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Products of Fermat or Mersenne numbers in some sequences

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Abstract. Let P_n be an n-th Padovan number, E_n an n-th Perrin number, and N_n an n-th Narayana number. In this paper, we solve the Diophantine equations

$$P_n = (2^a - 1)(2^b - 1),$$

$$E_n = (2^a - 1)(2^b - 1),$$

and

$$N_n = (2^a \pm 1)(2^b \pm 1),$$

in positive integers n, a and b. Therefore, we determine the Padovan or Perrin numbers that are products of two Mersenne numbers and the Narayana numbers that are products of two Mersenne or Fermat numbers.
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1. Introduction

Let (U_n) and (V_n) be two linear recurrent sequences. The problem of finding the common terms of (U_n) and (V_n) was treated in [10], [12], [15], [16], [19]. Their authors proved, under some assumption, that the Diophantine equation

$$U_n = V_m$$

has only finitely many integer solutions (m, n).

The Padovan sequence $(P_n)_{n \geq 0}$ is defined by

$$P_{n+3} = P_{n+1} + P_n,$$

for $n \geq 0$, where $P_0 = P_1 = P_2 = 1$. This is the sequence A000931 in the OEIS [20]. A few terms of this sequence are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, 265, 351, 465, 616, \ldots$$

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Let $(E_n)_{n\geq 0}$ be the Perrin sequence given by

$$E_{n+3} = E_{n+1} + E_n,$$

for $n \geq 0$, where $E_0 = 3$, $E_1 = 0$ and $E_2 = 2$. It is the OEIS [20] A001608 sequence. Its first few terms are

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, 277, \ldots$$

Narayana's cows sequence $(N_n)_{n\geq 0}$ originated from a herd of cows and calves problem, proposed by the Indian mathematician Narayana in his book *Ganita Kaumudi* [1]. It is the sequence A000930 in the OEIS [20] satisfying the recurrence relation

$$N_{n+3} = N_{n+2} + N_n, \tag{1}$$

for $n \geq 0$, with initial terms $N_0 = 0$ and $N_1 = N_2 = 1$. The first few terms of $(N_n)_{n\geq 0}$ are

$$0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, \ldots.$$

For $n \geq 0$, the n^{th} *Fermat number*, denoted by F_n, is a number of the form

$$F_n = 2^{2^n} + 1.$$

The first elements of its list are

$$3, 5, 17, 257, 65537, 4294967297, 18446744073709551617,$$

$$340282366920938463463374607431768211457, \ldots.$$

This sequence is indexed as A000125 in the OEIS [20]. It is known that every odd prime number of the form $2^k + 1$ is a Fermat number and such primes are called Fermat primes. It is conjectured that just the first five numbers in this sequence are primes. More details and properties of the Fermat numbers may be found in [11]. For $n \geq 0$, the n^{th} *Mersenne number*, denoted by M_n, is a number of the form

$$M_n = 2^n - 1.$$

The first elements of its list are

$$0, 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, \ldots$$

These numbers are named after Mersenne, who studied them, though the term usually refers to numbers of the form $2^p - 1$, where p is a prime.

It is known that if M_n is a prime, then n is also a prime. It is conjectured that there are infinitely many Mersenne primes, (see [11]).

In this paper, we are interested in solving equations involving Padovan, Perrin, Narayana, Fermat, and Mersenne numbers. The case of Padovan and Perrin numbers, which are products of two Fermat numbers, has been treated in [2]. Mainly, we will prove the following theorems.

Theorem 1. *All the solutions of the Diophantine equation*

$$P_n = (2^a - 1)(2^b - 1), \tag{2}$$

in nonnegative integers n, a and b, with $1 \leq a \leq b$, are given by

$$P_0 = P_1 = P_2 = (2^1 - 1)(2^1 - 1) = 1, \quad P_5 = (2^1 - 1)(2^2 - 1) = 3,$$

$$P_8 = (2^1 - 1)(2^3 - 1) = 7, \quad P_9 = (2^2 - 1)(2^2 - 1) = 9, \quad P_{12} = (2^2 - 1)(2^3 - 1) = 21,$$

$$P_{15} = (2^3 - 1)(2^3 - 1) = 49 \quad \textit{and} \quad P_{23} = (2^4 - 1)(2^5 - 1) = 465.$$

Theorem 2. *All the solutions of the Diophantine equation*

$$E_n = (2^a - 1)(2^b - 1), \tag{3}$$

in nonnegative integers n, a and b, with $0 \leq a \leq b$, are given by

$$E_0 = E_3 = (2^1 - 1)(2^2 - 1) = 3, \; E_1 = 0 \quad \textit{and} \quad E_7 = (2^1 - 1)(2^3 - 1) = 7.$$

Theorem 3. *All the solutions of the Diophantine equations*

$$N_n = (2^a \pm 1)(2^b \pm 1), \tag{4}$$

in nonnegative integers n, a and b, with $1 \leq a \leq b$, are given by

$$N_1 = N_2 = N_3 = (2^1 - 1)(2^1 - 1) = 1, \quad N_5 = (2^1 - 1)(2^2 - 1) = (2^1 - 1)(2^1 + 1) = 3,$$

$$N_8 = (2^1 + 1)(2^1 + 1) = (2^1 + 1)(2^2 - 1) = (2^1 - 1)(2^3 + 1) = (2^2 - 1)(2^2 - 1) = 9,$$

$$N_{15} = (2^1 - 1)(2^7 + 1) = 129$$

and

$$N_{16} = (2^1 + 1)(2^6 - 1) = (2^2 - 1)(2^6 - 1) = 189.$$

By theorems 1 and 2 one can easily deduce the following consequence.

Corollary 1. *The only Padovan and Perrin numbers which are Mersenne numbers are $P_0 = P_1 = P_2 = 1$, $P_5 = 3$, $P_8 = 7$ and $E_0 = E_3 = 3$, $E_1 = 0$, $E_7 = 7$.*

By Theorem 3 we deduce the following consequences.

Corollary 2. *The only product of two Fermat numbers in Narayana's cows sequence is $N_8 = 9$, and the only product of two Mersenne numbers in Narayana's cows sequence are $N_0 = 0$, $N_1 = N_2 = N_3 = 1$, $N_5 = 3$, $N_8 = 9$, $N_{16} = 189$.*

Corollary 3. *The only Mersenne numbers in Narayana's cows sequence are*

$$N_0 = 0, \quad N_1 = N_2 = N_3 = 1, \quad N_5 = 3.$$

Notice that the case $N_0 = 0$ gives infinitely many solutions.

Our proofs of theorems 1, 2 and 3 are mainly based on linear forms in logarithms of algebraic numbers and a reduction algorithm originally introduced by Baker and Davenport in [3]. Here, we use a modified version of the result due to Dujella-Pethő [7].

2. Useful properties of these sequences

2.1. Properties of Padovan and Perrin sequences

In this subsection, we recall some facts and properties of the Padovan and the Perrin sequences which will be used later. Their characteristic equation

$$x^3 - x - 1 = 0$$

has roots $\alpha, \beta, \gamma = \overline{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \quad \beta = \frac{-r_1 - r_2 + i\sqrt{3}(r_1 - r_2)}{12}$$

and

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \text{ and } r_2 = \sqrt[3]{108 - 12\sqrt{69}}.$$

Let

$$\begin{aligned} c_\alpha &= \frac{(1-\beta)(1-\gamma)}{(\alpha-\beta)(\alpha-\gamma)} = \frac{1+\alpha}{-\alpha^2+3\alpha+1}, \\ c_\beta &= \frac{(1-\alpha)(1-\gamma)}{(\beta-\alpha)(\beta-\gamma)} = \frac{1+\beta}{-\beta^2+3\beta+1}, \\ c_\gamma &= \frac{(1-\alpha)(1-\beta)}{(\gamma-\alpha)(\gamma-\beta)} = \frac{1+\gamma}{-\gamma^2+3\gamma+1} = \overline{c_\beta}. \end{aligned}$$

Binet's formula of P_n is

$$P_n = c_\alpha \alpha^n + c_\beta \beta^n + c_\gamma \gamma^n, \text{ for all } n \geq 0, \tag{5}$$

and that of E_n is

$$E_n = \alpha^n + \beta^n + \gamma^n, \text{ for all } n \geq 0. \tag{6}$$

Numerically, we have

$$\begin{aligned} &1.32 < \alpha < 1.33, \\ &0.86 < |\beta| = |\gamma| < 0.87, \\ &0.72 < c_\alpha < 0.73, \\ &0.24 < |c_\beta| = |c_\gamma| < 0.25. \end{aligned}$$

It is easy to check that

$$|\beta| = |\gamma| = \alpha^{-1/2}.$$

Furthermore, using induction, one can prove the following inequalities:

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1}, \quad \text{for all } n \geq 4, \tag{7}$$

and

$$\alpha^{n-2} \leq E_n \leq \alpha^{n+1}, \quad \text{for all } n \geq 2. \tag{8}$$

2.2. Properties of the Narayana sequence

The characteristic equation of sequence (1) is $x^3 - x^2 - 1 = 0$, which has roots ρ, δ, $\lambda = \overline{\delta}$, where

$$\rho = \frac{\sqrt[3]{116 + 12\sqrt{93}}}{6} + \frac{2}{3\sqrt[3]{116 + 12\sqrt{93}}} + \frac{1}{3}$$

and

$$\delta = -\frac{\sqrt[3]{116 + 12\sqrt{93}}}{12} - \frac{1}{3\sqrt[3]{116 + 12\sqrt{93}}} + \frac{1}{3} + i\frac{\sqrt{3}}{2}\left(\frac{\sqrt[3]{116 + 12\sqrt{93}}}{6} - \frac{2}{3\sqrt[3]{116 + 12\sqrt{93}}}\right).$$

Narayana's cows sequence has Binet's formula

$$N_n = C_\rho \rho^n + C_\delta \delta^n + C_\lambda \lambda^n, \quad \text{for all } n \geq 0, \tag{9}$$

where

$$C_\rho = \frac{\rho}{(\rho - \delta)(\rho - \lambda)}, \quad C_\delta = \frac{\delta}{(\delta - \rho)(\delta - \lambda)}, \quad C_\lambda = \frac{\lambda}{(\lambda - \rho)(\lambda - \delta)}.$$

Formula (9) can also be written in the form

$$N_n = c_\rho \rho^{n+2} + c_\delta \delta^{n+2} + c_\lambda \lambda^{n+2}, \quad \text{for all } n \geq 0, \tag{10}$$

where

$$c_\rho = \frac{1}{\rho^3 + 2}, \quad c_\delta = \frac{1}{\delta^3 + 2}, \quad c_\lambda = \frac{1}{\lambda^3 + 2}.$$

The coefficient c_ρ has the minimal polynomial $31x^3 - 31x^2 + 10x - 1$ over $\mathbb{Z}$ and all the zeros of this polynomial lie strictly inside the unit circle.

Numerically, we have

$$\begin{aligned} &1.46 < \rho < 1.47, \\ &0.82 < |\delta| = |\lambda| < 0.83, \\ &0.19 < c_\rho < 0.2, \\ &0.40 < |c_\delta| = |c_\lambda| < 0.41. \end{aligned}$$

Using the facts from the introduction, one can prove that the n^{th} Narayana number satisfies the following inequalities:

$$\rho^{n-2} \leq N_n \leq \rho^{n-1}, \tag{11}$$

for all $n \geq 1$, (see [4]).

3. The proof of Theorem 1

We take equation (2) with non-negative integers (n, a, b), where $1 \leq a \leq b$, and assume that $n \geq 4$.

Using inequality (7) and equation (2), we obtain

$$\alpha^{n-2} \leq P_n = (2^a - 1)(2^b - 1) < 2^{a+b}$$

and

$$2^{a+b-2} = (2^{a-1})(2^{b-1}) \leq (2^a - 1)(2^b - 1) = P_n \leq \alpha^{n-1}.$$

Hence, we get

$$(a + b - 2)\frac{\log 2}{\log \alpha} + 1 \leq n < (a + b)\frac{\log 2}{\log \alpha} + 2.$$

Thus, using $2.43 < \dfrac{\log 2}{\log \alpha} < 2.5$, we deduce that

$$2.43(a + b) - 3.86 < n < 2.5(a + b) + 2. \tag{12}$$

From Binet's formula (5), we rewrite equation (2) and obtain

$$\left|c_\alpha \alpha^n - 2^{a+b}\right| \leq 2^a + 2^b + 1 + 2|c_\beta| \cdot |\beta|^n < 2^a + 2^b + 2,$$

for $n \geq 4$. Dividing through by 2^{a+b}, we get

$$|\Gamma_1| < \frac{1}{2^b} + \frac{1}{2^a} + \frac{2}{2^{a+b}} < \frac{4}{2^a}, \tag{13}$$

where

$$\Gamma_1 := c_\alpha \alpha^n 2^{-(a+b)} - 1.$$

Before determining a lower bound of Γ_1, let us recall a useful result related to Baker's method.

Let α be an algebraic number of degree d, let $a > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\alpha = \alpha^{(1)}, \ldots, \alpha^{(d)}$ denote its conjugates. We denote by

$$h(\alpha) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log\left(\max\{|\alpha^{(i)}|, 1\}\right)\right)$$

the logarithmic height of α. This height has the following properties. For α, β algebraic numbers, we have

$$h(\alpha\beta) \leq h(\alpha) + h(\beta),$$
$$h(\alpha \pm \beta) \leq \log 2 + h(\alpha) + h(\beta).$$

Moreover, for any algebraic number $\alpha \neq 0$ and for any $n \in \mathbb{Z}$,

$$h(\alpha^n) \leq |n| h(\alpha).$$

Now, let $\mathbb{K}$ be an algebraic number field of degree $d_{\mathbb{K}}$. Let $\eta_1, \ldots, \eta_l \in \mathbb{K}$ and $d_1, \ldots, d_l$ be nonzero integers. Let $D \geq \max\{|d_1|, \ldots, |d_l|\}$ and

$$\Gamma = \prod_{i=1}^{l} \eta_i^{d_i} - 1.$$

Let $A_1, \ldots, A_l$ be real numbers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{K}} h(\eta_j), |\log \eta_j|, 0.16\}, \quad \text{for } j = 1, \ldots, l.$$

The first tool that we need is the following result due to Matveev [14]. But, we will use the version of Bugeaud, Mignotte and Siksek [[6], Theorem 9.4].

Theorem 4. *If $\Gamma \neq 0$, then*

$$\log |\Gamma| > -1.4 \cdot 30^{l+3} \cdot l^{4.5} \cdot d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log D) A_1 \ldots A_l.$$

Now, one can observe that $\Gamma_1 \neq 0$. To see this, we consider the $\mathbb{Q}$-automorphism σ of the Galois extension $\mathbb{Q}(\alpha, \beta)$ over $\mathbb{Q}$ given by $\sigma(\alpha) := \beta$ and $\sigma(\beta) := \alpha$. Thus, we have

$$1 < 2^{a+b} = |\sigma(c_\alpha \alpha^n)| = |c_\beta||\beta|^n < |c_\beta| < 0.25,$$

which is a contradiction. Hence $\Gamma_1 \neq 0$ and we can apply Theorem 4 to it. To do this, we consider

$$\eta_1 := c_\alpha, \ \eta_2 := \alpha, \ \eta_3 := 2, \quad d_1 := 1, \ d_2 := n, \ d_3 := -(a+b).$$

The algebraic numbers η_1, η_2, η_3 are elements of the field $\mathbb{K} = \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = 3$. We have that $h(\eta_2) = \log \alpha / 3$ and $h(\eta_3) = \log 2$. Thus, we can take

$$\max\{3h(\eta_2), |\log \eta_2|, 0.16\} < 0.3 := A_2$$

and

$$\max\{3h(\eta_3), |\log \eta_3|, 0.16\} < 2.08 := A_3.$$

On the other hand, the minimal polynomial of c_α is

$$23x^3 - 23x^2 + 6x - 1$$

and has roots c_α, c_β and c_γ. Since $c_\alpha < 1$ and $|c_\beta| = |c_\gamma| < 1$, then we get

$$h(\eta_1) = \frac{\log 23}{3}.$$

So, we can take

$$\max\{3h(\eta_1), |\log \eta_1|, 0.16\} < 3.14 := A_1.$$

Finally, inequality (12) implies that we can choose $D := n + 3.86 \geq \max\{1, n, a+b\}$.

From Theorem 4, we obtain

$$\begin{aligned}\log|\Gamma_1| &> -1.4\cdot 30^6\cdot 3^{4.5}\cdot 3^2(1+\log 3)(1+\log(n+3.86))\cdot 3.14\cdot 0.3\cdot 2.08\\ &> -5.30\cdot 10^{12}\cdot(1+\log(n+3.86)).\end{aligned}$$

By the fact $1+\log(n+3.86)<2.3\log n$, which holds for all $n\geq 4$, we obtain

$$\log|\Gamma_1| > -1.22\cdot 10^{13}\log n,$$

which, combined with (13), gives

$$a\log 2 < 1.23\cdot 10^{13}\log n. \tag{14}$$

We rewrite equation (2) as

$$\frac{P_n}{2^a-1}+1=2^b,$$

and consequently

$$\left|\frac{c_\alpha\alpha^n}{2^a-1}-2^b\right| = \left|-1-\frac{c_\beta\beta^n+c_\gamma\gamma^n}{2^a-1}\right| \leq 1+\frac{2|c_\beta|\cdot|\beta|}{2^a-1} < 1.5,$$

for $a\geq 1$. Dividing through by 2^b, we obtain

$$|\Gamma_2| < \frac{1.5}{2^b}, \tag{15}$$

where

$$\Gamma_2 := \frac{c_\alpha}{2^a-1}\alpha^n 2^{-b}-1.$$

Note that with a similar argument as above, it can be proved that $\Gamma_2\neq 0$. So, we can apply Theorem 4 to it. We consider

$$\eta_1 := \frac{c_\alpha}{2^a-1},\ \eta_2:=\alpha,\ \eta_3:=2,\quad d_1:=1,\ d_2:=n,\ d_3:=-b.$$

Thus, $D:=n+3.86$. The heights of η_2 and η_3 have already been calculated. From the properties of the heights we get that

$$h(\eta_1)\leq h(c_\alpha)+h(2^a-1)\leq 1.74+a\log 2+\log 2 < 1.24\cdot 10^{13}\log n,$$

where we have used inequality (14). We choose

$$\max\{3h(\eta_1),|\log\eta_1|,0.16\} < 3.72\cdot 10^{13}\log n := A_1,$$

A_2 and A_3 as above. Therefore, from Theorem 4 we obtain

$$\log|\Gamma_2| > -1.4\cdot 30^6\cdot 3^{4.5}\cdot 3^2(1+\log 3)(1+\log(n+3.86))\cdot 3.72\cdot 10^{13}\log n\cdot 0.3\cdot 2.08,$$

i.e.,

$$\log|\Gamma_2| > -1.45\cdot 10^{26}\cdot(\log n)^2, \tag{16}$$

where we used as above the inequality $1 + \log(n + 3.86) < 2.3 \log n$, for all $n \geq 4$. Using inequalities (16) and (15), we obtain

$$b < \frac{1.46 \cdot 10^{26} \cdot (\log n)^2}{\log 2} < 2.11 \cdot 10^{26} \cdot (\log n)^2,$$

which, combined with inequalities (12) and (14), gives

$$\frac{n}{(\log n)^2} < 5.3 \cdot 10^{26}.$$

To solve the above inequality, we will recall the following result of Guzmán and Luca [9], which will be very useful.

Lemma 1. *If* $m \geq 1$, $T > (4m^2)^m$ *and* $T > x/(\log x)^m$, *then*

$$x < 2^m T (\log T)^m.$$

Therefore, taking $m := 2$ and $T := 5.3 \cdot 10^{26}$ in Lemma 1, we get

$$n < 8.03 \cdot 10^{30}. \tag{17}$$

The above bound for n is very high. So, we will reduce it. To do this, we need to recall a variant of the reduction method of Baker and Davenport [3] due to Dujella and Pethö [7]. We use the one given by Bravo, Gómez, and Luca [5].

Lemma 2. *Let* M *be a positive integer, let* p/q *be a convergent of the continued fraction of the irrational* τ *such that* $q > 6M$, *and let* A, B, μ *be some real numbers with* $A > 0$ *and* $B > 1$. *Let further* $\varepsilon = ||\mu q|| - M \cdot ||\tau q||$, *where* $|| \cdot ||$ *denotes the distance from the nearest integer. If* $\varepsilon > 0$, *then there is no solution of the inequality*

$$0 < |m\tau - n + \mu| < AB^{-k},$$

in positive integers m, n *and* k *with*

$$m \leq M \text{ and } k \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

We first consider

$$\Lambda_1 := (a + b) \log 2 - n \log \alpha + \log \left(\frac{1}{c_\alpha} \right),$$

and go to inequality (13). Note that $e^{-\Lambda_1} - 1 = \Gamma_1 \neq 0$. Thus, $\Lambda_1 \neq 0$. If $\Lambda_1 < 0$, then

$$0 < |\Lambda_1| < e^{|\Lambda_1|} - 1 = |\Gamma_1| < \frac{4}{2^a}.$$

If $\Lambda_1 > 0$, we have $1 - e^{-\Lambda_1} = \left| e^{-\Lambda_1} - 1 \right| < 1/2$. Hence, $e^{\Lambda_1} < 2$. Thus, we get

$$0 < \Lambda_1 < e^{\Lambda_1} - 1 = e^{\Lambda_1} |\Gamma_1| < \frac{8}{2^a}.$$

So, in both cases we have

$$0 < |\Lambda_1| < \frac{8}{2^a}.$$

Dividing through by $\log\alpha$ we get

$$|m\tau - n + \mu| < \frac{29}{2^a},$$

where

$$m := a + b,\ \tau := \frac{\log 2}{\log\alpha} \text{ and } \mu := \frac{\log\left(\frac{1}{c_\alpha}\right)}{\log\alpha}.$$

We apply Lemma 2. Put $M := 8.03\cdot 10^{30}$, $A := 29$ and $B := 2$. Since $m = a+b < n$, from (17) we have that M is the upper bound on m. A quick computation with Maple reveals that the convergent

$$\frac{p_{73}}{q_{73}} = \frac{2823950171188783250614630890707 67}{1145634878592520868112972034593 50}$$

of τ is such that $q_{73} > 6M$ and $\varepsilon \geq 0.25250824 > 0$. Therefore, we get

$$a < \frac{1}{\log 2}\log\left(\frac{29\cdot 114563487859252086811297203459350}{\varepsilon}\right) \leq 114.$$

Next, we consider $1 \leq a \leq 113$ and

$$\Lambda_2 := b\log 2 - n\log\alpha + \log\left(\frac{2^a - 1}{c_\alpha}\right),$$

and go to inequality (15). Note that $e^{-\Lambda_2} - 1 = \Gamma_2 \neq 0$. Thus, $\Lambda_2 \neq 0$. With an argument similar to the above one, we get

$$0 < |\Lambda_2| < \frac{3}{2^b}.$$

Dividing through by $\log\alpha$ we get

$$|m\tau - n + \mu_a| < \frac{11}{2^b},$$

where

$$m := b,\ \tau := \frac{\log 2}{\log\alpha} \text{ and } \mu_a := \frac{\log\left(\frac{2^a-1}{c_\alpha}\right)}{\log\alpha} \quad (1 \leq a \leq 113).$$

We apply Lemma 2. Put $M := 8.03\cdot 10^{30}$, $A := 11$ and $B := 2$. Since $m = b < n$, from (17) we have that M is the upper bound on m. A quick computation with Maple proves that the convergent

$$\frac{p_{76}}{q_{76}} = \frac{259223773249806722990650449635952 3}{105163256428962117966701853572376 8}$$

of τ is such that $q_{78} > 6M$ and $\varepsilon_a \geq 0.0004236 > 0$, for all $1 \leq a \leq 113$. Therefore, we get that the maximum value of $\frac{\log(Aq/\varepsilon)}{\log B}$ over all $1 \leq a \leq 113$ is $124.360687\ldots$ which, according to Lemma 2, is an upper bound of b.

Hence, we deduce that the possible solutions (n, a, b) of equation (2) satisfy $1 \leq a \leq b \leq 124$. Therefore, we use inequality (12) to obtain $n \leq 621$.

Finally, we used Maple to compare P_n and $(2^a - 1)(2^b - 1)$, for $0 \leq n \leq 621$ and $1 \leq a \leq b \leq 124$. We checked that all the solutions of equation (2) in nonnegative integers n, a and b, with $1 \leq a \leq b$, are given by

$$P_0 = P_1 = P_2 = (2^1 - 1)(2^1 - 1) = 1, \quad P_5 = (2^1 - 1)(2^2 - 1) = 3,$$

$$P_8 = (2^1 - 1)(2^3 - 1) = 7, \quad P_9 = (2^2 - 1)(2^2 - 1) = 9, \quad P_{12} = (2^2 - 1)(2^3 - 1) = 21,$$

$$P_{15} = (2^3 - 1)(2^3 - 1) = 49 \text{ and } P_{23} = (2^4 - 1)(2^5 - 1) = 465.$$

This completes the proof of Theorem 1.

4. Proof of Theorem 2

We start the study of equation (3) in non-negative integers (n, a, b), where $1 \leq a \leq b$, and we may assume that $n \geq 4$. The proof of Theorem 2 is similar to that of Theorem 1. But for the sake of completeness, we will give most of the details.

Using inequality (8) and equation (3), we obtain

$$\alpha^{n-2} \leq E_n = (2^a - 1)(2^b - 1) < 2^{a+b},$$

and

$$2^{a+b-2} \leq (2^a - 1)(2^b - 1) = E_n \leq \alpha^{n+1}.$$

Hence, we get

$$(a + b - 2)\frac{\log 2}{\log \alpha} - 1 < n < (a + b)\frac{\log 2}{\log \alpha} + 2.$$

Thus, using $2.43 < \dfrac{\log 2}{\log \alpha} < 2.5$, we deduce that

$$2.43(a + b) - 5.86 < n < 2.5(a + b) + 2. \tag{18}$$

From Binet's formula (6), we rewrite equation (3) and obtain

$$\left|\alpha^n - 2^{a+b}\right| \leq 2^a + 2^b + 1 + 2|\beta|^n < 2^a + 2^b + 3.$$

Dividing through by 2^{a+b}, we get

$$|\Gamma_3| < \frac{1}{2^b} + \frac{1}{2^a} + \frac{3}{2^{a+b}} < \frac{5}{2^a}, \tag{19}$$

where

$$\Gamma_3 := \alpha^n 2^{-(a+b)} - 1.$$

Observe that $\Gamma_3 \neq 0$. To see this, we consider the $\mathbb{Q}$-automorphism σ of the Galois extension $\mathbb{Q}(\alpha, \beta)$ over $\mathbb{Q}$ given by $\sigma(\alpha) := \beta$ and $\sigma(\beta) := \alpha$. Thus, we obtain

$$1 < 2^{a+b} = |\sigma(\alpha^n)| = |\beta|^n < 0.87,$$

which is a contradiction. Hence $\Gamma_3 \neq 0$ and we can apply Theorem 4 to it. To do this, we consider

$$\eta_1 := \alpha,\ \eta_2 := 2, \quad d_1 := n,\ d_2 := -(a+b).$$

As previously, we have

$$A_1 := 0.3, \ \ A_2 := 2.08.$$

Furthermore, inequality (18) implies that we can choose $D := n + 5.86 \geq \max\{n, a+b\}$. Using Theorem 4, we obtain

$$\begin{aligned}\log|\Gamma_3| &> -1.4 \cdot 30^{2+3} \cdot 2^{4.5} \cdot 3^2 (1+\log 3)(1+\log(n+5.86)) \cdot 0.3 \cdot 2.08 \\ &> -9.08 \cdot 10^9 \cdot (1+\log(n+5.86)).\end{aligned}$$

As $1 + \log(n + 5.86) < 2.4 \log n$, for all $n \geq 4$, one can see that

$$\log|\Gamma_3| > -2.18 \cdot 10^{10} \log n.$$

Combining this with (19), gives

$$a \log 2 < 2.19 \cdot 10^{10} \log n. \tag{20}$$

We go back to equation (3) and rewrite it as

$$\frac{E_n}{2^a - 1} + 1 = 2^b.$$

Consequently, we obtain

$$\left| \frac{\alpha^n}{2^a - 1} - 2^b \right| = \left| -1 - \frac{\beta^n + \gamma^n}{2^a - 1} \right| \leq 1 + \frac{2\,|\beta|}{2^a - 1} < 2.74,$$

for $a \geq 1$. Dividing through by 2^b, we obtain

$$|\Gamma_4| < \frac{2.74}{2^b}, \tag{21}$$

where

$$\Gamma_4 := \frac{1}{2^a - 1} \alpha^n 2^{-b} - 1.$$

Note that with a similar argument as above, it can be proved that $\Gamma_4 \neq 0$. So, we can apply Theorem 4 to it. To do this, we consider

$$\eta_1 := \frac{1}{2^a - 1},\ \eta_2 := \alpha,\ \eta_3 := 2, \quad d_1 := 1,\ d_2 := n,\ d_3 := -b.$$

Again here, one can easily see that

$$A_2 := 0.3, \quad A_3 := 2.08, \quad D := n + 5.86.$$

Using the properties of the heights, we get

$$h(\eta_1) = h(2^a - 1) \leq 0.7 + a \log 2 < 2.2 \cdot 10^{10} \log n,$$

where we have used inequality (20). We choose

$$\max\{3h(\eta_1), |\log \eta_1|, 0.16\} < 6.6 \cdot 10^{10} \log n := A_1.$$

Therefore, from Theorem 4, we obtain

$$\log |\Gamma_4| > -1.4 \cdot 30^{3+3} \cdot 3^{4.5} \cdot 3^2 (1 + \log 3)(1 + \log(n + 5.86)) \cdot 6.6 \cdot 10^{10} \log n \cdot 0.3 \cdot 2.08,$$

i.e.,

$$\log |\Gamma_4| > -2.69 \cdot 10^{23} \cdot (\log n)^2, \tag{22}$$

knowing that $1 + \log(n + 5.86) < 2.4 \log n$, for all $n \geq 4$. Using inequalities (22) and (21), one can see that

$$b < \frac{2.7 \cdot 10^{23} \cdot (\log n)^2}{\log 2} < 3.9 \cdot 10^{23} \cdot (\log n)^2.$$

Combining this with inequalities (18) and (20), gives

$$\frac{n}{(\log n)^2} < 9.76 \cdot 10^{23}.$$

Taking $m := 2$ and $T := 9.76 \cdot 10^{23}$ in Lemma 1, we get

$$n < 1.2 \cdot 10^{28}. \tag{23}$$

Now, we will reduce the above bound on n. To do this, we first consider

$$\Lambda_3 := (a + b) \log 2 - n \log \alpha,$$

and go back to inequality (19). Note that $e^{-\Lambda_3} - 1 = \Gamma_3 \neq 0$. Thus, $\Lambda_3 \neq 0$. If $\Lambda_3 < 0$, then

$$0 < |\Lambda_3| < e^{|\Lambda_3|} - 1 = |\Gamma_3| < \frac{5}{2^a}.$$

If $\Lambda_3 > 0$, then we have $1 - e^{-\Lambda_3} = \left|e^{-\Lambda_3} - 1\right| < 1/2$. Hence, $e^{\Lambda_3} < 2$. Thus, we get

$$0 < \Lambda_3 < e^{\Lambda_3} - 1 = e^{\Lambda_3} |\Gamma_3| < \frac{10}{2^a}.$$

So, in both cases we have

$$0 < |\Lambda_3| < \frac{10}{2^a}.$$

Dividing through by $\log 2$, we get

$$\left| n\frac{\log\alpha}{\log 2} - (a+b)\right| < \frac{15}{2^a}. \tag{24}$$

To obtain a lower bound for n, unfortunately, we cannot apply Lemma 2 when μ is a linear combination of 1 and τ since then $\varepsilon < 0$. In this case, we use the following property of continued fractions (see Theorem 8.2.4 and the top of page 263 in [17]).

Lemma 3. *Let p_i/q_i be the convergents of the continued fraction $[a_0, a_1, \ldots]$ of the irrational number τ. Let M be a positive integer and put $a_M := \max\{a_i; 0 \le i \le N+1\}$, where $N \in \mathbb{N}$ is such that $q_N \le M \le q_{N+1}$. If $x, y \in \mathbb{Z}$ with $x > 0$, then*

$$|x\tau - y| > \frac{1}{(a_M+2)x}, \quad \textit{for all } x < M.$$

Therefore, here we will apply Lemma 3 to the left-hand side of inequality (24), with

$$x := n,\ \tau := \frac{\log\alpha}{\log 2} \ \text{ and } y := a+b.$$

Since $n < 1.2 \cdot 10^{28}$, then we can take $M := 1.2 \cdot 10^{28}$. Let

$$[a_0, a_1, a_2, a_3, a_4, \ldots] = [0, 2, 2, 6, 1, 1, 1, 2, 1, 13, 3, 1, 1, 1, 1, 1, 8, 1, 3, 2, \ldots]$$

be the continued fraction of τ. Using Maple, one can see that $q_{64} < M < q_{65}$, and since $\max\{a_i : 1 \le i \le 65\} = a_{42} = 80$, then by Lemma 3, we get

$$\left| n\frac{\log\alpha}{\log 2} - (a+b)\right| > \frac{1}{82n}. \tag{25}$$

Using inequalities (24), (25) and (23), we obtain

$$a < \frac{\log(15 \cdot 82 \cdot 1.2 \cdot 10^{28})}{\log 2} < 104.$$

Next, we consider $1 \le a \le 103$,

$$\Lambda_4 := b\log 2 - n\log\alpha + \log\left(2^a - 1\right),$$

and go to inequality (21). Note that $e^{-\Lambda_4} - 1 = \Gamma_4 \neq 0$. Thus, $\Lambda_4 \neq 0$. With an argument similar to the above one we get

$$0 < |\Lambda_4| < \frac{5.48}{2^b}. \tag{26}$$

For $a = 1$,

$$\Lambda_4 := b\log 2 - n\log\alpha.$$

Hence, we get

$$\left| n\frac{\log\alpha}{\log 2} - b\right| < \frac{8}{2^b}. \tag{27}$$

To obtain a lower bound for the left-hand side of inequality (27), we will apply Lemma 3 with

$$x := n,\ \tau := \frac{\log\alpha}{\log 2} \ \text{ and } y := b.$$

Since $n < 1.2\cdot 10^{28}$, then we can take $M := 1.2\cdot 10^{28}$. Let

$$[a_0, a_1, a_2, a_3, a_4, \ldots] = [0, 2, 2, 6, 1, 1, 1, 2, 1, 13, 3, 1, 1, 1, 1, 1, 8, 1, 3, 2, \ldots]$$

be the continued fraction of τ. Using Maple, we see that $q_{64} < M < q_{65}$ and, since $\max\{a_i : 1 \le i \le 65\} = a_{42} = 80$, then, by Lemma 3, we get

$$\left| n\frac{\log\alpha}{\log 2} - b \right| > \frac{1}{82n}. \tag{28}$$

Using inequalities (27), (28) and (23), we obtain

$$b < \frac{\log(8\cdot 82\cdot 1.2\cdot 10^{28})}{\log 2} < 103.$$

For $2 \le a \le 103$, dividing inequality (26) by $\log\alpha$, we get

$$|m\tau - n + \mu_a| < \frac{20}{2^b},$$

where

$$m := b,\ \tau := \frac{\log 2}{\log\alpha} \ \text{and}\ \ \mu_a := \frac{\log\left(2^a - 1\right)}{\log\alpha} \quad (2 \le a \le 103).$$

Now we apply Lemma 2. Put $M := 1.2\cdot 10^{28}$, $A := 20$ and $B := 2$. Since $m = b < n$, from (23) we have that M is the upper bound on m. We use a quick computation with Maple to see that the convergent

$$\frac{p_{69}}{q_{69}} = \frac{12136119213285503727037560321 19}{49234443310463193396894273881 7}$$

of τ is such that $q_{69} > 6M$ and $\varepsilon_a \ge 0.00065221 > 0$, for all $2 \le a \le 103$. Therefore, we get that the maximum value of $\frac{\log(Aq/\varepsilon)}{\log B}$ over all $2 \le a \le 103$ is 113.539868, which, according to Lemma 2, is an upper bound of b.

Hence, we deduce that the possible solutions (n, a, b) of equation (3) satisfy $1 \le a \le b \le 113$. Therefore, we use inequality (18) to obtain $n \le 566$.

Finally, we used Maple to compare E_n and $(2^a - 1)(2^b - 1)$, for $0 \le n \le 566$ and $0 \le a \le b \le 113$, and checked that all the solutions of equation (3) in nonnegative integers n, a and b with $0 \le a \le b$, are given by

$$E_0 = E_3 = (2^1 - 1)(2^2 - 1) = 3,\ E_1 = 0, \ \text{ and }\ E_7 = (2^1 - 1)(2^3 - 1) = 7.$$

Notice that the case $E_1 = 0$ gives infinitely many solutions. This completes the proof of Theorem 2.

5. The proof of Theorem 3

In this section, we will prove Theorem 3 in three steps.

Before, we prove the following useful lemma.

Lemma 4. *Let (n, a, b) be a solution in integers of equation (4), with $1 \le a \le b$, and assume that $n \ge 0$. Then, we have the following inequalities:*

$$1.79(a+b) - 0.58 < n + 2 < 1.84(a+b) + 7.68.$$

Proof. Using inequality (11) and equation (4), we obtain

$$\rho^{n-2} \le N_n = (2^a \pm 1)(2^b \pm 1) \le 2^{a+b+2}$$

and

$$2^{a+b-2} = 2^{a-1} \cdot 2^{b-1} \le (2^a \pm 1)(2^b \pm 1) = N_n \le \rho^{n-1}.$$

Hence, we get

$$(a+b-2)\frac{\log 2}{\log \rho} + 1 \le n \le (a+b+2)\frac{\log 2}{\log \rho} + 2.$$

Thus, using $1.79 < \dfrac{\log 2}{\log \rho} < 1.84$, we deduce that

$$1.79(a+b) - 0.58 < n + 2 < 1.84(a+b) + 7.68.$$

□

Step A: In this step, we take equation (4) with non-negative integers (n, a, b), where $1 \le a \le b$, and we search for solutions to this equation using Maple for $n \le 500$ and find

$$N_1 = N_2 = N_3 = (2^1-1)(2^1-1) = 1, \quad N_5 = (2^1-1)(2^2-1) = (2^1-1)(2^1+1) = 3,$$

$$N_8 = (2^1+1)(2^1+1) = (2^1+1)(2^2-1) = (2^1-1)(2^3+1) = (2^2-1)(2^2-1) = 9,$$
$$N_{15} = (2^1-1)(2^7+1) = 129$$

and

$$N_{16} = (2^1+1)(2^6-1) = (2^2-1)(2^6-1) = 189.$$

Step B: Now, we assume that $n > 500$. Substituting Binet's formula (10) in equation (4), we obtain

$$\left|c_\rho \rho^{n+2} - 2^{a+b}\right| \le 2^a + 2^b + 1 + 2|c_\delta| \cdot |\delta|^{n+2} < 2^a + 2^b + 2.$$

Dividing through by 2^{a+b}, we get

$$|\Gamma_5| < \frac{1}{2^b} + \frac{1}{2^a} + \frac{2}{2^{a+b}} < \frac{4}{2^a}, \tag{29}$$

where

$$\Gamma_5 := c_\rho \rho^{n+2} 2^{-(a+b)} - 1.$$

Now, we need to check that $\Gamma_5 \neq 0$. If $\Gamma_5 = 0$, then

$$c_\rho \rho^{n+2} = 2^{a+b}. \tag{30}$$

Applying σ on both sides of (30), where σ is a $\mathbb{Q}$-automorphism of the Galois group of the extension $\mathbb{Q}(\rho, \delta)$ over $\mathbb{Q}$ given by $\sigma(\rho) := \delta$ and $\sigma(\delta) := \rho$, we have

$$1 < 2^{a+b} = \left|\sigma(c_\rho \rho^{n+2})\right| = |c_\delta||\delta|^{n+2} < |c_\delta| < 0.41,$$

which is a contradiction. Hence, $\Gamma_5 \neq 0$, and we can apply Theorem 4 to it. To do this, we consider

$$\eta_1 := c_\rho,\ \eta_2 := \rho,\ \eta_3 := 2, \quad d_1 := 1,\ d_2 := n+2,\ d_3 := -(a+b).$$

The algebraic numbers η_1, η_2, η_3 are elements of the field $\mathbb{K} = \mathbb{Q}(\rho)$ and $d_{\mathbb{K}} = 3$. We have that $h(\eta_1) = \frac{\log 31}{3}$, $h(\eta_2) = \log \rho/3$ and $h(\eta_3) = \log 2$. Thus, we can take

$$\max\{3h(\eta_1), |\log \eta_1|, 0.16\} < 3.44 := A_1,$$

$$\max\{3h(\eta_2), |\log \eta_2|, 0.16\} < 0.39 := A_2,$$

and

$$\max\{3h(\eta_3), |\log \eta_3|, 0.16\} < 2.08 := A_3.$$

Finally, Lemma 4 implies that we can choose $D := n+2 = \max\{1, n+2, a+b\}$. From Theorem 4, we get

$$\begin{aligned}\log|\Gamma_5| &> -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2 (1 + \log 3)(1 + \log(n+2)) \cdot 3.44 \cdot 0.39 \cdot 2.08 \\ &> -7.55 \cdot 10^{12} \cdot (1 + \log(n+2)).\end{aligned}$$

By the fact $1 + \log(n+2) < 1.7 \log(n+2)$, which holds for all $n \geq 3$, we obtain

$$\log|\Gamma_5| > -1.29 \cdot 10^{13} \log(n+2),$$

which combined with (29), gives

$$a \log 2 < 1.3 \cdot 10^{13} \log(n+2). \tag{31}$$

We go back to equation (4) and rewrite it as

$$\frac{N_n}{2^a \pm 1} \mp 1 = 2^b,$$

and consequently

$$\left|\frac{c_\rho \rho^{n+2}}{2^a \pm 1} - 2^b\right| = \left|\pm 1 - \frac{c_\delta \delta^{n+2} + c_\lambda \lambda^{n+2}}{2^a \pm 1}\right| \leq 1 + \frac{2|c_\delta| \cdot |\delta|}{2^a \pm 1} < 1.7.$$

Dividing through by 2^b, we obtain

$$|\Gamma_6| < \frac{1.7}{2^b}, \tag{32}$$

where

$$\Gamma_6 := \frac{c_\rho}{2^a \pm 1}\rho^{n+2}2^{-b} - 1.$$

Note that with a similar argument to the above one, it can be proved that $\Gamma_6 \neq 0$. So, we can apply Theorem 4 to it. We consider

$$\eta_1 := \frac{c_\rho}{2^a \pm 1},\ \eta_2 := \rho,\ \eta_3 := 2, \quad d_1 := 1,\ d_2 := n+2,\ d_3 := -b.$$

Thus, $D := n+2$. The heights of η_2 and η_3 have already been calculated. From the properties of the heights we get that

$$h(\eta_1) \leq h(c_\rho) + h(2^a \pm 1) \leq \frac{\log 31}{3} + a\log 2 + \log 2 < 1.31 \cdot 10^{13}\log(n+2),$$

where we have used inequality (31). We choose

$$\max\{3h(\eta_1), |\log \eta_1|, 0.16\} < 3.93 \cdot 10^{13}\log(n+2) := A_1,$$

A_2, and A_3 as above. Therefore, from Theorem 4 we obtain

$$\log|\Gamma_6| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2(1+\log 3)(1+\log(n+2)) \cdot 3.93 \cdot 10^{13}\log(n+2) \cdot 0.3 \cdot 2.08,$$

i.e.,

$$\log|\Gamma_6| > -1.47 \cdot 10^{26} \cdot (\log(n+2))^2, \tag{33}$$

where we used as above the inequality $1+\log(n+2) < 1.7\log(n+2)$, for all $n \geq 3$. By inequalities (33) and (32), we get

$$b < \frac{1.48 \cdot 10^{26} \cdot (\log(n+2))^2}{\log 2} < 2.14 \cdot 10^{26} \cdot (\log(n+2))^2,$$

which, combined with Lemma 4, gives

$$\frac{n+2}{(\log(n+2))^2} < 3.94 \cdot 10^{26}.$$

Taking $m := 2$ and $T := 3.94 \cdot 10^{26}$ in Lemma 1, we get

$$n+2 < 5.92 \cdot 10^{30}. \tag{34}$$

Step C: The aim of the last step is to reduce the above bound on n. To do this, we first consider

$$\Lambda_5 := (a+b)\log 2 - (n+2)\log\rho + \log\left(\frac{1}{c_\rho}\right).$$

Note that $e^{-\Lambda_5} - 1 = \Gamma_5 \neq 0$. Thus, $\Lambda_5 \neq 0$. If $\Lambda_5 < 0$, then

$$0 < |\Lambda_5| < e^{|\Lambda_5|} - 1 = |\Gamma_5| < \frac{4}{2^a},$$

according to inequality (29). If $\Lambda_5 > 0$, then we have

$$1 - e^{-\Lambda_5} = \left|e^{-\Lambda_5} - 1\right| < 1/2.$$

Hence, $e^{\Lambda_5} < 2$. Thus, we get

$$0 < \Lambda_5 < e^{\Lambda_5} - 1 = e^{\Lambda_5}|\Gamma_5| < \frac{8}{2^a}.$$

So, in both cases we have

$$0 < |\Lambda_5| < \frac{8}{2^a}.$$

Dividing through by $\log\rho$, we see that

$$|m\tau - (n+2) + \mu| < \frac{22}{2^a},$$

where

$$m := a + b,\ \tau := \frac{\log 2}{\log\rho} \text{ and } \ \mu := \frac{\log\left(\frac{1}{c_\rho}\right)}{\log\rho}.$$

Now, we apply Lemma 2. Put $M := 5.92 \cdot 10^{30}$, $A := 22$ and $B := 2$. Since $m = a + b < n + 2$, from (34) we have that M is the upper bound on m. A quick computation with Maple reveals that the convergent

$$\frac{p_{73}}{q_{73}} = \frac{2396951239422507242109069211202 96}{1321830136461470590645183083884 83}$$

of τ is such that $q_{73} > 6M$ and $\varepsilon \geq 0.39937103 > 0$. Therefore, we get

$$a < \frac{1}{\log 2}\log\left(\frac{22 \cdot 132183013646147059064518308388483}{\varepsilon}\right) \leq 113.$$

Second, we consider $1 \leq a \leq 112$ and set

$$\Lambda_6 := b\log 2 - (n+2)\log\rho + \log\left(\frac{2^a \pm 1}{c_\rho}\right),$$

and go to inequality (32). Note that $e^{-\Lambda_6} - 1 = \Gamma_6 \neq 0$. Thus, $\Lambda_6 \neq 0$. With an argument similar to the above one, we get

$$0 < |\Lambda_6| < \frac{3.4}{2^b}.$$

Dividing through by $\log\rho$, we obtain

$$|m\tau - (n+2) + \mu_a| < \frac{9}{2^b},$$

where

$$m := b,\ \tau := \frac{\log 2}{\log\rho} \text{ and } \ \mu_a := \frac{\log\left(\frac{2^a\pm 1}{c_\rho}\right)}{\log\rho}, \quad (1 \leq a \leq 112).$$

Now we apply Lemma 2. Put $M := 5.92 \cdot 10^{30}$, $A := 9$ and $B := 2$. Since $m = b < n + 2$, from (34) we have that M is the upper bound on m. A quick computation with Maple proves that the convergent

$$\frac{p_{76}}{q_{76}} = \frac{112228729673401306189988273208995 9}{61890002023917086552040204862007 5}$$

of τ is such that $q_{76} > 6M$ and $\varepsilon_a \geq 0.00543643 > 0$, for all $1 \leq a \leq 112$. Therefore, we get that the maximum value of $\frac{\log(Aq/\varepsilon)}{\log B}$ over all $1 \leq a \leq 112$ is $119.624454\ldots$ which, according to Lemma 2, is an upper bound of b.

Hence, we deduce that the possible solutions (n, a, b) of equation (4) satisfy $1 \leq a \leq b \leq 119$. Therefore, we use Lemma 4 to obtain $n \leq 443$, which is a contradiction, and Theorem 3 is proved.

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