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Stable convergence in law in approximation of stochastic integrals with respect to diffusions*

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Abstract. We assume that the one-dimensional diffusion X satisfies a stochastic differential equation of the form: $dX_t = \mu(X_t)dt + \nu(X_t)dW_t$, $X_0 = x_0$, $t \geq 0$. Let $(X_{i\Delta_n}, 0 \leq i \leq n)$ be discrete observations along a fixed time interval $[0,T]$. We prove that random vectors whose j-th component is $\frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} g_j(X_s)(f_j(X_s) - f_j(X_{t_{i-1}}))dW_s$, for $j = 1, \ldots, d$, converge stably in law to a mixed normal random vector with a covariance matrix which depends on the path $(X_t, 0 \leq t \leq T)$, when $n \to \infty$. We use this result to prove stable convergence in law for $\frac{1}{\sqrt{\Delta_n}}(\int_0^T f(X_s)dX_s - \sum_{i=1}^n f(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}))$.

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1. Introduction

Let $W = (W_t, t \geq 0)$ be a one-dimensional standard Brownian motion defined on filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We take the filtration $(\mathcal{F}_t)_{t\geq 0}$ to be the smallest one such that it satisfies the usual conditions and to which W is adapted. Let $X = (X_t, t \geq 0)$ be a one-dimensional diffusion which satisfies Itô's stochastic differential equation (SDE) of the form

$$dX_t = \mu(X_t)dt + \nu(X_t)dW_t, \quad X_0 = x_0, \quad t \geq 0, \tag{1}$$

where μ and ν are real functions and $x_0 \in \mathbb{R}$ is the initial value of X (a deterministic one).

Let $T \in \mathbb{R}$ be a fixed number such that $T > 0$ and $0 =: t_0 < t_1 < \cdots < t_n := T$, and let $n \in \mathbb{N}$ be an equidistant deterministic subdivision of the segment $[0,T]$, i.e. $t_i = i\Delta_n, i = 0, \ldots, n$, $\Delta_n = \frac{T}{n}$. Let $(X_{t_i}, 0 \leq i \leq n)$ be a discrete observation of the trajectory $(X_t, t \in [0,T])$.

If we approximate the stochastic integral $\int_0^T f(X_s)dX_s$ with $\sum_{i=1}^n f(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}})$, we would like to identify the limiting distribution of the error of that approximation when $n \to \infty$. For a two-times continuously differentiable function f we will prove that the difference between $\int_0^T f(X_s)dX_s$ and its approximation

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$\sum_{i=1}^{n} f(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}})$ divided by $\sqrt{\Delta_n}$ has an approximately mixed normal distribution. In the case when $f(x) = x$, this result can be found in [8]. In the case when $dX_t = dW_t$, similar results can be found in [5] and [15]. In [11], the authors looked at the normed differences between some integral θ and its estimators $\hat{\theta}_n$, where θ was of the form $\int_0^T \nu^r(X_s)ds$, $r > 0$, and they got stable convergence in law for them in the case when $n \to \infty$. We believe that the approach from the article mentioned last can also be used in the case of the integral $\int_0^T f(X_s)dX_s$ and its approximations (estimators) $\sum_{i=1}^{n} f(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}})$ (see Theorem 5., Section 4.5 in [11] for $\alpha = \frac{1}{2}$), but the calculations for getting limiting distribution can be complicated.

We present here a unifying approach to a series of well-known results that were previously proved using individual approaches. We hope that our approach will lead to possible further generalizations. Here, in Theorem 2, we prove that random vectors whose j-th component is $\frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} g_j(X_s)(f_j(X_s) - f_j(X_{t_{i-1}}))dW_s$, for $j = 1, \ldots, d$, converge stably in law to a mixed normal random vector. All other interesting limiting distributions in this paper will be easily calculated based on this general result. Let us point out that in the last couple of years, many authors are interested in a limiting distribution of the terms $\sum_{i=1}^{n} f(X_{t_i} - X_{t_{i-1}})$ when $n \to \infty$, for a different kind of processes X and for a suitably chosen function f (for example, $f(x) = |x|^r, r > 0$), and they found a limiting distribution when an appropriate normalized constant is used (see, for example, [7, 2, 11]). However, this kind of approximations for diffusion is not discussed in this paper.

In our case, stable convergence in law plays an important role because the variance in a limiting distribution depends on the path $(X_t, 0 \leq t \leq T)$, i.e. it is a random variable. Stable convergence in law allows us to divide the difference with the square root of some approximation of variance and we will still have convergence to normal distribution. These results can be used to find approximate confidence intervals for stochastic integrals with respect to diffusions and to estimate the approximate standard error of the approximations of stochastic integrals.

The paper is organized in the following way. In the next section, we introduce notations and definitions we need. The main results and proofs are presented in Section 3. In the last section, we will present three interesting examples. In the first example, we deal with the approximation of Itô's integral, in the second we do a simulation study for the geometric Brownian motion, and in the third example, we present one theoretical result which is proved in the [10] as the straight consequence of our main Theorem 2.

2. Preliminaries

We denote by $\|\cdot\|$ the norm induced by the scalar product $\langle\cdot,\cdot\rangle$ in the d-dimensional Euclidean space $\mathbb{R}^d$. We will denote the $L^1(\mathbb{P})$-norm by $\|\cdot\|_1 := \mathbb{E}[|\cdot|]$ and the $L^2(\mathbb{P})$-norm by $\|\cdot\|_2 := \sqrt{\mathbb{E}[(\cdot)^2]}$. We will denote by E an open interval, $E \subseteq \mathbb{R}$. For fixed $T > 0$, let $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a given filtered probability space. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}_T, \tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{0 \leq t \leq T}, \tilde{\mathbb{P}})$ be an *extension* of $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ (for details, see [6]). The extension is called *very good* if all martingales on the space

$(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ are also martingales on $(\tilde{\Omega}, \tilde{\mathcal{F}}_T, \tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{0\le t\le T}, \tilde{\mathbb{P}})$. Let A be some Polish space. Let (Z_n) be a sequence of A-valued random vectors, all defined on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, and let Z be an A-valued random vector defined on the extension $(\tilde{\Omega}, \tilde{\mathcal{F}}_T, \tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{0\le t\le T}, \tilde{\mathbb{P}})$. We will say that (Z_n) *converges stably in law* to Z, and write $Z_n \overset{st}{\Rightarrow} Z$, if

$$\lim_{n\to\infty} \mathbb{E}[Yf(Z_n)] = \tilde{\mathbb{E}}[Yf(Z)],$$

for all bounded continuous functions $f\colon A \to \mathbb{R}$ and all bounded random variables Y on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$. For more details about this kind of convergence, see [13] and [1].

We will say that an $\mathbb{R}^d$-valued random vector Y has *a mixed normal distribution* with an $\mathcal{F}_T$-measurable random covariance matrix $C = (C^{jk})_{j,k=1,\dots,d}$, and we write $Y \sim MN(0, C)$ if $\mathbb{E}[e^{i\langle t,Y\rangle}|\mathcal{F}_T] = e^{-\frac{1}{2}\sum_{j,k=1,\dots,d} t_j t_k C^{jk}}$ (see [1]). An $\mathbb{R}^d$-valued process $(Y_t)_{0\le t\le T}$ is *a centered Gaussian process* if for all $0 \le s_1 < s_2 < \cdots < s_k \le T$ a random matrix $(Y_{s_1}, \dots, Y_{s_k}) \in \mathbb{R}^{dk}$ has a multivariate normal distribution and $\mathbb{E}[Y_t] = 0$, $t \in [0,T]$.

For $n \in \mathbb{N}$, let $t_i := i\Delta_n, i = 0, \dots, n$ be an equidistant subdivision of the segment $[0,T]$, $\Delta_n = \frac{T}{n}$. Let $A_n^t := \max\{j\colon t_j \le t\}$ and $\mathcal{F}_{n,i} := \mathcal{F}_{t_i}, i = 0, \dots, n$, $n \in \mathbb{N}$. The following theorem is a version of Theorem 3-2 from [6] which we need for proof.

Theorem 1. *Let W be a one-dimensional Brownian motion on $(\Omega, \mathcal{F}_T, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, and let χ_i^n be $\mathcal{F}_{t_i}$-measurable square integrable $\mathbb{R}^d$-valued random vectors. Let $C = (C^{jk})$ be a continuous adapted process defined on $(\Omega, \mathcal{F}_T, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ such that C_t is a positive semidefinite symmetric $d \times d$ matrix for all $t \in [0,T]$. Assume that the following conditions hold:*

$$\sup_{0\le t\le T} \|\sum_{i=1}^{A_n^t} \mathbb{E}[\chi_i^n|\mathcal{F}_{n,i-1}]\| \xrightarrow{\mathbb{P}} 0, \tag{2}$$

$$\sum_{i=1}^{A_n^t} (\mathbb{E}[\chi_i^{n,j}\chi_i^{n,k}|\mathcal{F}_{n,i-1}] - \mathbb{E}[\chi_i^{n,j}|\mathcal{F}_{n,i-1}]\mathbb{E}[\chi_i^{n,k}|\mathcal{F}_{n,i-1}]) \xrightarrow{\mathbb{P}} C_t^{j,k}, \quad \forall t \in [0,T], j,k = 1,\dots,d, \tag{3}$$

$$\sum_{i=1}^{A_n^t} \mathbb{E}[\chi_i^n(W_{t_i} - W_{t_{i-1}})|\mathcal{F}_{n,i-1}] \xrightarrow{\mathbb{P}} 0, \qquad \forall t \in [0,T], \tag{4}$$

$$\sum_{i=1}^{n} \mathbb{E}[\|\chi_i^n\|^2 1_{\{\|\chi_i^n\|>\epsilon\}}|\mathcal{F}_{n,i-1}] \xrightarrow{\mathbb{P}} 0, \qquad \forall \epsilon > 0, \tag{5}$$

$$\sum_{i=1}^{A_n^t} \mathbb{E}[\chi_i^n(N_{t_i} - N_{t_{i-1}})|\mathcal{F}_{n,i-1}] \xrightarrow{\mathbb{P}} 0, \qquad \forall t \in [0,T], \tag{6}$$

where N is a bounded $\mathcal{F}_t$-martingale orthogonal to W.

Then we have

$$\sum_{i=1}^{A_n^t} \chi_i^n \overset{st}{\Rightarrow} Y \qquad \text{on } D([0,T],\mathbb{R}^d),$$

where Y *is a continuous process defined on a very good filtered extension* $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{0\le t\le T},\tilde{\mathbb{P}})$ *of* $(\Omega,\mathcal{F}_T,\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T},\mathbb{P})$ *and which, conditionally on the* σ*-field* $\mathcal{F}_T$*, is a centered Gaussian* $\mathbb{R}^d$*-valued process with independent increments satisfying* $\tilde{\mathbb{E}}[Y_t^j Y_t^k|\mathcal{F}_T] = C_t^{jk}, t\in[0,T], j,k = 1,\dots,d$.

3. Main results

We assume that state space E is an open interval in $\mathbb{R}$.

Let the following assumptions be satisfied:

(A1) There exists a strong solution X of SDE (1) on the time interval $[0,+\infty\rangle$. This solution X has continuous paths with values in E and if X' is any other solution of SDE (1), with the same Brownian motion W, then the law of X' is identical to the law of X.

(A2) Functions $x\mapsto\mu(x)$ and $x\mapsto\nu(x)$ are two-times continuously differentiable on E, and $\nu(x) > 0$ for all $x\in E$.

(A3) Functions $x\mapsto g_j(x)$ and $x\mapsto f_j(x)$ are two-times continuously differentiable on E, for all $j = 1,\dots,d$.

Let Σ be a $d\times d$ random matrix whose jk-th component is defined by

$$\Sigma^{jk} = \frac{1}{2}\int_0^T \nu^2(X_s)g_j(X_s)g_k(X_s)f_j'(X_s)f_k'(X_s)ds.$$

Theorem 2. *Assume that (A1)-(A3) hold. Then*

$$\frac{1}{\sqrt{\Delta_n}}\begin{bmatrix}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} g_1(X_s)(f_1(X_s)-f_1(X_{t_{i-1}}))dW_s\\ \vdots\\ \sum_{i=1}^n \int_{t_{i-1}}^{t_i} g_d(X_s)(f_d(X_s)-f_d(X_{t_{i-1}}))dW_s\end{bmatrix} \overset{st}{\Rightarrow} MN(0,\Sigma).$$

Proof. First, let us assume that all of the functions μ,ν,g_j,f_j and their derivatives are uniformly bounded. Using Itô's formula for $j = 1,\dots,d$ we get

$$\begin{aligned}&\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^n\int_{t_{i-1}}^{t_i} g_j(X_s)(f_j(X_s)-f_j(X_{t_{i-1}}))dW_s\\ =&\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^n\int_{t_{i-1}}^{t_i} g_j(X_s)\left(\int_{t_{i-1}}^s (f_j'(X_u)\mu(X_u)+\frac{1}{2}f_j''(X_u)\nu^2(X_u))du\right)dW_s \qquad (7)\\ &+\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^n\int_{t_{i-1}}^{t_i} g_j(X_s)\left(\int_{t_{i-1}}^s f_j'(X_u)\nu(X_u)dW_u\right)dW_s. \qquad (8)\end{aligned}$$

With notation $l_j(X_u) = f_j'(X_u)\mu(X_u) + \frac{1}{2}f_j''(X_u)\nu^2(X_u)$, the square of the L^2-norm of term (7) can be written in the form

$$
\begin{aligned}
&\| \frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} g_j(X_s) \int_{t_{i-1}}^{s} l_j(X_u) du dW_s \|_2^2 \\
=& \frac{1}{\Delta_n} \sum_{i=1}^{n} \mathbb{E}[(\int_{t_{i-1}}^{t_i} g_j(X_s) \int_{t_{i-1}}^{s} l_j(X_u) du dW_s)^2] \\
&+ \frac{2}{\Delta_n} \sum_{1\le i<k\le n} \mathbb{E}[\left(\int_{t_{i-1}}^{t_i} g_j(X_s) \int_{t_{i-1}}^{s} l_j(X_u) du dW_s \right) \\
&\cdot \left(\int_{t_{k-1}}^{t_k} g_j(X_s) \int_{t_{k-1}}^{s} l_j(X_u) du dW_s \right)].
\end{aligned}
$$

The processes $(\int_0^t g_j(X_s)dW_s)_{0\le t\le T}$ and $(\int_0^t g_j(X_s)\int_0^s l_j(X_u)dudW_s)_{0\le t\le T}$ are martingales with respect to the given filtration $(\mathcal{F}_t)_{0\le t\le T}$, hence for $1 \le i < k \le n$ there holds:

$$
\begin{aligned}
&\mathbb{E}[\int_{t_{k-1}}^{t_k} g_j(X_s) \int_{t_{k-1}}^{s} l_j(X_u) du dW_s | \mathcal{F}_{n,k-1}] \\
=&\mathbb{E}[\int_{t_{k-1}}^{t_k} g_j(X_s) (\int_0^s l_j(X_u) du - \int_0^{t_{k-1}} l_j(X_u) du) dW_s | \mathcal{F}_{n,k-1}] \\
=&\mathbb{E}[\int_0^{t_k} g_j(X_s) \int_0^s l_j(X_u) du dW_s - \int_0^{t_{k-1}} g_j(X_s) \int_0^s l_j(X_u) du dW_s \\
&-(\int_0^{t_{k-1}} l_j(X_u) du) \cdot \int_0^{t_k} g_j(X_s) dW_s \\
&+(\int_0^{t_{k-1}} l_j(X_u) du) \cdot \int_0^{t_{k-1}} g_j(X_s) dW_s | \mathcal{F}_{n,k-1}] = 0,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\mathbb{E}[(\int_{t_{i-1}}^{t_i} g_j(X_s) \int_{t_{i-1}}^{s} l_j(X_u) du dW_s) \cdot (\int_{t_{k-1}}^{t_k} g_j(X_s) \int_{t_{k-1}}^{s} l_j(X_u) du dW_s)] \\
&= \mathbb{E}[\mathbb{E}[(\int_{t_{i-1}}^{t_i} g_j(X_s) \int_{t_{i-1}}^{s} l_j(X_u) du dW_s) \cdot (\int_{t_{k-1}}^{t_k} g_j(X_s) \int_{t_{k-1}}^{s} l_j(X_u) du dW_s) | \mathcal{F}_{n,k-1}]] \\
&= \mathbb{E}[(\int_{t_{i-1}}^{t_i} g_j(X_s) \int_{t_{i-1}}^{s} l_j(X_u) du dW_s) \mathbb{E}[\int_{t_{k-1}}^{t_k} g_j(X_s) \int_{t_{k-1}}^{s} l_j(X_u) du dW_s | \mathcal{F}_{n,k-1}]] = 0.
\end{aligned}
$$

Hence, we can conclude that for $j = 1, \dots, d$ there exists a constant k_1 such that

$$
\begin{aligned}
&\|\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} g_j(X_s)\left(\int_{t_{i-1}}^{s}(f_j'(X_u)\mu(X_u)+\frac{1}{2}f_j''(X_u)\nu^2(X_u))du\right)dW_s\|_2^2\\
&=\frac{1}{\Delta_n}\sum_{i=1}^{n}\mathbb{E}[\left(\int_{t_{i-1}}^{t_i} g_j(X_s)\left(\int_{t_{i-1}}^{s}(f_j'(X_u)\mu(X_u)+\frac{1}{2}f_j''(X_u)\nu^2(X_u))du\right)dW_s\right)^2]\\
&=\frac{1}{\Delta_n}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\mathbb{E}[\left(g_j(X_s)\left(\int_{t_{i-1}}^{s}(f_j'(X_u)\mu(X_u)+\frac{1}{2}f_j''(X_u)\nu^2(X_u))du\right)\right)^2]ds\\
&\le\frac{\|g_j^2\|_\infty}{\Delta_n}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(s-t_{i-1})\int_{t_{i-1}}^{s}\mathbb{E}[\left(f_j'(X_u)\mu(X_u)+\frac{1}{2}f_j''(X_u)\nu^2(X_u)\right)^2]duds\\
&\le k_1\Delta_n T,
\end{aligned}
$$

so (7) converges in the L^2-norm to zero when n goes to infinity. Therefore, (7) converges in probability to zero when $n \to \infty$. If we denote by

$$
Z_{n,j}=\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} g_j(X_s)\left(\int_{t_{i-1}}^{s}(f_j'(X_u)\mu(X_u)+\frac{1}{2}f_j''(X_u)\nu^2(X_u))du\right)dW_s,
$$

and with Z_n a random vector whose j-th component is $Z_{n,j}$, then it holds:

$$
\begin{aligned}
&\frac{1}{\sqrt{\Delta_n}}\begin{bmatrix}\sum_{i=1}^n\int_{t_{i-1}}^{t_i} g_1(X_s)(f_1(X_s)-f_1(X_{t_{i-1}}))dW_s\\ \vdots\\ \sum_{i=1}^n\int_{t_{i-1}}^{t_i} g_d(X_s)(f_d(X_s)-f_d(X_{t_{i-1}}))dW_s\end{bmatrix}\\
&\quad=Z_n+\frac{1}{\sqrt{\Delta_n}}\begin{bmatrix}\sum_{i=1}^n\int_{t_{i-1}}^{t_i} g_1(X_s)(\int_{t_{i-1}}^{s} f_1'(X_u)\nu(X_u)dW_u)dW_s\\ \vdots\\ \sum_{i=1}^n\int_{t_{i-1}}^{t_i} g_d(X_s)(\int_{t_{i-1}}^{s} f_d'(X_u)\nu(X_u)dW_u)dW_s\end{bmatrix},
\end{aligned}
$$

with $Z_n \overset{P}{\Rightarrow} 0$ when $n \to \infty$.

With a notation $R_t^j := f_j'(X_t)\nu(X_t)$, we have

$$
\Sigma^{jk}=\frac{1}{2}\int_0^T g_j(X_s)g_k(X_s)R_s^jR_s^k ds, \qquad j,k=1,\dots,d.
$$

Let $C = (C_t^{jk})_{0\le t\le T}$ be a continuous adapted process defined by

$$
C_t^{jk}=\frac{1}{2}\int_0^t g_j(X_s)g_k(X_s)R_s^jR_s^k ds, \qquad j,k=1,\dots,d,\ t\in[0,T],
$$

and let $\chi_i^n, i = 1,\dots,n$ be $\mathcal{F}_{t_i}$-measurable random vectors defined by

$$
\chi_i^n=\begin{bmatrix}\chi_i^{n,1}\\ \vdots\\ \chi_i^{n,d}\end{bmatrix}=\frac{1}{\sqrt{\Delta_n}}\begin{bmatrix}\int_{t_{i-1}}^{t_i} g_1(X_s)\int_{t_{i-1}}^{s} R_u^1 dW_u dW_s\\ \vdots\\ \int_{t_{i-1}}^{t_i} g_d(X_s)\int_{t_{i-1}}^{s} R_u^d dW_u dW_s\end{bmatrix}.
$$

We will prove that Theorem 1 holds for C and χ_i^n. Since all functions are bounded, we conclude that random vectors χ_i^n are square integrable. By its definition, C_t is a symmetric positive semidefinite random matrix, for all $t \in [0,T]$. Using a notation $\mathcal{F}_{n,i} := \mathcal{F}_{t_i}, i = 1, \dots, n$, it holds:

$$\mathbb{E}[\chi_i^{n,j} | \mathcal{F}_{n,i-1}] = 0, \qquad \forall j = 1, \dots, d, \forall i = 1, \dots, n,$$

hence (2) is satisfied.
Let $\epsilon > 0$. There exists a constant $k_2 > 0$ such that

$$\| \sum_{i=1}^{n} \mathbb{E}[\|\chi_i^n\|^2 1_{\{\|\chi_i^n\| > \epsilon\}} | \mathcal{F}_{n,i-1}]\|_1 \leq \frac{d}{\epsilon^2} \sum_{i=1}^{n} \mathbb{E}[\sum_{j=1}^{d} (\chi_i^{n,j})^4] \leq k_2 \Delta_n T,$$

hence (5) is satisfied.

Let $N = (N_t)_{0 \leq t \leq T}$ be any bounded $\mathcal{F}_t$-martingal orthogonal to W. Since $(\mathcal{F}_t)_{0 \leq t \leq T}$ is generated by the Brownian motion W, it follows from the martingale representation theorem (see [9, Theorem III.4.33]) that N_t is equal to the constant, so (6) is satisfied.

For $j = 1, \dots, d$, first using the integration by parts formula (see [14, Chapter 3, Proposition 3.1]) and that Itô's integrals are martingals, and then using Itô's formula for the function g_j, we get

$$\begin{aligned}
&\sum_{i=1}^{A_n^t} \mathbb{E}[\chi_i^{n,j} (W_{t_i} - W_{t_{i-1}}) | \mathcal{F}_{n,i-1}] \\
=&\frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{A_n^t} \mathbb{E}[\int_{t_{i-1}}^{t_i} g_j(X_s) \int_{t_{i-1}}^{s} R_u^j dW_u dW_s \cdot \int_{t_{i-1}}^{t_i} dW_s | \mathcal{F}_{n,i-1}] \\
=&\frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{A_n^t} \mathbb{E}[\int_{t_{i-1}}^{t_i} (\int_{t_{i-1}}^{s} dW_u) \cdot \left(g(X_s) \int_{t_{i-1}}^{s} R_u^j dW_u \right) dW_s | \mathcal{F}_{n,i-1}] \\
&+ \frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{A_n^t} \mathbb{E}[\int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} g(X_u) (\int_{t_{i-1}}^{u} R_l^j dW_l) dW_u dW_s | \mathcal{F}_{n,i-1}] \\
&+ \frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{A_n^t} \mathbb{E}[\int_{t_{i-1}}^{t_i} g_j(X_s) \int_{t_{i-1}}^{s} R_u^j dW_u ds | \mathcal{F}_{n,i-1}] \\
=&\frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{A_n^t} \mathbb{E}[\int_{t_{i-1}}^{t_i} g_j(X_s) \int_{t_{i-1}}^{s} R_u^j dW_u ds | \mathcal{F}_{n,i-1}] \\
=&\frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{A_n^t} \mathbb{E}[\int_{t_{i-1}}^{t_i} \left(\int_{t_{i-1}}^{s} \left(g_j'(X_u) \mu(X_u) + \frac{1}{2} g_j''(X_u) \nu^2(X_u) \right) du \right) \\
&\cdot \left(\int_{t_{i-1}}^{s} R_u^j dW_u \right) ds | \mathcal{F}_{n,i-1}]
\end{aligned} \tag{9}$$

$$+\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^{A_n^t}\mathbb{E}[\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}g_j'(X_u)\nu(X_u)dW_u\right)\cdot\left(\int_{t_{i-1}}^{s}R_u^j dW_u\right)ds|\mathcal{F}_{n,i-1}] \quad (10)$$

$$+\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^{A_n^t}\mathbb{E}[\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}g_j(X_{t_{i-1}})R_u^j dW_u ds|\mathcal{F}_{n,i-1}]. \quad (11)$$

Term (11) equals zero since by the integration by parts formula and the martingale property of Itô's integral it holds:

$$\begin{aligned}&\mathbb{E}[\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}g_j(X_{t_{i-1}})R_u^j dW_u ds|\mathcal{F}_{n,i-1}]\\&=g_j(X_{t_{i-1}})\mathbb{E}[t_i\int_{t_{i-1}}^{t_i}R_s^j dW_s|\mathcal{F}_{n,i-1}]-g_j(X_{t_{i-1}})\mathbb{E}[\int_{t_{i-1}}^{t_i}sR_s^j dW_s|\mathcal{F}_{n,i-1}]=0.\end{aligned}$$

It can be shown that there exist constants k_3, k_4 such that

$$\begin{aligned}&\|\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^{A_n^t}\mathbb{E}[\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}\left(g_j'(X_u)\mu(X_u)+\frac{1}{2}g_j''(X_u)\nu^2(X_u)\right)du\right)\\&\cdot\left(\int_{t_{i-1}}^{s}R_u^j dW_u\right)ds|\mathcal{F}_{n,i-1}]\|_1\le k_3\sqrt{\Delta_n}T,\\&|\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^{A_n^t}\mathbb{E}[\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s}g_j'(X_u)\nu(X_u)dW_u\right)\cdot\left(\int_{t_{i-1}}^{s}R_u^j dW_u\right)ds|\mathcal{F}_{n,i-1}]|\\&=|\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^{A_n^t}\mathbb{E}[\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}g_j'(X_u)\nu(X_u)R_u^j du ds|\mathcal{F}_{n,i-1}]|\\&\le k_4\sqrt{\Delta_n}T,\end{aligned}$$

so (9) converges in the L^1-norm to zero and (10) converges almost surely to zero, hence (4) is satisfied.

It remains to prove that (3) is satisfied. Let $1\le j\le k\le d$. Then

$$\sum_{i=1}^{A_n^t}(\mathbb{E}[\chi_i^{n,j}\chi_i^{n,k}|\mathcal{F}_{n,i-1}]-\mathbb{E}[\chi_i^{n,j}|\mathcal{F}_{n,i-1}]\mathbb{E}[\chi_i^{n,k}|\mathcal{F}_{n,i-1}])$$

$$=\frac{1}{\Delta_n}\sum_{i=1}^{A_n^t}\mathbb{E}[\int_{t_{i-1}}^{t_i}g_j(X_s)g_k(X_s)\int_{t_{i-1}}^{s}J_u^{(i,j)}R_u^k dW_u ds|\mathcal{F}_{n,i-1}] \quad (12)$$

$$+\frac{1}{\Delta_n}\sum_{i=1}^{A_n^t}\mathbb{E}[\int_{t_{i-1}}^{t_i}g_j(X_s)g_k(X_s)\int_{t_{i-1}}^{s}J_u^{(i,k)}R_u^j dW_u ds|\mathcal{F}_{n,i-1}] \quad (13)$$

$$+\frac{1}{\Delta_n}\sum_{i=1}^{A_n^t}\mathbb{E}[\int_{t_{i-1}}^{t_i}g_j(X_s)g_k(X_s)\int_{t_{i-1}}^{s}R_u^j R_u^k du ds|\mathcal{F}_{n,i-1}], \quad (14)$$

where $J_s^{(i,j)} := \int_{t_{i-1}}^s R_u^j dW_u$. It can be shown that there exist constants k_5, k_6, such that

$$\|\frac{1}{\Delta_n}\sum_{i=1}^{A_n^t}\mathbb{E}[\int_{t_{i-1}}^{t_i} g_j(X_s)g_k(X_s)\int_{t_{i-1}}^{s} J_u^{(i,j)}R_u^k dW_u ds|\mathcal{F}_{n,i-1}]\|_2 \le k_5\sqrt{\Delta_n}(T+\sqrt{T}),$$

$$\|\frac{1}{\Delta_n}\sum_{i=1}^{A_n^t}\mathbb{E}[\int_{t_{i-1}}^{t_i} g_j(X_s)g_k(X_s)\int_{t_{i-1}}^{s} J_u^{(i,k)}R_u^j dW_u ds|\mathcal{F}_{n,i-1}]\|_2 \le k_6\sqrt{\Delta_n}(T+\sqrt{T}),$$

so (12) and (13) converge in probability to zero. At the end, we need to prove that (14) converges in probability to C_t^{jk}. There exists a constant k_7 such that

$$\begin{aligned}
&\frac{1}{\Delta_n}\sum_{i=1}^{A_n^t}\mathbb{E}[\int_{t_{i-1}}^{t_i} g_j(X_s)g_k(X_s)\int_{t_{i-1}}^{s} R_u^j R_u^k du ds|\mathcal{F}_{n,i-1}]\\
=&\frac{1}{\Delta_n}\sum_{i=1}^{A_n^t}(\mathbb{E}[\int_{t_{i-1}}^{t_i} g_j(X_s)g_k(X_s)\int_{t_{i-1}}^{s} R_u^j R_u^k du ds|\mathcal{F}_{n,i-1}]\\
&-\int_{t_{i-1}}^{t_i} g_j(X_s)g_k(X_s)\int_{t_{i-1}}^{s} R_u^j R_u^k du ds) \qquad (15)\\
&+\frac{1}{\Delta_n}\sum_{i=1}^{A_n^t}\int_{t_{i-1}}^{t_i} g_j(X_s)g_k(X_s)\int_{t_{i-1}}^{s} R_u^j R_u^k du ds, \qquad (16)
\end{aligned}$$

and

$$\begin{aligned}
\|&\frac{1}{\Delta_n}\sum_{i=1}^{A_n^t}(\mathbb{E}[\int_{t_{i-1}}^{t_i} g_j(X_s)g_k(X_s)\int_{t_{i-1}}^{s} R_u^j R_u^k du ds|\mathcal{F}_{n,i-1}]\\
&-\int_{t_{i-1}}^{t_i} g_j(X_s)g_k(X_s)\int_{t_{i-1}}^{s} R_u^j R_u^k du ds)\|_2 \le k_7\sqrt{\Delta_n}\sqrt{T},
\end{aligned}$$

so (15) converges in probability to zero.

To prove that (16) converges in probability to C_t^{jk}, we use the idea from the proof of Theorem 5.2. in [10]. For the sake of completness, we will write down the steps. Let

$$l(u,s)(\omega) := (g_j(X_s)g_k(X_s)R_u^j R_u^k)(\omega)$$

be a function defined on $[0,T]\times[0,T]\times\Omega$. For fixed $\omega\in\Omega$, the function $l(u,s)(\omega)$ is a bounded continuous function on $[0,T]\times[0,T]$, which means that there exist $u_i^*(\omega), s_i^*(\omega) \in [t_{i-1}, t_i]$ such that $u_i^*(\omega) \le s_i^*(\omega)$ and

$$\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s} l(u,s)du ds = \frac{\Delta_n^2}{2} l(u_i^*, s_i^*), \qquad i\in\{1,\dots,n\}.$$

Hence,

$$
\begin{aligned}
&\frac{1}{\Delta_n}\sum_{i=1}^{A_n^t}\int_{t_{i-1}}^{t_i} g_j(X_s)g_k(X_s)\int_{t_{i-1}}^{s} R_u^j R_u^k du ds - C_t^{jk}\\
&=\sum_{i=1}^{A_n^t}\frac{\Delta_n}{2} l(u_i^*,u_i^*) - \frac{1}{2}\int_0^t g_j(X_s)g_k(X_s)R_s^jR_s^k ds \qquad (17)\\
&\quad+\sum_{i=1}^{A_n^t}\frac{\Delta_n}{2}(l(u_i^*,s_i^*) - l(u_i^*,u_i^*)). \qquad (18)
\end{aligned}
$$

Since the integrand function is continuous, there exist some $t^*(\omega) \in [t_{A_n^t}, t]$, such that $\frac{1}{2}\int_{t_{A_n^t}}^t g_j(X_s)g_k(X_s)R_s^j(\theta)R_s^k(\theta)ds = \frac{1}{2}(t-t_{A_n^t})g_j(X_{t^*})g_k(X_{t^*})R_{t^*}^jR_{t^*}^k$, so for (17) we have

$$
\begin{aligned}
&\sum_{i=1}^{A_n^t}\frac{\Delta_n}{2} l(u_i^*,u_i^*) - \frac{1}{2}\int_0^t g_j(X_s)g_k(X_s)R_s^jR_s^k ds\\
&=\sum_{i=1}^{A_n^t}\frac{\Delta_n}{2} g_j(X_{u_i^*})g_k(X_{u_i^*})R_{u_i^*}^jR_{u_i^*}^k + \frac{1}{2}(t-t_{A_n^t})g_j(X_{u_i^*})g_k(X_{u_i^*})R_{t^*}^jR_{t^*}^k\\
&\quad-\frac{1}{2}\int_0^t g_j(X_s)g_k(X_s)R_s^jR_s^k ds - \frac{1}{2}\int_{t_{A_n^t}}^t g_j(X_s)g_k(X_s)R_s^jR_s^k ds \xrightarrow{a.s.} 0,
\end{aligned}
$$

which gives us that (17) converges in probability to zero. For (18) there exists a constant k_8 such that

$$
\begin{aligned}
&\Big|\sum_{i=1}^{A_n^t}\frac{\Delta_n}{2}(l(u_i^*,s_i^*) - l(u_i^*,u_i^*))\Big| = \Big|\sum_{i=1}^{A_n^t}\frac{\Delta_n}{2}(g_j(X_{s_i^*})g_k(X_{s_i^*}) - g_j(X_{u_i^*})g_k(X_{u_i^*}))R_{u_i^*}^jR_{u_i^*}^k\Big|\\
&\le k_8\sum_{i=1}^{A_n^t}\frac{\Delta_n}{2}|g_j(X_{s_i^*})g_k(X_{s_i^*}) - g_j(X_{u_i^*})g_k(X_{u_i^*})|.
\end{aligned}
$$

Let $\epsilon > 0$. For fixed $\omega \in \Omega$, the function $t \mapsto g_j(X_t(\omega))g_k(X_t(\omega))$ is continuous on $[0,T]$, so it is uniformly continuous. It means that

$$(\exists\delta>0)(\forall s,t\in[0,T])(|s-t|<\delta) \Rightarrow |g_j(X_s(\omega))g_k(X_s(\omega)) - g_j(X_t(\omega))g_k(X_t(\omega))| < \epsilon.$$

Because ω is fixed, we omitted writing it. Since $\lim_{n\to\infty}\Delta_n = 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, holds $\Delta_n < \delta$. Then, for all $n \ge n_0$ we have

$$\sum_{i=1}^{A_n^t}\frac{\Delta_n}{2}|g_j(X_{s_i^*})g_k(X_{s_i^*}) - g_j(X_{u_i^*})g_k(X_{u_i^*})| \le \epsilon\frac{T}{2},$$

which means that (18) converges almost surely to zero 0, so it converges in probability to zero.

Therefore, conditions of Theorem 1 are satisfied provided that the assumption about uniform boundedness of functions holds. If we denote by π_T a projection function $\pi_T\colon D([0,T],\mathbb{R}^d)\to\mathbb{R}^d$, defined by $\pi_T((X_s,s\in[0,T])):=X_T$, then by [3, Theorem 12.5] projection π_T is a continuous function. Notice that

$$\frac{1}{\sqrt{\Delta_n}}\begin{bmatrix}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}g_1(X_s)(f_1(X_s)-f_1(X_{t_{i-1}}))dW_s\\ \vdots\\ \sum_{i=1}^n\int_{t_{i-1}}^{t_i}g_d(X_s)(f_d(X_s)-f_d(X_{t_{i-1}}))dW_s\end{bmatrix}$$
$$=\pi_T\Big(\Big(\sum_{i=1}^{A_n^t}\chi_i^n,t\in[0,T]\Big)\Big)+Z_n,$$

hence by definition and properties of stable convergence in law, our theorem holds under the assumption that all of the functions are uniformly bounded.

In a general case (i.e. without the assumption that functions are uniformly bounded), we again use the idea presented in [10]. For the completeness of the proof we will write down some steps, but for details, see [10]. Let $(E_N,N\in\mathbb{N})$ be a sequence of open and relatively compact subsets of E such that $x_0\in E_1$, $Cl(E_N)\subseteq E_{N+1},\forall N\in\mathbb{N}$ and $\cup_{N=1}^{\infty}E_N=E$. For the solution $(X_t,t\ge 0)$ of our SDE (1) define

$$T_N:=\inf\{t\ge 0\colon X_t\in E_N^c\},N\in\mathbb{N},$$

where $\inf\emptyset=+\infty$. Let $(\Phi_N,N\in\mathbb{N})$ be a sequence of $C^\infty(E)$-functions such that $\Phi_N=1$ on $Cl(E_N)$, and $\Phi_N=0$ on $Cl(E_{N+1})^c$. Let us define the functions $\mu_N(x):=\Phi_N(x)\mu(x),f_{j,N}(x):=\Phi_N(x)f_j(x),g_{j,N}(x):=\Phi_N(x)g_j(x),x\in E$, and let ν_N be continuous functions on E such that $\nu_N(x)=\nu(x)$ for $x\in Cl(E_N)$ and $\nu_N(x)=const$ for $x\in E\backslash Cl(E_{N+1})$ (ν_N can be defined as $\nu_N(x)=\Phi_N(x)\nu(x)+(1-\Phi_N(x))c_N$, where $c_N:=\min_{x\in Cl(E_{N+1})}\nu(x)$). For a fixed $N\in\mathbb{N}$, let the process $X^N=(X_t^N,0\le t\le T)$ be a unique strong solution of the SDE (for details see [12])

$$dX_t^N=\mu_N(X_t^N)dt+\nu_N(X_t^N)dW_t,\quad X_0^N=x_0,x_0\in E.$$

Let Y be a random vector such that $Y=\sqrt{\Sigma}Z$, where Z is a standard normal random vector independent of $\mathcal{F}_T$. Let Y_N be a random vector such that $Y_N=\sqrt{\Sigma_N}Z$, where Σ_N is a random matrix Σ from the first part of this proof which we apply on the process X^N and functions ν_N, $\mu_N,g_{j,N}$ and $f_{j,N}$. From the first part of the proof, with the notation

$$V_{n,N}:=\frac{1}{\sqrt{\Delta_n}}\begin{bmatrix}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}g_{1,N}(X_s^N)(f_{1,N}(X_s^N)-f_{1,N}(X_{t_{i-1}}^N))dW_s\\ \vdots\\ \sum_{i=1}^n\int_{t_{i-1}}^{t_i}g_{1,N}(X_s^N)(f_{d,N}(X_s^N)-f_{d,N}(X_{t_{i-1}}^N))dW_s\end{bmatrix},$$

we have $V_{n,N}\overset{st}{\Rightarrow}Y_N$.

Let us denote by $V_n:=\frac{1}{\sqrt{\Delta_n}}\begin{bmatrix}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}g_1(X_s)(f_1(X_s)-f_1(X_{t_{i-1}}))dW_s\\ \vdots\\ \sum_{i=1}^n\int_{t_{i-1}}^{t_i}g_d(X_s)(f_d(X_s)-f_d(X_{t_{i-1}}))dW_s\end{bmatrix}$. We

want to show that $V_n \overset{st}{\Rightarrow} Y$, $n \to \infty$.
Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a bounded continuous function, and let U be a bounded $\mathcal{F}_T$-measurable random variable. Let $k_9, k_{10} > 0$ be constants such that $|f| \le k_9$ and $|U| \le k_{10}$. Now, we have:

$$\begin{aligned}&|\mathbb{E}[f(V_n)U] - \tilde{\mathbb{E}}[f(Y)U]|\\ &\quad \le |\mathbb{E}[f(V_{n,N})U1_{\{T_N>T\}}] - \tilde{\mathbb{E}}[f(Y_N)U1_{\{T_N>T\}}]| + 2k_9k_{10}\mathbb{P}(T_N \le T).\end{aligned}$$

$U1_{\{T_N>T\}}$ is a bounded, $\mathcal{F}_T$-measurable random variable, and $V_{n,N} \overset{st}{\Rightarrow} Y_N$; hence

$$\overline{\lim_n} |\mathbb{E}[f(V_n)U] - \tilde{\mathbb{E}}[f(Y)U]| \le 2k_9k_{10}\mathbb{P}(T_N \le T),$$

and by letting $N \to \infty$ we have

$$\overline{\lim_n} |\mathbb{E}[f(V_n)U] - \tilde{\mathbb{E}}[f(Y)U]| = 0,$$

which implies

$$\lim_n |\mathbb{E}[f(V_n)U] - \tilde{\mathbb{E}}[f(Y)U]| = 0,$$

and this proves our theorem. □

Let us now look at the case when $d = 1$. Let us denote $f \equiv f_1$.

Corollary 1. *Assume that (A1)-(A2) hold and that f is a two-times continuously differentiable function on E. Then*

$$\frac{1}{\sqrt{\Delta_n}}\Big(\int_0^T f(X_s)dW_s - \sum_{i=1}^n f(X_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})\Big) \overset{st}{\Rightarrow} MN\Big(0, \frac{1}{2}\int_0^T \nu^2(X_s)(f')^2(X_s)ds\Big).$$

Proof. Since

$$\begin{aligned}&\frac{1}{\sqrt{\Delta_n}}\Big(\int_0^T f(X_s)dW_s - \sum_{i=1}^n f(X_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})\Big)\\ &= \frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (f(X_s) - f(X_{t_{i-1}}))dW_s,\end{aligned}$$

the result follows from Theorem 2 for $d = 1$ and $g_1 \equiv 1$. □

The result about stable convergence in law for difference between Itô's integral with respect to diffusion and its approximation is stated in the following lemma.

Lemma 1. *Assume that (A1)-(A2) hold and that f is a two-times continuously differentiable function on E. Then*

$$\frac{1}{\sqrt{\Delta_n}}\Big(\int_0^T f(X_s)dX_s - \sum_{i=1}^n f(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}})\Big) \overset{st}{\Rightarrow} MN\Big(0, \frac{1}{2}\int_0^T \nu^4(X_s)(f')^2(X_s)ds\Big).$$

Proof. It holds

$$\begin{aligned}
&\frac{1}{\sqrt{\Delta_n}}(\int_0^T f(X_s)dX_s - \sum_{i=1}^n f(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}))\\
=&\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^n \int_{t_{i-1}}^{t_i}(f(X_s) - f(X_{t_{i-1}}))dX_s\\
=&\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^n \int_{t_{i-1}}^{t_i}(f(X_s) - f(X_{t_{i-1}}))\mu(X_s)ds\\
&+\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^n \int_{t_{i-1}}^{t_i}(f(X_s) - f(X_{t_{i-1}}))\nu(X_s)dW_s.
\end{aligned}$$

It can be shown that $\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^n \int_{t_{i-1}}^{t_i}(f(X_s) - f(X_{t_{i-1}}))\mu(X_s)ds$ converges in probability to 0 when $n \to \infty$, and from Theorem 2 it follows that $\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^n \int_{t_{i-1}}^{t_i}(f(X_s) - f(X_{t_{i-1}}))\nu(X_s)dW_s$ converges stably in law to a mixed normal random variable, with variance $\frac{1}{2}\int_0^T \nu^4(X_s)(f')^2(X_s)ds$.

Hence, $\frac{1}{\sqrt{\Delta_n}}(\int_0^T f(X_s)dX_s - \sum_{i=1}^n f(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}))$ converges stably in law to a mixed normal random variable with variance $\frac{1}{2}\int_0^T \nu^4(X_s)(f')^2(X_s)ds$. □

Remark 1. *The result from Lemma 1 for the case when $f(x) = x$ can be found in [8] (see Theorem 5.5. for $d = 1$).*

Corollary 2. *Assume that (A1)-(A2) hold and that f is a two-times continuously differentiable function on E. Then*

$$\frac{\sqrt{2}}{\Delta_n}\frac{\int_0^T f(X_s)dX_s - \sum_{i=1}^n f(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}})}{\sqrt{\sum_{i=1}^n \nu^4(X_{t_{i-1}})(f'^2)(X_{t_{i-1}})}} \overset{st}{\Rightarrow} N(0,1).$$

Proof. Notice that $\frac{1}{2}\sum_{i=1}^n \nu^4(X_{t_{i-1}})(f'^2)(X_{t_{i-1}})\Delta_n$ converges almost surely (and consequently in probability) to $\frac{1}{2}\int_0^T \nu^4(X_s)(f')^2(X_s)ds$ when $n \to \infty$, so the result follows from Lemma 1 and properties of stable convergence in law. □

Remark 2. *In Corollary 2, we could use any sequence v_n which converges in probability to random variance $\frac{1}{2}\int_0^T \nu^4(X_s)(f')^2(X_s)ds$, and the result would be the same, i.e.*

$$\frac{1}{\sqrt{\Delta_n}}\frac{\int_0^T f(X_s)dX_s - \sum_{i=1}^n f(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}})}{\sqrt{v_n}} \overset{st}{\Rightarrow} N(0,1).$$

4. Applications

Example 1. Let us look at the simple case when $dX_t = dW_t$, i.e. X_t is a standard Brownian motion. In that case, $\mu \equiv 0, \nu \equiv 1$. For a two-times continuously differentiable function f, from Corollary 1, we conclude that

$$\frac{1}{\sqrt{\Delta_n}}\Big(\int_0^T f(W_s)dW_s - \sum_{i=1}^n f(W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})\Big) \overset{st}{\Rightarrow} N(0, \frac{1}{2}\int_0^T (f')^2(W_s)ds).$$

This result can be found in [5] for the function f which satisfies some boundness condition. We proved the same result but in the case when $f \in C^2(E)$. Similar result with weak convergence can be found in Theorem 2.1. in [15]. In the special case when $f(x) = x$, this result can be found in Theorem 5.1. in [8].
Let us see a simulation study where $f(x) = x, dX_t = dW_t$. In that case

$$\frac{1}{\sqrt{\Delta_n}}\Big(\int_0^T W_s dW_s - \sum_{i=1}^n W_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})\Big) \overset{st}{\Rightarrow} N(0, \frac{T}{2}),$$

where $\int_0^T W_s dW_s = \frac{1}{2}(W_T^2 - T)$. Let $-z_{\frac{\alpha}{2}}$ be a $\frac{\alpha}{2}$-quantile of the standard normal distribution. We simulated M realizations of a discrete random sample $W_{t_1}, \ldots, W_{t_n}$ over equidistant points $t_i = i\Delta_n$, where $n = 10, 50, 100, 200, 300, 500, 1000, T = 1$ and $\Delta_n = \frac{1}{n}$. Then we calculated how many times the values of the variable

$$\frac{1}{\sqrt{\Delta_n}}\Big(\frac{1}{2}(W_{t_n}^2 - T) - \sum_{i=1}^n W_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})\Big)$$

are in the interval $[-z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{T}{2}}, z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{T}{2}}]$ and present that number as a percentage. The results are given in Table 1, 2, 3 and 4. The simulations show that percentages are close to $1 - \alpha$, even for small n.
Example 2. Let X_t be a geometric Brownian motion with parameters μ and σ. It means that X_t is a process which satisfies the SDE

$$dX_t = \mu X_t dt + \sigma X_t dW_t, X_0 = x_0. \tag{19}$$

We suppose that $x_0 > 0$. It is well known that the solution to (19) is given by

$$X_t = x_0 \exp\{(\mu - \frac{\sigma^2}{2})t + \sigma W_t\}.$$

Let $f(x) = x$. Then from Corollary 2 we have

$$\frac{\sqrt{2}}{\sigma^2 \Delta_n} \frac{\int_0^T X_s dX_s - \sum_{i=1}^n X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})}{\sqrt{\sum_{i=1}^n X_{t_{i-1}}^4}} \overset{st}{\Rightarrow} N(0, 1).$$

We simulated M realizations of a discrete random sample $(X_{t_1}, \ldots, X_{t_n})$ with $x_0 = 1$ and parameters $\mu = 2, \sigma = 1$, over equidistant points $t_i = i\Delta_n$, where $n = 2^k$, $T = 1$ and $\Delta_n = \frac{T}{n}$. We simulated in the way that $n' = 2^l > 2^k$ and we use all of these points to estimate $\int_0^T X_s dX_s$ with $\sum_{i=1}^{n'} X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})$. Then we take a subsample of length $n = 2^k$ and calculate $\sum_{i=1}^n X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})$. Then we calculate percentage of values

$$\frac{\sqrt{2}}{\sigma^2 \Delta_n} \frac{\sum_{i=1}^{n'} X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}) - \sum_{i=1}^n X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})}{\sqrt{\sum_{i=1}^n X_{t_{i-1}}^4}},$$

which are in the interval $[-z_{\frac{\alpha}{2}}, z_{\frac{\alpha}{2}}]$, where $z_{\frac{\alpha}{2}}$ is a $(1-\alpha)$-quantile of standard normal distribution. Results are presented in tables 5, 6 and 7. Simulations showed that the percentage, even for small k, is a value close to $1-\alpha$.
Let $f(x) = x^2$. Then from Corollary 2 we have

$$\frac{1}{\sigma^2 \Delta_n \sqrt{2}} \frac{\int_0^T X_s^2 dX_s - \sum_{i=1}^n X_{t_{i-1}}^2 (X_{t_i} - X_{t_{i-1}})}{\sqrt{\sum_{i=1}^n X_{t_{i-1}}^6}} \overset{st}{\Rightarrow} N(0,1).$$

We simulate in the same way as before, and calculate the percentage of values

$$\frac{1}{\sigma^2 \Delta_n \sqrt{2}} \frac{\sum_{i=1}^{n'} X_{t_{i-1}}^2 (X_{t_i} - X_{t_{i-1}}) - \sum_{i=1}^n X_{t_{i-1}}^2 (X_{t_i} - X_{t_{i-1}})}{\sqrt{\sum_{i=1}^n X_{t_{i-1}}^6}},$$

which are in the interval $[-z_{\frac{\alpha}{2}}, z_{\frac{\alpha}{2}}]$. Results are presented in tables 8, 9 and 10. Again, simulations showed that the percentage, even for small k, is a value close to $1-\alpha$.
Example 3. Let $X = (X_t, t \geq 0)$ be a one-dimensional diffusion which satisfies Itô's stochastic differential equation (SDE) of the form

$$dX_t = \mu(X_t, \theta)dt + \sigma_0 \nu(X_t) dW_t, \quad X_0 = x_0, \quad t \geq 0, \tag{20}$$

where ν and μ are real functions and x_0 is a given deterministic initial value of X.

Let θ_0 and σ_0 be true values of the drift parameter and the diffusion coefficient parameter. We assume that $\sigma_0 > 0$ is known. We assume that θ belongs to the parameter space Θ which is a relatively compact, open and convex set in Euclidean space $\mathbb{R}^d$. Let $T \in \mathbb{R}$ be a fixed number such that $T > 0$ and $0 =: t_0 < t_1 < \cdots < t_n := T$, $n \in \mathbb{N}$ be an equidistant deterministic subdivision of the segment $[0, T]$, i.e. $t_i = i\Delta_n, i = 0, \ldots, n$, $\Delta_n = \frac{T}{n}$. Let $(X_{t_i}, 0 \leq i \leq n)$ be a discrete observation of the trajectory $(X_t, t \in [0, T])$. If $(x, \theta) \mapsto f(x, \theta)$ is a real function, then we will denote by $D_\theta^m f$ the m-th partial derivatives with respect to θ of the function f, $m \in \mathbb{N}$. If $\theta \mapsto f(\theta)$ is a real-valued function defined on an open subset of $\mathbb{R}^d$, then we will denote by $Df(\theta), D^2 f(\theta)$ its first and second derivatives with respect to θ. Let K be a relatively compact set in $\mathbb{R}^d$. We say that a partial derivative $D_\theta^m f$ of a real function $f : E \times Cl(K) \to \mathbb{R}$ exists on $E \times Cl(K)$ if there exists an open set $U^K \subseteq \mathbb{R}^d$, such that $E \times Cl(K) \subseteq E \times U^K$, and an extension $\tilde{f}$ of f, defined on $E \times U^K$, such that $D_\theta^m \tilde{f}$ exists. Let the following assumptions be satisfied:

(A1') For all $\theta \in \Theta$, there exists a strong solution X of SDE (20) on the time interval $[0, +\infty\rangle$. This solution X has continuous paths with values in E and if X' is any other solution of SDE (20), with the same Brownian motion W, then the law of X' is identical to the law of X.

(A2') Function $x \mapsto \nu(x)$ is two-times continuously differentiable on E, and $\nu(x) > 0$ for all $x \in E$. For all $\theta \in Cl(\Theta)$, function $\mu(\cdot, \theta) \colon E \to \mathbb{R}$ is continuously differentiable on E. Functions $(x, \theta) \mapsto \mu(x, \theta)$, $(x, \theta) \mapsto \frac{\partial^2}{\partial x^2} \mu(x, \theta)$, $(x, \theta) \mapsto \frac{\partial}{\partial x} \mu(x, \theta)$, are continuous on $E \times Cl(\Theta)$.

(A3') Functions $x \mapsto g_j(x)$ are two-times continuously differentiable on E, for all $j = 1, \dots, d$. Functions $(x, \theta) \mapsto f_j(x, \theta)$, $(x, \theta) \mapsto \frac{\partial^2}{\partial x^2} f_j(x, \theta)$, $(x, \theta) \mapsto \frac{\partial}{\partial x} f_j(x, \theta)$, are continuous on $E \times Cl(\Theta)$, for all $j = 1, \dots, d$.

(A4') Functions $(x, \theta) \mapsto \frac{\partial^3}{\partial x^2 \partial \theta_j} \mu(x, \theta)$, $(x, \theta) \mapsto \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial x} \frac{\partial}{\partial \theta_j} \mu(x, \theta)$ are continuous on $E \times Cl(\Theta)$, for all $i, j = 1, \dots, d$. For all $m \le d + 3$, there exist partial derivatives $D_\theta^m \mu(x, \theta)$ and $\frac{\partial}{\partial x} D_\theta^m \mu(x, \theta)$ on $E \times Cl(\Theta)$. Furthermore, functions $(x, \theta) \mapsto D_\theta^m \mu(x, \theta)$, $(x, \theta) \mapsto \frac{\partial}{\partial x} D_\theta^m \mu(x, \theta)$, $m \le d + 3$, are continuous on $E \times Cl(\Theta)$.

For each $\theta \in \Theta$ denote by $\Sigma(\theta)$ a $d \times d$ random matrix whose jk-th component is defined by

$$\Sigma(\theta)^{jk} = \frac{\sigma_0^2}{2} \int_0^T \nu^2(X_s) g_j(X_s) g_k(X_s) f_j'(X_s, \theta) f_k'(X_s, \theta) ds,$$

where f_j' is the first derivative of the function f_j with respect to x, $j = 1, \dots, d$. From Theorem 2 it follows:

Theorem 3. *Assume that (A1')-(A3') hold. Then, for arbitrary fixed $\theta \in \Theta$ it holds:*

$$\frac{1}{\sqrt{\Delta_n}} \begin{bmatrix} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} g_1(X_s)(f_1(X_s, \theta) - f_1(X_{t_{i-1}}, \theta)) dW_s \\ \vdots \\ \sum_{i=1}^n \int_{t_{i-1}}^{t_i} g_d(X_s)(f_d(X_s, \theta) - f_d(X_{t_{i-1}}, \theta)) dW_s \end{bmatrix} \overset{st}{\Rightarrow} MN(0, \Sigma(\theta)).$$

Note that the covariance matrix in this case depends on parameter θ.

Let

$$L_T(\theta) = \int_0^T \frac{\mu(X_s, \theta)}{\sigma_0^2 \nu^2(X_s)} dX_s - \frac{1}{2} \int_0^T \frac{\mu^2(X_s, \theta)}{\sigma_0^2 \nu^2(X_s)} ds$$

be a continuous-time log-likelihood function (see [4]) and define a contrast function by (for details, see [10])

$$L_n(\theta) = \sum_{i=1}^n \left(\frac{(X_{t_i} - X_{t_{i-1}}) \mu(X_{t_{i-1}}, \theta)}{\sigma_0^2 \nu^2(X_{t_{i-1}})} - \frac{1}{2} \frac{\mu^2(X_{t_{i-1}}, \theta)(t_i - t_{i-1})}{\sigma_0^2 \nu^2(X_{t_{i-1}})} \right).$$

For $\theta \in \Theta$, let $\Sigma(\theta)$ be a $d \times d$ random matrix whose jk-th component is defined by

$$\Sigma(\theta)^{jk} = \frac{1}{2} \int_0^T \nu^4(X_s) \frac{\partial}{\partial x} \frac{\frac{\partial}{\partial \theta_j} \mu(X_s, \theta)}{\nu^2(X_s)} \frac{\partial}{\partial x} \frac{\frac{\partial}{\partial \theta_k} \mu(X_s, \theta)}{\nu^2(X_s)} ds.$$

Under assumptions (A1')-(A4') it is proved in [10] that

$$\frac{1}{\sqrt{\Delta_n}} (DL_T(\theta) - DL_n(\theta)) \overset{st}{\Rightarrow} Y(\theta),$$

where $Y(\theta) \sim MN(0, \Sigma(\theta))$. This result allowed us to prove that the difference between the approximate maximum likelihood estimator and the maximum likelihood estimator of drift parameters is an asymptotically mixed normal (for details see [10]). We will show that this result follows from Theorem 3. With a notation $\partial\theta_j := \frac{\partial}{\partial\theta_j}, j = 1, \ldots, d$, for the j-th component of $\frac{1}{\sqrt{\Delta_n}}(DL_T(\theta) - DL_n(\theta))$ we have

$$\begin{aligned}
&\frac{1}{\sqrt{\Delta_n}}(DL_T(\theta) - DL_n(\theta))_j \\
=&\frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^{n}\Big(\int_{t_{i-1}}^{t_i}\big(\frac{1}{\sigma_0^2\nu^2(X_s)}\partial\theta_j\mu(X_s,\theta)\mu(X_s,\theta_0) \\
&- \frac{1}{\sigma_0^2\nu^2(X_{t_{i-1}})}\partial\theta_j\mu(X_{t_{i-1}},\theta)\mu(X_s,\theta_0)\big)ds\Big) \qquad (21)\\
&+ \frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^{n}\Big(\int_{t_{i-1}}^{t_i}\big(\frac{1}{\sigma_0^2\nu^2(X_{t_{i-1}})}\partial\theta_j\mu(X_{t_{i-1}},\theta)\mu(X_{t_{i-1}},\theta) \\
&- \frac{1}{\sigma_0^2\nu^2(X_s)}\partial\theta_j\mu(X_s,\theta)\mu(X_s,\theta)\big)ds\Big) \qquad (22)\\
&+ \frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{\nu(X_s)}{\sigma_0}\big(\frac{\partial\theta_j\mu(X_s,\theta)}{\nu^2(X_s)} - \frac{\partial\theta_j\mu(X_{t_{i-1}},\theta)}{\nu^2(X_{t_{i-1}})}\big)dW_s. \qquad (23)
\end{aligned}$$

In article [10], is proved that (21) and (22) converge in probability to zero when $n \to \infty$. If we define

$$g_j(x) := \frac{\nu(X_s)}{\sigma_0}, f_j(x,\theta) := \frac{\partial\theta_j\mu(X_s,\theta)}{\nu^2(X_s)}, j = 1, \ldots, d,$$

then the result follows from Theorem 3.

n	10	50	100	200	300	500	1000
%	0.93	0.94	0.94	0.96	1.00	0.96	0.98

Table 1: *Example 1, M=100,* $\alpha = 0.05$

n	10	50	100	200	300	500	1000
%	0.97	0.99	0.98	1	0.99	1	0.98

Table 2: *Example 1, M=100,* $\alpha = 0.01$

n	10	50	100	200	300	500	1000
%	0.953	0.947	0.938	0.967	0.960	0.932	0.951

Table 3: *Example 1, M=1000,* $\alpha = 0.05$

n	10	50	100	200	300	500	1000
%	0.988	0.983	0.989	0.995	0.986	0.988	0.993

Table 4: *Example 1, M=1000,* $\alpha = 0.01$

k	4	5	6	7	8	9	10	11	12
%	0.92	0.96	0.97	0.94	0.98	0.98	0.96	0.98	0.96

Table 5: *Example 2,* $f(x) = x$*,, M=100,* α*=0.05, l=14*

k	4	5	6	7	8	9	10	11	12	13	14
%	0.93	0.95	0.95	0.95	0.95	0.95	0.95	0.97	0.94	0.96	0.99

Table 6: *Example 2,* $f(x) = x$*, M=100,* α*=0.05, l=16*

k	4	5	6	7	8	9	10	11	12
%	0.922	0.938	0.943	0.956	0.96	0.955	0.954	0.961	0.981

Table 7: *Example 2,* $f(x) = x$*, M=1000,* α*=0.05, l=14*

k	4	5	6	7	8	9	10	11	12
%	0.92	0.97	0.97	0.95	0.96	0.95	0.93	0.98	1

Table 8: *Example 2,* $f(x) = x^2$*, M=100,* α*=0.05, l=14*

k	4	5	6	7	8	9	10	11	12	13	14
%	0.94	0.99	0.96	0.94	0.91	0.92	0.98	0.97	0.95	0.99	0.98

Table 9: *Example 2,* $f(x) = x^2$*, M=100,* α*=0.05, l=16*

k	4	5	6	7	8	9	10	11	12
%	0.947	0.945	0.944	0.95	0.962	0.961	0.958	0.968	0.981

Table 10: *Example 2,* $f(x) = x^2$*, M=1000,* α*=0.05, l=14*

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References

[1] D. J. Aldous, G. K. Eagleson, *On mixing and stability of limit theorems*, Ann. Probab. **6**(1978), 325–331.

[2] O. E. Barndorff-Nielsen, J. M. Gorcuera, M. Podolskij, *Power variation for Gaussian processes with stationary increments*, Stochastic Process. Appl. **119**(2009), 1845–1865.

[3] P. Billingsley, *Convergence of probability measures*, 2nd ed., Wiley and Sons, New York, 1999.

[4] P. D. Feigin, *Maximum likelihood estimation for continuous-time stochastic processes*, Adv. Appl. Prob. **8**(1976), 712–736.

[5] S. Grira, *On Stable Convergence in the Central Limit Theorem*, Global Journal of Pure and Applied Mathematics **12**(2016), 5379–5385.

[6] J. Jacod, *On continuous conditional Gaussian martingales and stable convergence in law*, Séminaire de probabilitiés, Strasbourg, tome 31, 1997, 232–246.

[7] J. Jacod, *Asymptotic properties of realized power variations and related functionals of semimartingales*, Stochastic Process. Appl. **118**(2008), 517–559.

[8] J. Jacod, P. Protter, *Asymptotic error distributions for the Euler method for stochastic differential equations*, Ann. Probab. **26**(1998), 267–307.

[9] J. Jacod, A. N. Shiryaev, *Limit theorems for stochastic processes*, 2nd ed., Springer-Verlag, New York, 2003.

[10] S. Lubura Strunjak, M. Huzak, *Local asymptotic mixed normality of approximate maximum likelihood estimator of drift parameters in diffusion model*, Glas. Mat. Ser. III **52**(2017), 377–410.

[11] P. A. Mykland, L. Zhang, *Inference for continuous semimartingales observed at high frequency*, Econometrica **77**(2009), 1403–1445.

[12] P. Protter, *Stochastic integration and differential equations*, 2nd ed., Springer-Verlag, Berlin, 2004.

[13] A. Rényi, *On stable sequences of events*, Sankhya A **25**(1963), 293–302.

[14] D. Revuz, M. Yor, *Continuous martingales and Brownian motion*, 3rd ed., Springer-Verlag, Berlin, 1999.

[15] H. Rootzén, *Limit distributions for the error in the approximations of stochastic integrals*, Ann. Probab. **8**(1980), 241–251.