be replaced by $\frac{1}{2\pi}\int_{\partial B(0,1)} g(x,y,z)\, dydz$. Considering the boundary conditions $T_0^0(0) = \theta_l$ and $T_0^0(1) = \theta_r$, we obtain a solution

$$\begin{aligned} T_0^0(x) = {} & \sqrt{\frac{2\alpha}{\kappa}} \int_0^x \sinh \frac{\sqrt{2\alpha}(t-x)}{\sqrt{\kappa}} \cdot g(t)\, dt + \theta_l \cosh \frac{\sqrt{2\alpha}x}{\sqrt{\kappa}} \\ & + \frac{\sinh \frac{\sqrt{2\alpha}x}{\sqrt{\kappa}}}{\sinh \sqrt{\frac{2\alpha}{\kappa}}} \left(\theta_r - \theta_l \cosh \sqrt{\frac{2\alpha}{\kappa}} - \sqrt{\frac{2\alpha}{\kappa}} \int_0^1 \sinh \frac{\sqrt{2\alpha}(t-1)}{\sqrt{\kappa}} \cdot g(t)\, dt \right). \end{aligned} \tag{20}$$

Note that, if truly $g = g(x)$, in view of (19), problem (17)–(18) can be solved by taking $T_0^2(x,\rho,\tau) = -\frac{1}{4}(\rho^2+\tau^2)\frac{d^2 T_0^0}{dx^2}(x)$.

By continuing the asymptotic procedure, we find the system for T_0^3:

$$\begin{aligned} -\triangle_{\rho,\tau} T_0^3 - \frac{\partial^2 T_0^1}{\partial x^2} &= 0 \quad \text{in } B_1^1 \\ \kappa \frac{\partial T_0^3}{\partial \boldsymbol{n}} &= -\alpha T_0^1 \quad \text{on } \Gamma_1^1. \end{aligned}$$

The compatibility conditions lead to homogenous linear ODE

$$\frac{d^2 T_0^1}{dx^2} = \frac{2\alpha}{\kappa} T_0^1 \quad \text{in } (0,1).$$

Due to the boundary conditions $T_0^1(0) = 0$ and $T_0^1(1) = 0$, we deduce $T_0^1(x) = 0$. Therefore, we approximate T_0 with $\mathcal{T}_0 = T_0^0 + \delta T_0^1 = T_0^0$.

Thus, the approximation $\mathcal{T}_0$ for T^ϵ was derived in explicit form. To obtain an approximation θ_0 for the solution θ^ϵ of the starting problem, the variables must be changed. Namely,

$$\theta_0(\xi_0) = \mathcal{T}_0(x), \tag{21}$$

where in accordance with (6) and (7), we define

$$x = \xi_0 - \epsilon \int_0^{\xi_0} \langle \theta_0 \rangle(s)\, ds = \xi_0 - \epsilon \int_0^x \frac{\mathcal{T}_0(s)}{1 - \epsilon \mathcal{T}_0(s)}\, ds. \tag{22}$$

Recall that the last equation has a unique solution $x \in (0, 1)$ for every $\xi \in (0, l_0)$, where $l_0 = 1 + \epsilon \int_0^1 \frac{\mathcal{T}_0(s)}{1-\epsilon\mathcal{T}_0(s)}\, ds$, because the variable ξ_0 increases continuously in x. Since this is a nonlinear equation that also has an integral that is difficult to solve exactly, we will discuss the approximation for a selected numerical example.

Assuming that the considered fluid is water, we take the following constant values: the thermal conductivity $\kappa = 0.6\mathrm{W}/(\mathrm{mK})$, the rate $\alpha{=}1.29$, the temperature at the pipe's ends $\theta_l = 25°\mathrm{C}$ and $\theta_r = 20°\mathrm{C}$. The small parameter ϵ is of order 10^{-5} for different pipe materials, so we set $\epsilon = 10^{-5}$. In the engineering literature, linearity of g is usually assumed so we take $g(x) = x + 1$. Because of the choice of the function g, it is possible to calculate exactly the integrals in expression (20). We skip this technical part and prefer to present the results graphically. All numerical calculations were performed in MATLAB.

For the approximation of the length of the pipe we have $l_0 = 1 + 1.72331 \cdot 10^{-4}$. The dependence of the variable ξ on x is shown in Fig. 2.

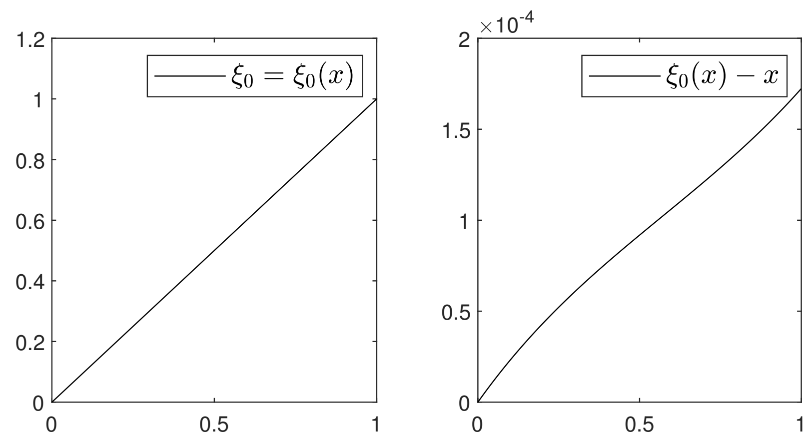

Figure 2: *Variable change for zero order approximation*

One can notice that visually the dependence is almost linear. By including different constants of the problem, it can be seen that the linearity is almost preserved for all linear functions g. However, in the case of a larger difference between the temperature of the fluid at the ends of the pipe, a slightly larger dilation is observed at the end with a higher temperature.

Now the approximation θ_0 can be determined according to (21) and (22). Fig. 3 shows the obtained approximation θ_0 and the approximation T_0 defined on the fixed domain $(0,1)$. Since the graphs of these functions almost coincide, their difference is also shown. As expected, the error is larger at the right end of the pipe due to the chosen notation of the domain. The error itself is of order 10^{-3}, which may not be significant compared to the values of the fluid temperature, but it is open to discussion whether it is negligible compared to the small parameter ϵ.

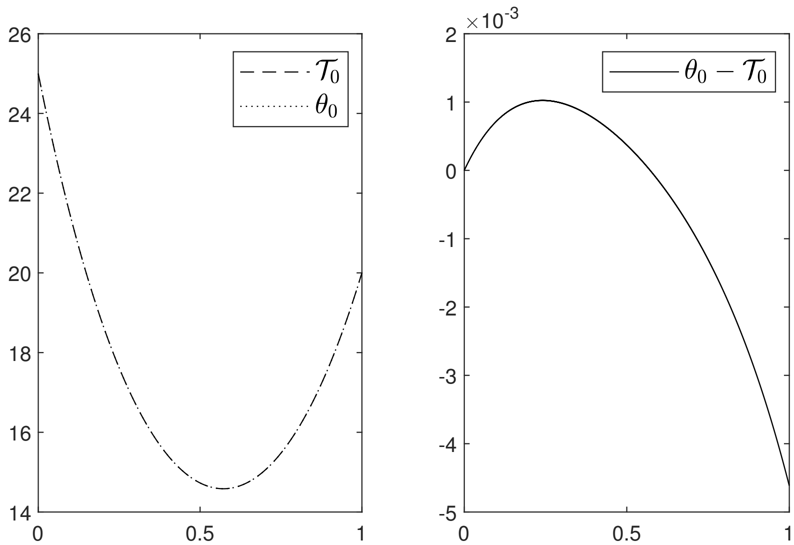

Figure 3: *Comparison of approximations on dilated and fixed domain*

To obtain an approximation for T_1, we consider the following problem formulated in terms of fast variables:

$$-\kappa\left(\frac{1}{\delta^2}\triangle_{\rho,\tau}T_1+\frac{\partial^2 T_1}{\partial x^2}\right)+u(\rho,\tau)\frac{\partial T_1}{\partial x}$$
$$=-2\kappa\langle T_0\rangle\frac{\partial^2 T_0}{\partial x^2}-\kappa\left\langle\frac{\partial T_0}{\partial x}\right\rangle\frac{\partial T_0}{\partial x}+u(\rho,\tau)\langle T_0\rangle\frac{\partial T_0}{\partial x}\quad\text{in } B_1^1$$
$$\kappa\frac{\partial T_1}{\partial \boldsymbol{n}}=\alpha\delta^2 T_1\quad\text{on }\Gamma_1^1.$$

To simplify the notation of the solution, let us now assume that $g=0$. If we had used this fact earlier, we would have obtained that the zero-order approximation length of the dilated pipe is $l_0=1+1.68565\cdot 10^{-4}$.

By using (20) we get

$$-\kappa\left(\frac{1}{\delta^2}\triangle_{\rho,\tau}T_1+\frac{\partial^2 T_1}{\partial x^2}\right)+u(\rho,\tau)\frac{\partial T_1}{\partial x}=\frac{-2\alpha}{\sinh^2\sqrt{\frac{2\alpha}{\kappa}}}$$

$$\cdot\left(\left(\theta_r\cosh\frac{\sqrt{2\alpha}x}{\sqrt{\kappa}}-\theta_l\cosh\frac{\sqrt{2\alpha}(1-x)}{\sqrt{\kappa}}\right)^2+2\left(\theta_r\sinh\frac{\sqrt{2\alpha}x}{\sqrt{\kappa}}+\theta_l\sinh\frac{\sqrt{2\alpha}(1-x)}{\sqrt{\kappa}}\right)^2\right)$$

$$+\mathcal{O}(\delta^2)\quad\text{in } B_1^1. \tag{23}$$

We expand $T_1=T_1^0+\delta T_1^1+\delta^2T_1^2+\cdots$. Using exactly the same reasoning as before, we obtain that $T_1^0=T_1^0(x)$ and $T_1^1=T_1^1(x)$. The function T_1^2 is the solution of problem

$$-\kappa\left(\triangle_{\rho,\tau}T_1^2+\frac{\partial^2T_1^2}{\partial x^2}\right)=\beta(x)\quad\text{in } B_1^1$$

$$\kappa\frac{\partial T_1^2}{\partial \boldsymbol{n}}=-\alpha T_1^0\quad\text{on } \Gamma_1^1,$$

where $\beta(x)$ is an x-dependent expression on the right-hand side of (23). A necessary compatibility condition gives ODE for T_1^0:

$$\frac{d^2T_1^0}{dx^2}-\frac{2\alpha}{\kappa}=-\frac{\beta(x)}{\kappa}.$$

By imposing the boundary conditions $T_1^0(0)=T_1^0(1)=0$ we get

$$T_1^0(x)=-\frac{\sinh\frac{\sqrt{2\alpha}x}{\sqrt{\kappa}}}{\sqrt{2\alpha\kappa}\sinh\sqrt{\frac{2\alpha}{\kappa}}}\int_0^1\sinh\frac{\sqrt{2\alpha}(t-1)}{\sqrt{\kappa}}\cdot\beta(t)\,dt$$

$$+\frac{1}{\sqrt{2\alpha\kappa}}\int_0^x\sinh\frac{\sqrt{2\alpha}(t-x)}{\sqrt{\kappa}}\cdot\beta(t)\,dt.$$

The compatibility condition for the problem for T_1^3 together with the boundary conditions $T_1^1(0)=T_1^1(1)=0$ yields $T_1^1(x)=0$ as before. Thus the approximation $\mathcal{T}_1=T_1^0+\delta T_1^1=T_1^0$ is determined.

The graphical results are presented in Fig. 4. On the left side, the graph of $\mathcal{T}_1$ is shown. It can be seen that the function is of order 10^2, so its contribution to the approximation $\mathcal{T}=\mathcal{T}_0+\epsilon\mathcal{T}_1$ is of order 10^{-2}. The difference $\theta_1-\mathcal{T}$ is shown on the right, where θ_1 is defined analogously to (21) and (22):

$$\theta_1(\xi_1)=\mathcal{T}(x),\quad x=\xi_1-\epsilon\int_0^{\xi_1}\langle\theta_1\rangle(s)\,ds=\xi_1-\epsilon\int_0^x\frac{\mathcal{T}(s)}{1-\epsilon\mathcal{T}(s)}\,ds.$$

We see that this difference has the same order as the difference $\theta_0-\mathcal{T}_0$. This suggests the importance of returning to the original domain of the extended pipe. Namely, taking into account $\mathcal{T}_0\in H^2(0,1)$ (by the standard theory of elliptic PDEs), for all $\eta\in(0,l_0)\cap(0,1)$, we have

$$|\theta_0(\eta)-\mathcal{T}_0(\eta)|=|\mathcal{T}_0(\xi_0^{-1}(\eta))-\mathcal{T}_0(\eta)|=|\mathcal{T}_0'(\tilde\eta)|\cdot|\xi_0^{-1}(\eta)-\eta|=\epsilon|\mathcal{T}_0'(\tilde\eta)|\cdot\left|\int_0^\eta\langle\theta_0\rangle(s)\,ds\right|.$$

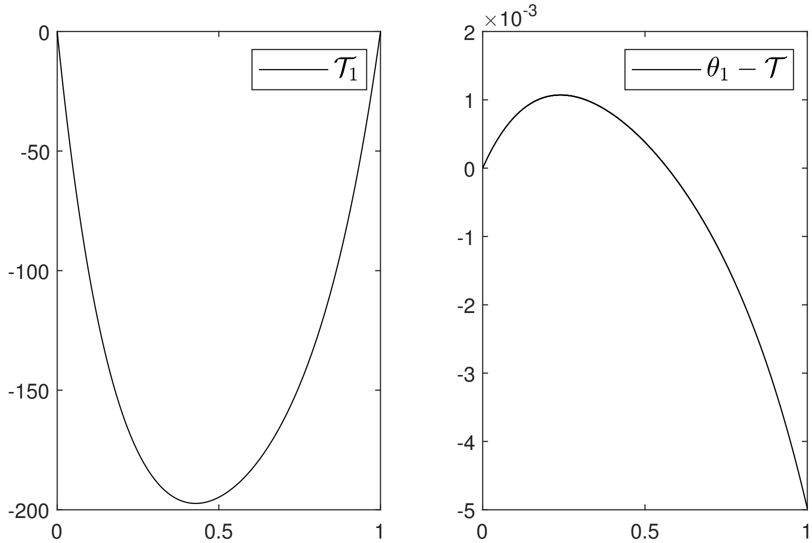

Figure 4: *Comparison of approximations on dilated and fixed domain*

Analogously, it turns out that $\theta_1 - \mathcal{T}$ is also of the same order, i.e., an error of order ϵ is to be expected, especially at the right boundary of the pipe. From this we conclude that the search for a corrective term is meaningless unless the original temperature θ^ϵ is approximated by a function on the dilated domain.

Let us also state that with the approximation $\mathcal{T}$ we arrive at the length of the pipe $l_1 = 1 + 1.68551 \cdot 10^{-4}$ compared to the previously obtained approximation $l_0 = 1 + 1.68565 \cdot 10^{-4}$.

4. Error estimate

Having emphasized the importance of approximation on a dilated domain, let us now evaluate the approximation obtained with such approach. The usual procedure of defining a weak formulation of the problem and using a test function equal to the difference between the solution and the approximation (see [9] for details) yields

$$\begin{aligned} \|\langle T_0\rangle - \mathcal{T}_0\|_{H^1(0,1)} &\leq C\delta^2 \\ \|\langle T_1\rangle - \mathcal{T}\|_{H^1(0,1)} &\leq C\delta^2, \end{aligned} \tag{24}$$

for small enough δ. Note that the functions $\mathcal{T}_0$ and $\mathcal{T}$ depend only on x, so the above estimates are actually estimates in the entire flow domain.

Theorem 1. *Let θ^ϵ be the solution of problem (2)–(4). Its approximations θ_0 and θ_1 are defined as in the previous section. Then the following estimates hold:*

$$\|\langle\theta^\epsilon\rangle - \theta_0\|_{H^1(0,l_{\theta^\epsilon})} \leq C(\epsilon + \delta^2) \qquad \|\langle\theta^\epsilon\rangle - \theta_1\|_{H^1(0,l_{\theta^\epsilon})} \leq C(\epsilon^2 + \delta^2),$$

where ϵ and δ are small enough and C is a constant independent of ϵ and δ.

Remark 1. *The approximation θ_0 is not necessarily defined on the whole domain B_{θ^ϵ}, but only for the pipe's length l_0. As both θ^ϵ and θ_0 are L^∞ functions and the length difference $l_{\theta^\epsilon} - l_0$ is bounded by $C\epsilon(\epsilon+\delta^2)$ analogously to (14), the extension by a constant is sufficient to have no influence on the error estimate.*

Remark 2. *Some constants in Section 2 depend on δ. Derivations with respect to the variables y and z and bounds depending on the dimensions of the domain are affected. All this has no effect on the theorem stated.*

Proof of Theorem 1. Let ξ, ξ_0, and ξ_1, as before, be the functions of the change of the variable x in the extended domain corresponding respectively to the solutions θ^ϵ, θ_0, θ_1 (or T^ϵ, $\mathcal{T}_0$, $\mathcal{T}$). Keeping in mind that ξ mappings connect the points of the original pipe and the dilated ones, we conclude: $x \mapsto \xi$ connects the position x with the point of the extended pipe by the temperature θ^ϵ. Then $\xi \mapsto^{\xi_0^{-1}} \tilde{x}$ finds the point which would be the starting point if the pipe has been extended by the temperature θ_0. We start with the variable change and use the fact that ξ functions as well as their derivatives are bounded:

$$\begin{aligned}\|\langle\theta^\epsilon\rangle - \theta_0\|_{L^2(0,l_{\theta^\epsilon})} =&\|d\xi/dx\|^{1/2}_{L^\infty(0,1)}\|\langle T^\epsilon\rangle - \mathcal{T}_0\circ\xi_0^{-1}\circ\xi\|_{L^2(0,1)}\\ \le& C(\|\langle T^\epsilon - T_0\rangle\|_{L^2(0,1)} + \|\langle T_0\rangle - \mathcal{T}_0\|_{L^2(0,1)}\\ &+\|\mathcal{T}_0 - \mathcal{T}_0\circ\xi_0^{-1}\circ\xi\|_{L^2(0,1)}).\end{aligned}$$

The first two terms are bounded by (13) and (24). Recall again that $\mathcal{T}_0 \in H^2(0,1)$. Also, $\xi_0 = \xi_0(x)$ is smooth with derivative $1/(1-\epsilon\theta_0)$. Due to (5), the inverse ξ_0^{-1} is derivable and its derivative is bounded independently of ϵ. By the same reasoning as in (15), we have $|\xi_0(x) - \xi(x)| \le C\epsilon\|\mathcal{T}_0 - \langle T^\epsilon\rangle\|_{L^2(0,1)}$. Therefore for all $x \in B(0,1)$:

$$\begin{aligned}|\mathcal{T}_0(x) - \mathcal{T}_0(\xi_0^{-1}(\xi(x)))| \le& C|x - \xi_0^{-1}(\xi(x))| = C|\xi_0^{-1}(\xi_0(x)) - \xi_0^{-1}(\xi(x))|\\ \le& C\|(\xi_0^{-1})'\|_{L^\infty(0,1)}|\xi_0(x) - \xi(x)|\\ \le& C\epsilon\|\mathcal{T}_0 - \langle T^\epsilon\rangle\|_{L^2(0,1)}.\end{aligned}\tag{25}$$

We deduce

$$\|\langle\theta^\epsilon\rangle - \theta_0\|_{L^2(0,l_{\theta^\epsilon})} \le C(\epsilon+\delta^2+\epsilon(\epsilon+\delta^2)) \le C(\epsilon+\delta^2).$$

For the second inequality, we start from

$$\begin{aligned}\|\frac{d}{d\xi}\langle\theta^\epsilon\rangle - \theta_0'\|_{L^2(0,l_{\theta^\epsilon})} =&\|\frac{d}{dx}\langle T^\epsilon\rangle - \frac{d}{dx}\mathcal{T}_0\circ\xi_0^{-1}\circ\xi\|_{L^2(0,1)}\\ \le&\|\langle T^\epsilon - T_0\rangle\|_{H^1(0,1)} + \|\langle T_0\rangle - \mathcal{T}_0\|_{H^1(0,1)}\\ &+\|\mathcal{T}_0' - \frac{d}{dx}\mathcal{T}_0\circ\xi_0^{-1}\circ\xi\|_{L^2(0,1)}.\end{aligned}$$

The third term can be treated as follows:

$$
\begin{aligned}
&\|\mathcal{T}_0' - \frac{d}{dx}\mathcal{T}_0 \circ \xi_0^{-1} \circ \xi\|_{L^2(0,1)} \\
&\qquad =\|\mathcal{T}_0' - \mathcal{T}_0' \circ \xi_0^{-1} \circ \xi \cdot (\xi_0^{-1})' \circ \xi \cdot \xi'\|_{L^2(0,1)} \\
&\qquad \leq\|\mathcal{T}_0' - \mathcal{T}_0' \cdot (\xi_0^{-1})' \circ \xi \cdot \xi'\|_{L^2(0,1)} \\
&\qquad \quad + \|\mathcal{T}_0' \cdot (\xi_0^{-1})' \circ \xi \cdot \xi' - \mathcal{T}_0' \circ \xi_0^{-1} \circ \xi \cdot (\xi_0^{-1})' \circ \xi \cdot \xi'\|_{L^2(0,1)} \\
&\qquad \leq\|\mathcal{T}_0'\|_{L^2(0,1)}\|1 - (\xi_0^{-1})' \circ \xi \cdot \xi'\|_{L^\infty(0,1)} \\
&\qquad \quad + \|(\xi_0^{-1})' \circ \xi \cdot \xi'\|_{L^\infty(0,1)}\|\mathcal{T}_0' - \mathcal{T}_0' \circ \xi_0^{-1} \circ \xi\|_{L^2(0,1)} \\
&\qquad \leq C\|1 - (1 - \epsilon\mathcal{T}_0 \circ \xi_0^{-1} \circ \xi) \cdot \frac{1}{1 - \epsilon\langle T^\epsilon\rangle}\|_{L^\infty(0,1)} \\
&\qquad \quad + C\|\mathcal{T}_0\|_{H^2(0,1)}\| \operatorname{id}_{(0,1)} - \xi_0^{-1} \circ \xi\|_{L^\infty(0,1)} \\
&\qquad \leq C\epsilon\|\mathcal{T}_0 \circ \xi_0^{-1} \circ \xi - \langle T^\epsilon\rangle\|_{L^2(0,1)} + C\|(\xi_0^{-1})'\|_{L^\infty(0,1)}\|\xi_0 - \xi\|_{L^\infty(0,1)}.
\end{aligned}
$$

The first term is bounded by (25), so that we finally have

$$
\|\frac{d}{d\xi}\langle\theta^\epsilon\rangle - \theta_0'\|_{L^2(0,l_{\theta^\epsilon})} \leq C(\epsilon + \delta^2 + \epsilon^2(\epsilon + \delta^2) + \epsilon(\epsilon + \delta^2)) \leq C(\epsilon + \delta^2).
$$

The proof of the second inequality is analogous. □

5. Conclusion

In this paper, a study of the flow of fluid through a thin pipe whose surroundings have a different temperature than the fluid inside the pipe is presented. A nonlinear model is used to describe the coupling of the temperature of the fluid and the longitudinal expansion of the tube. Using asymptotic analysis with respect to small parameters ϵ (heat expansion coefficient) and δ (ratio between pipe thickness and length), an asymptotic solution was observed. Although the approximation is initially defined on a fixed domain (defined by the change in variables), the importance of considering the approximate solution on the original extended domain is presented. By proving the error estimate for the approximation on the extended domain, the justification of the effective model is given, which is our main contribution.

The asymptotic behavior of the heat conduction problem has already been studied (e.g. [1, 5, 7, 8]), but to our knowledge, research on temperature-variable domains has been neglected. This paper shows that for a good approximation it is necessary to take into account the change of the domain, because it is not enough just to formulate the problem on the temperature dependent domain, but it is also advisable to determine the approximation on the extended domain. We believe that the presented paper can improve engineering practice, especially in numerical simulations, considering that variable substitution functions are simple and numerically convenient.

References

[1] M. Beneš, I. Pažanin, *Rigorous derivation of the effective model describing a non-isothermal fluid flow in a vertical pipe filled with porous medium*, Cont. Mech. Therm. **30**(2018), 301–317.

[2] S. Čanić, A. Mikelić, *Effective equations describing the flow of a viscous incompressible fluid through a long elastic tube*, C. R. Acad. Sci. Paris, Sér. IIb **330**(2002), 661–666.

[3] A. Ćurković, E. Marušić-Paloka, *Asymptotic analysis of a thin fluid layer-elastic plate interaction problem*, Appl. Anal. **98**(2019), 2118–2143.

[4] D. Gilbarg, N. S. Trudinger *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2001.

[5] M. Ljulj, E. Marušić-Paloka, I. Pažanin, J. Tambača *Mathematical model of heat transfer through a conductive pipe*, ESAIM Math. Model. Numer. Anal. **55**(2021), 627–658.

[6] V. Mácha, B. Muha, Š. Nečasová, A. Roy, S. Trifunović, *Existence of a weak solution to a nonlinear fluid-structure interaction problem with heat exchange*, Comm. Partial Differential Equations, **47**(2022), 1591–1635.

[7] E. Marušić-Paloka, I. Pažanin, *Non-isothermal fluid flow through a thin pipe with cooling*, Appl. Anal. **88**(2009), 495–515.

[8] E. Marušić-Paloka, I. Pažanin, *On the effects of curved geometry on heat conduction through a distorted pipe*, Nonlinear Anal. Real World Appl. **11**(2010), 4554–4564.

[9] E. Marušić-Paloka, I. Pažanin, M. Prša, *Asymptotic analysis of the heat conduction problem in a dilated pipe*, Appl. Math. Comput. **355**(2019), 135–150.

[10] E. Marušić-Paloka, I. Pažanin, M. Prša, *Heat Conduction Problem in a Dilated Pipe: Existence and Uniqueness Result*, Mediterr. J. Math. **14**(2017), 97.

[11] G. Panasenko, *Asymptotic expansion of the solution of Navier-Stokes equation in a tube structure*, C. R. Math. Acad. Sci. Paris, Sér. IIb **326**(1998), 867–872.

[12] G. Panasenko, K. Pileckas, *Periodic in time flow in thin structure: Equation on the graph*, J. Math. Anal. Appl **490**(2020), 124335.