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Partial Fibonomial and Lucanomial sums through the extended q-Kummer formula

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Abstract. By employing identities for extended q-Kummer theorems, we examine *partial* sums involving products of two Fibonomial/Lucanomial coefficients. Seven remarkable closed formulae are established.
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1. Introduction

Define the two Fibonacci–like sequences $\{U_n\}$ and $\{V_n\}$ by second order linear recursions

$$U_n = pU_{n-1} + U_{n-2} \quad \text{and} \quad V_n = pV_{n-1} + V_{n-2}$$

with the initial conditions $U_0 = 0,\ U_1 = 1$, and $V_0 = 2,\ V_1 = p$, respectively. They are related by the equality $V_n = U_{n+1}+U_{n-1}$ and represented by the following Binet forms:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n(1+q^n), \tag{1}$$

where $q = \beta/\alpha = -\alpha^{-2}$, which implies $\alpha\beta = -1$ and $\alpha = \mathbf{i}/\sqrt{q}$ with $\mathbf{i} = \sqrt{-1}$ being the imaginary unit. Two special cases of these sequences are recorded here:

p	α	$\{U_n\}$	$\{V_n\}$
1	$\frac{1+\sqrt{5}}{2}$	Fibonacci sequence $\{F_n\}$	Lucas sequence $\{L_n\}$
2	$1+\sqrt{2}$	Pell sequence $\{P_n\}$	Pell–Lucas sequence $\{Q_n\}$

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Denote by $\mathbb{Z}$ and $\mathbb{N}$ the sets of integers and natural numbers with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, respectively. For $n, m, k \in \mathbb{N}$ with $n \geq k \geq 1$, define the generalized Fibonomial and Lucanomial coefficients by

$$\left\{ {n \atop k} \right\}_{U_m} = \prod_{j=1}^{k} \frac{U_{(n-j+1)m}}{U_{jm}} = \frac{U_{nm}U_{(n-1)m}\cdots U_{(n-k+1)m}}{U_m U_{2m}\cdots U_{km}},$$
$$\left\{ {n \atop k} \right\}_{V_m} = \prod_{j=1}^{k} \frac{V_{(n-j+1)m}}{U_{jm}} = \frac{V_{nm}V_{(n-1)m}\cdots V_{(n-k+1)m}}{V_m V_{2m}\cdots V_{km}};$$

with the boundary conditions

$$\left\{ {n \atop 0} \right\}_{U_m} = \left\{ {n \atop 0} \right\}_{V_m} = \left\{ {n \atop n} \right\}_{U_m} = \left\{ {n \atop n} \right\}_{V_m} = 1.$$

In particular, for $m = 1$, the reduced coefficients $\left\{ {n \atop k} \right\}_{U_1}$ and $\left\{ {n \atop k} \right\}_{V_1}$ will be denoted briefly by $\left\{ {n \atop k} \right\}_{U}$ and $\left\{ {n \atop k} \right\}_{V}$, respectively. When $U_n = F_n$ and $V_n = L_n$, these coefficients reduce to the usual Fibonomial coefficients $\left\{ {n \atop k} \right\}_{F}$ and Lucanomial coefficients $\left\{ {n \atop k} \right\}_{L}$, respectively. For details about the Fibonomial and Lucanomial coefficients and their properties, the interested reader may refer to [7, 8, 9, 10, 17, 18].

Let p and q be two indeterminates. The q-Pochhammer symbol is defined by

$$(p; q)_0 = 1 \quad \text{and} \quad (p; q)_n = (1-p)(1-pq)\cdots(1-pq^{n-1}) \quad \text{for} \quad n \in \mathbb{N}.$$

Then for $n, k \in \mathbb{N}_0$, the generalized (p, q)-binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{cases} \dfrac{(p;q)_n}{(p;q)_k (p;q)_{n-k}}, & \text{if } k \leq n, \\ \dfrac{(p;q)_n (pq^{n-k};q)_{k-n}}{(p;q)_k}, & \text{if } k > n. \end{cases}$$

When $p = q$, this becomes the usual Gaussian q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

The objective of this paper is to establish a few formulae for partial sums involving generalized Fibonomial and Lucanomial coefficients. Our approach will mainly be based on the following expressions of the generalized Fibonomial and Lucanomial coefficients in terms of Gaussian q-binomial coefficients:

$$\left\{ {n \atop k} \right\}_{U_m} = \alpha^{mk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^m}, \tag{2}$$
$$\left\{ {n \atop k} \right\}_{V_m} = \alpha^{mk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{-q^m, q^m}; \tag{3}$$

where $n \geq k \geq 0$. Instead, for $k > n$, a particular care should be taken because

$$\begin{bmatrix} n \\ k \end{bmatrix}_{-q,q} = 2(-1)^{\binom{k-n}{2}} \frac{\alpha^{k(k-n)}}{V_{k-n}} \left\{ {k \atop n} \right\}_{V}^{-1}.$$

There are numerous summation identities of Gaussian q-binomial coefficients with certain weight functions in the literature. According to (2) and (3), they can be transformed into sums about generalized Fibonomial and Lucanomial coefficients (cf. [4, 8, 13, 14, 17, 18]) and then evaluated in closed form by employing known q-series results.

However, there exist very few formulae for partial sums in the q-series theory. For example, three sets of weighted partial sums of the Gaussian binomial q-binomial coefficients were derived in [11]. One of them is recorded below:

$$\sum_{k=0}^{2n+1}\begin{bmatrix}2n+2m+4\\ k+m\end{bmatrix}_q\left(1-q^{2n+m+4-k}\right)(-1)^{\binom{k}{2}+kn}q^{\frac{1}{2}k(k-3)-kn}$$
$$=q^{-n-1}\begin{bmatrix}2n+2m+4\\ m+1\end{bmatrix}_q\begin{cases}\left(1+q^{n+1}\right)\left(1-q^{m+n+2}\right), & \text{if } n \text{ is even;}\\ -\left(1-q^{n+1}\right)\left(1+q^{m+n+2}\right), & \text{if } n \text{ is odd.}\end{cases}$$

Another example can be found in [12], which states that for even n, it holds

$$\sum_{j=0}^{n}\begin{bmatrix}2n\\ j\end{bmatrix}_q\mathbf{i}^{j^2}(-1)^{j(n-1)}q^{\frac{j(j-2n+2)}{2}}(1-q^{2n-2j})\eta^{[2\nmid j]}$$
$$=(-1)^{n+1}\,\mathbf{i}^{-n^2}q^{-\frac{1}{2}n^2+n-1}\frac{(1-q^n)(1-q^{n+1})}{1-q^{2n-1}}\begin{bmatrix}2n\\ n\end{bmatrix}_q,$$

where $\eta=-\mathbf{i}q^{-1/2}(1+q)$ and "$[\cdots]$" stands for the Iverson notation defined by [true] = 1 and [false] = 1. As a consequence, this was converted, when n is even, into the following partial sum of the Fibonomial coefficients:

$$\sum_{j=0}^{n}\begin{Bmatrix}2n\\ j\end{Bmatrix}_U p^{[2\nmid j]}U_{2n-2j}=\frac{U_nU_{n+1}}{U_{2n-1}}\begin{Bmatrix}2n\\ n\end{Bmatrix}_U.$$

In this paper, we shall examine a new kind of partial sums including the product of the Gaussian q-binomial coefficients or the generalized (p,q)-binomial coefficients, which are substantially different from the previous ones in two aspects. The main difference lies in the fact that these sums contain two q-binomial coefficients instead of one. Furthermore, the q-coefficients involved are either both q-based, or one is q-based, while another is "$-q$"-based. The rest of the paper will be organized as follows. As a preliminary, two summation theorems for partial sums of generalized Gaussian q-binomial coefficients will be proved in the next section. Then in Section 3, they will be utilized to deduce 7 new summation formulae about generalized Fibonomial and Lucanomial coefficients.

2. Preliminary q-series formulae

For an indeterminate x and an integer $n\in\mathbb{Z}$, the shifted factorial in the base q with $0<|q|<1$ is defined by

$$(x;q)_\infty=\prod_{k=0}^{\infty}(1-q^kx)\quad\text{and}\quad(x;q)_n=\frac{(x;q)_\infty}{(q^nx;q)_\infty}.$$

The quotient form with multi–parameters is abbreviated to

$$\begin{bmatrix} \alpha, \ \beta, \ \cdots, \ \gamma \\ A, \ B, \ \cdots, \ C \end{bmatrix} q \Big]_n = \frac{(\alpha;q)_n\,(\beta;q)_n\cdots(\gamma;q)_n}{(A;q)_n\,(B;q)_n\cdots(C;q)_n}.$$

Following Bailey [1] and Gasper-Rahman [6], the basic hypergeometric series (or shortly q-series) reads:

$${}_{1+\lambda}\phi_\lambda \left[\begin{matrix} a_0, a_1, \cdots, a_\lambda \\ b_1, \cdots, b_\lambda \end{matrix}\Big| q; z\right] = \sum_{n=0}^{\infty} \frac{(a_0;q)_n (a_1;q)_n \cdots (a_\lambda;q)_n}{(q;q)_n\ (b_1;q)_n \cdots (b_\lambda;q)_n}\ z^n.$$

This series terminates if one of its numerator parameters is of the form q^{-m} with $m \in \mathbb{N}_0$. Otherwise, the series is said to be nonterminating. In the latter case, the base q will be restricted, for convergence, to $|q| < 1$.

There exist numerous q-series identities in the literature. One of them is called the q-Kummer formula established independently by Bailey [2] and Daum [5] (see also Gasper–Rahman [6, II-9]):

$${}_2\phi_1 \left[\begin{matrix} a, \ c \\ qa/c \end{matrix}\Big| q; -q/c\right] = \frac{(qa;q^2)_\infty}{(qa/c^2;q^2)_\infty} \left[\begin{matrix} qa/c^2, -q \\ qa/c, -q/c \end{matrix}\Big| q\right]_\infty. \tag{4}$$

By making use of the linearization method [3], Li and Chu [16] extended the above formulae by introducing two integer parameters. Two of their results are recorded as follows:

$${}_2\phi_1 \left[\begin{matrix} a, \ c \\ qa/c \end{matrix}\Big| q; -q^2/c\right] = \left[\begin{matrix} qa/c^2, -q \\ qa/c, -1/c \end{matrix}\Big| q\right]_\infty \left\{\frac{(1+a/c)(qa;q^2)_\infty}{a(qa/c^2;q^2)_\infty} - \frac{(a;q^2)_\infty}{a(q^2a/c^2;q^2)_\infty}\right\}, \tag{5}$$

$${}_2\phi_1 \left[\begin{matrix} a, \ c \\ qa/c \end{matrix}\Big| q; -1/c\right] = \left[\begin{matrix} qa/c^2, -q \\ qa/c, -1/c \end{matrix}\Big| q\right]_\infty \left\{\frac{(1+a/c)(qa;q^2)_\infty}{(qa/c^2;q^2)_\infty} + \frac{(a;q^2)_\infty}{(q^2a/c^2;q^2)_\infty}\right\}. \tag{6}$$

Now letting $a \to -q^{-m}$ and $c \to q^{-n}$ with $m, n \in \mathbb{N}_0$ in (4), (5), (6), and then expressing the factorial quotient as a q-binomial product (where $\varepsilon = \pm 1$),

$$\left[\begin{matrix} q^{-m}\varepsilon, \ q^{-n} \\ q, q^{1+\lambda-m+n}\varepsilon \end{matrix}\Big| q\right]_k q^{k(n+1)} = q^{k(k-m)}\varepsilon^k \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} \lambda+n \\ m-k \end{bmatrix}_{q\varepsilon,q} \begin{bmatrix} \lambda+n \\ m \end{bmatrix}_{q\varepsilon,q}^{-1}, \tag{7}$$

we deduce, after some simplifications, the following q-binomial summation formulae.

Theorem 1 $(m, n \in \mathbb{N}_0)$**.**

$$\text{(a)}\quad \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_{-q,q} q^{k(k-m)} = 2\frac{q^{-\binom{m+1}{2}}(-q;q)_n^2}{(-q^{-m};q^2)_{n+1}};$$

$$\text{(b)}\quad \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_{-q,q} q^{k(k-m+1)} = 2\frac{q^{-\binom{m}{2}}(-q;q)_n^2}{(1+q^n)(-q^{1-m};q^2)_n} \times \left\{1 - \frac{(q^m-q^n)(-q^{1-m};q^2)_n}{(1+q^m)(-q^{2-m};q^2)_n}\right\};$$

$$\text{(c)}\quad \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_{-q,q} q^{k(k-m-1)} = 2\frac{q^{-\binom{m+1}{2}}(-q;q)_n^2}{(1+q^n)(-q^{1-m};q^2)_n} \times \left\{1 + \frac{(q^m-q^n)(-q^{1-m};q^2)_n}{(1+q^m)(-q^{2-m};q^2)_n}\right\}.$$

Alternatively, by specifying $a \to q^{-m}$ and $c \to q^{-n}$ with $m, n \in \mathbb{N}_0$ in (4), (5), (6), and then applying (7) again, we deduce, after some simplifications, the three alternating q-binomial identities.

Theorem 2 $(m, n \in \mathbb{N}_0)$**.**

$$\text{(a)}\quad \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_q q^{k(k-m)} = \begin{cases} (-q)^{-\frac{m^2}{4}} \begin{bmatrix} n \\ \frac{m}{2} \end{bmatrix}_{q^2}, & m\text{–}even; \\ 0, & m\text{–}odd; \end{cases}$$

$$\text{(b)}\quad \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_q q^{k(k-m+1)} = \begin{cases} (-q)^{m-\frac{m^2}{4}} \begin{bmatrix} n \\ m/2 \end{bmatrix}_{q^2} \dfrac{1+q^{n-m}}{1+q^n}, & m\text{–}even; \\ (-q)^{-\frac{(m-1)^2}{4}} \begin{bmatrix} n \\ \frac{m-1}{2} \end{bmatrix}_{q^2} \dfrac{1-q^{1+2n-m}}{1+q^n}, & m\text{–}odd; \end{cases}$$

$$\text{(c)}\quad \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_q q^{k(k-m-1)} = \begin{cases} (-q)^{-\frac{m^2}{4}} \begin{bmatrix} n \\ m/2 \end{bmatrix}_{q^2} \dfrac{1+q^{n-m}}{1+q^n}, & m\text{–}even; \\ (-q)^{-m-\frac{(m-1)^2}{4}} \begin{bmatrix} n \\ \frac{m-1}{2} \end{bmatrix}_{q^2} \dfrac{1-q^{1+2n-m}}{1+q^n}, & m\text{–}odd. \end{cases}$$

3. Applications to the Fibonomial-Lucanomial sums

As consequences of the theorems proved in the last section, we present three corollaries in this section that evaluate Fibonomial and Lucanomial sums in closed form. As showcases, we will limit to provide a detailed proof only for one of the identities displayed in each corollary since the remaining ones can be done in a similar manner. For a given equation "(x)", we shall denote its left-hand and right-hand sides by $\mathcal{L}(x)$ and $\mathcal{R}(x)$, respectively. Throughout this section, we shall make use of the notation $\Delta = 4 + p^2$ or in q-form $\Delta = -q^{-1}(1-q)^2$ for brevity.

Corollary 1 ($X \in \{U, V\}$).

(a) *Let $m, n \in \mathbb{N}_0$, with m being odd. Then*

$$\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_U \begin{Bmatrix} n \\ m-k \end{Bmatrix}_U U_k X_k = X_m \frac{U_{2n-m+1}}{V_n} \begin{Bmatrix} n \\ \frac{m-1}{2} \end{Bmatrix}_{U_2}.$$

(b) *Let $m, n \in \mathbb{N}_0$, with m being odd. Then*

$$\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_U \begin{Bmatrix} n \\ m-k \end{Bmatrix}_U V_k X_k = \frac{U_{2n-m+1}}{V_n} \begin{Bmatrix} n \\ \frac{m-1}{2} \end{Bmatrix}_{U_2} \begin{cases} \Delta U_m, & X = V; \\ V_m, & X = U. \end{cases}$$

(c) *Let $m, n \in \mathbb{N}_0$, with m being even. Then*

$$\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_U \begin{Bmatrix} n \\ m-k \end{Bmatrix}_U (-1)^k U_k X_k = X_{n-m} \frac{U_m}{V_n} \begin{Bmatrix} n \\ m/2 \end{Bmatrix}_{U_2} \begin{cases} 1, & X = V; \\ -1, & X = U. \end{cases}$$

Proof of (c). As an example, we are going to show only the case $X = U$. The case $X = V$ can be treated similarly. By means of (1) and (2), we first convert the sum in (c) to the q-binomial sum

$$\begin{aligned} \mathcal{L}(c) &= \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_U \begin{Bmatrix} n \\ m-k \end{Bmatrix}_U (-1)^k U_k^2 \\ &= \alpha^{mn-m^2-2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_q \alpha^{2k+2km-2k^2} \frac{(1-q^k)^2}{(1-q)^2} (-1)^k \\ &= \frac{\alpha^{nm-2-m^2}}{(1-q)^2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_q (-1)^k q^{k^2-km-k} (1 - 2q^k + q^{2k}). \end{aligned}$$

Then write the right-hand side of (c) in terms of q-binomial coefficients

$$\begin{aligned} \mathcal{R}(c) &= -U_{n-m} \frac{U_m}{V_n} \begin{Bmatrix} n \\ m/2 \end{Bmatrix}_{U_2} \\ &= -\alpha^{nm-2-\frac{m^2}{2}} \frac{(1-q^{n-m})}{(1-q)(1+q^n)} \frac{(1-q^m)}{(1-q)} \begin{bmatrix} n \\ m/2 \end{bmatrix}_{q^2}. \end{aligned}$$

By equating these two expressions $\mathcal{L}(c) = \mathcal{R}(c)$ and then making some simplifications, we find that the resulting identity is equivalent to the following one:

$$\begin{aligned}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_q (-1)^k q^{k^2-km-k}(1-2q^k+q^{2k}) \\ = -\alpha^{\frac{1}{2}m^2} \frac{(1-q^{n-m})(1-q^m)}{(1+q^n)} \begin{bmatrix} n \\ m/2 \end{bmatrix}_{q^2} \\ = -(-q)^{-\frac{m^2}{4}} \frac{(1-q^{n-m})(1-q^m)}{(1+q^n)} \begin{bmatrix} n \\ m/2 \end{bmatrix}_{q^2}.\end{aligned}$$

The above identity can be confirmed by applying Theorem 2 as follows:

$$\begin{aligned}&\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_q (-1)^k q^{k^2-km-k}(1-2q^k+q^{2k}) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_q (-1)^k (q^{k^2-km-k} - 2q^{k^2-km} + q^{k^2-km+k}) \\ &= (-q)^{-\frac{m^2}{4}} \begin{bmatrix} n \\ m/2 \end{bmatrix}_{q^2} \frac{1+q^{n-m}}{1+q^n} - 2(-q)^{-\frac{m^2}{4}} \begin{bmatrix} n \\ m/2 \end{bmatrix}_{q^2} \\ &\quad + (-q)^{m-\frac{m^2}{4}} \begin{bmatrix} n \\ m/2 \end{bmatrix}_{q^2} \frac{1+q^{n-m}}{1+q^n} \\ &= (-q)^{-\frac{m^2}{4}} \begin{bmatrix} n \\ m/2 \end{bmatrix}_{q^2} \left(\frac{1+q^{n-m}}{1+q^n} - 2 + q^m \frac{1+q^{n-m}}{1+q^n} \right) \\ &= -(-q)^{-\frac{m^2}{4}} \begin{bmatrix} n \\ m/2 \end{bmatrix}_{q^2} (1-q^{n-m}) \frac{(1-q^m)}{1+q^n}.\end{aligned}$$

This completes the proof of identity (c) when $X = U$. □

Corollary 2. *Let $X \in \{U, V\}$ and $m, n \in \mathbb{N}_0$, with m being odd. Then the following two summation formulae hold:*

$$\text{(a)} \quad \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_U \begin{Bmatrix} n \\ m-k \end{Bmatrix}_U (-1)^k U_{n-k} X_k = \frac{U_{2n-m+1}}{V_n} \begin{Bmatrix} n \\ \frac{m-1}{2} \end{Bmatrix}_{U_2} \begin{cases} V_{n-m}, & X = V; \\ -U_{n-m}, & X = U. \end{cases}$$

$$\text{(b)} \quad \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_U \begin{Bmatrix} n \\ m-k \end{Bmatrix}_U (-1)^k V_{n-k} X_k = \frac{U_{2n-m+1}}{V_n} \begin{Bmatrix} n \\ \frac{m-1}{2} \end{Bmatrix}_{U_2} \begin{cases} \Delta U_{n-m}, & X = V; \\ -V_{n-m}, & X = U. \end{cases}$$

Proof of (b). We give a proof for identity (b) when $X = V$. In this case, the corresponding sum can be reformulated by invoking (1) and (2) as follows:

$$\begin{aligned}\mathcal{L}(b) &= \alpha^{mn-m^2+n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n\\m-k\end{bmatrix}_q(-1)^k(1+q^k)(1+q^{n-k})q^{k^2-mk}\\ &= \alpha^{mn-m^2+n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n\\m-k\end{bmatrix}_q(-1)^k q^{k^2-mk}\left[(1+q^n)+q^k+q^{n-k}\right].\end{aligned}$$

Analogously, we write the right-hand side of (b) in terms of q-binomial coefficients

$$\mathcal{R}(b) = \alpha^{n-m+mn-\frac{1}{2}m^2+\frac{1}{2}}\frac{\left(1-q^{n-m}\right)\left(1-q^{2n-m+1}\right)}{(1+q^n)}\begin{bmatrix}n\\\frac{m-1}{2}\end{bmatrix}_{q^2}.$$

By equating these two expressions $\mathcal{L}(b) = \mathcal{R}(b)$ and then making some simplifications, we find that the resulting identity is equivalent to the following one:

$$\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n\\m-k\end{bmatrix}_q(-1)^k\, q^{k^2-km}[(1+q^n)+q^k+q^{n-k}]\\ &\quad= \alpha^{\frac{1}{2}(m-1)^2}\frac{\left(1-q^{n-m}\right)\left(1-q^{2n-m+1}\right)}{(1+q^n)}\begin{bmatrix}n\\\frac{m-1}{2}\end{bmatrix}_{q^2}\\ &\quad= (-q)^{-\frac{1}{4}(m-1)^2}\frac{\left(1-q^{n-m}\right)\left(1-q^{2n-m+1}\right)}{(1+q^n)}\begin{bmatrix}n\\\frac{m-1}{2}\end{bmatrix}_{q^2}.\end{aligned}$$

The above identity is confirmed by applying Theorem 2 as follows:

$$\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n\\m-k\end{bmatrix}_q(-1)^k\, q^{k^2-km}[(1+q^n)+q^k+q^{n-k}]\\ &\quad= \Bigg\{0(1+q^n)+(-q)^{-\frac{(m-1)^2}{4}}\begin{bmatrix}n\\\frac{m-1}{2}\end{bmatrix}_{q^2}\frac{1-q^{1+2n-m}}{1+q^n}\\ &\qquad+q^n(-q)^{-m-\frac{(m-1)^2}{4}}\begin{bmatrix}n\\\frac{m-1}{2}\end{bmatrix}_{q^2}\frac{1-q^{1+2n-m}}{1+q^n}\Bigg\}\\ &\quad= (-q)^{-\frac{(m-1)^2}{4}}(1-q^{n-m})\begin{bmatrix}n\\\frac{m-1}{2}\end{bmatrix}_{q^2}\frac{1-q^{1+2n-m}}{1+q^n}.\end{aligned}$$

This completes the proof of identity (b) for the case $X = V$. □

Corollary 3 ($X \in \{U, V\}$).

(a) *Let* $m, n \in \mathbb{N}_0$*, with* $m \le n$*. Then*

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\}_U \left\{ {n \atop m-k} \right\}_V X_{2k} (-1)^{km} = 2(-1)^{\binom{m+1}{2}} \prod_{j=1}^{n} \frac{V_j^2}{V_{2j-1-m}}$$

$$\times \frac{U_m}{V_n} \begin{cases} 1 + (-1)^m \dfrac{U_{n-m}}{U_m} \displaystyle\prod_{j=1}^{n} \frac{V_{2j-1-m}}{V_{2j-m}}, & X = U; \\ 1 + (-1)^m \sqrt{\Delta} \dfrac{U_n U_{n-m}}{U_m V_n} \displaystyle\prod_{j=1}^{n} \frac{V_{2j-1-m}}{V_{2j-m}}, & X = V. \end{cases}$$

(b) *Let* $m, n \in \mathbb{N}_0$*, with* $m \ge 2n+1$*. Then*

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\}_U \left\{ {m-k \atop n+1} \right\}_V^{-1} X_{2k} (-1)^{\binom{k+1}{2} - kn} = (-1)^{(n-2m)(n+1)/2} \prod_{j=1}^{n} \frac{V_j^2}{V_{2j-1-m}}$$

$$\times \frac{U_m V_{n+1}}{V_n} \begin{cases} 1 + (-1)^m \dfrac{U_{n-m}}{U_m} \displaystyle\prod_{j=1}^{n} \frac{V_{2j-1-m}}{V_{2j-m}}, & X = U; \\ \sqrt{\Delta} + \Delta(-1)^m \dfrac{U_{n-m}}{U_m} \displaystyle\prod_{j=1}^{n} \frac{V_{2j-1-m}}{V_{2j-m}}, & X = V. \end{cases}$$

Proof of (a). We offer a sample proof for the case $X = U$. By making use of (1), (2) and (3), we can express the left-hand side of the claimed identity as:

$$\begin{aligned} \mathcal{L}(a) &= \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_U \left\{ {n \atop m-k} \right\}_V U_{2k} (-1)^{km} \\ &= \frac{\alpha^{nm-1-m^2}}{(1-q)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_{-q,q} (-1)^{km} (1 - q^{2k}) \alpha^{2k+2km-2k^2} \\ &= \frac{\alpha^{nm-1-m^2}}{(1-q)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_{-q,q} (1 - q^{2k}) q^{k^2-km-k}. \end{aligned}$$

Now, evaluating the above q-binomial sum by applying formulae (b) and (c) given in Theorem 1

$$\begin{aligned} &\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_{-q,q} (1 - q^{2k}) q^{k^2-km-k} \\ &\quad= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ m-k \end{bmatrix}_{-q,q} (q^{k^2-km-k} - q^{k^2-km+k}) \\ &\quad= 2 \frac{q^{-\binom{m+1}{2}} (-q;q)_n^2}{(1+q^n)(-q^{1-m};q^2)_n} \left\{ 1 + \frac{(q^m - q^n)(-q^{1-m};q^2)_n}{(1+q^m)(-q^{2-m};q^2)_n} \right\} \\ &\quad\quad - 2 \frac{q^{-\binom{m}{2}} (-q;q)_n^2}{(1+q^n)(-q^{1-m};q^2)_n} \left\{ 1 - \frac{(q^m - q^n)(-q^{1-m};q^2)_n}{(1+q^m)(-q^{2-m};q^2)_n} \right\}, \end{aligned}$$

we find, after some simplifications, the following closed expression:

$$\mathcal{L}(a) = 2(-1)^{\binom{m+1}{2}} \frac{\alpha^{m+nm-1}(1-q^m)(-q;q)_n^2}{(1-q)(1+q^n)(-q^{1-m};q^2)_n} \left\{1 + q^m \frac{1-q^{n-m}}{1-q^m} \frac{(-q^{1-m};q^2)_n}{(-q^{2-m};q^2)_n}\right\}.$$

This can further be written in terms of general Fibonacci–Lucas sequences. In fact, by utilizing the three equalities

$$(-q;q)_n = \alpha^{-\frac{n}{2}(n+1)} \prod_{j=1}^{n} V_j,$$
$$(-q^{1-m};q^2)_n = \alpha^{n(m-n)} \prod_{j=1}^{n} V_{2j-1-m},$$
$$(-q^{2-m};q^2)_n = \alpha^{n(m-n-1)} \prod_{j=1}^{n} V_{2j-m};$$

we can proceed further with the following reformulation:

$$\mathcal{L}(a) = (-1)^{\binom{m+1}{2}} \frac{2U_m}{V_n} \prod_{j=1}^{n} \frac{V_j^2}{V_{2j-1-m}} \left\{1 + (-1)^m \frac{U_{n-m}}{U_m} \prod_{j=1}^{n} \frac{V_{2j-1-m}}{V_{2j-m}}\right\},$$

which is exactly the expression $\mathcal{R}(a)$ for the case $X = U$. □

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