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Dynamical analysis on the discrete pentagon fractal

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Abstract. In this study, we aim to define a new chaotic dynamical system family on a discrete pentagon fractal, P^d, a totally disconnected fractal set. One of the ways to define dynamical systems on the discrete $n-$flake fractal is to use the elements of its symmetry group. Thus, with the help of the elements of the symmetry group of the equilateral pentagon D_5 and the shift map (σ), we obtain different dynamical systems via code representations of the points on P^d. Moreover, we investigate Devaney's chaos conditions for this family of dynamical systems.

AMS subject classifications: 28A80, 37D45, 37B10, 37C15

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1. Introduction

There exist numerous self-similar sets based on various types of structures, while the Cantor set, the Sierpinski triangle, the Sierpinski carpet, ferns, the Koch snowflake are the geometrical shapes that often come to mind when discussing fractals. One such example is a fractal formed by starting with any $n-$gon, which is referred to as an $n-$flake or polygonal fractal. For instance, the Koch snowflake can be defined as a $3-$flake fractal that originates from an equilateral triangle. Similarly, a $5-$flake fractal can be based on a pentagon, formed by scaling down each side of the pentagon with a specific scale (contraction) factor. It is possible to generate different variations of $5-$flake fractals using different contractions.

Throughout this study, we focus on a discrete pentagon fractal, in other words, a discrete Sierpinski pentagon or discrete pentagasket, denoted as P^d, which is a totally disconnected set. This fractal emerges as an attractor of the iterated function system consisting of five corresponding contraction mappings. In the literature, while there have been studies on various dynamical systems related to classical fractals like the Sierpinski triangle, the Sierpinski tetrahedron, the Cantor dust, the Vicsek fractal using expanding, folding mappings and symmetries [2, 3, 4, 6, 15, 16], there has not been much research into dynamical systems on the pentagon fractal via the elements of symmetry group D_5 defined by code representations of the points.

Although there are different ways to obtain fractals such as L-systems, the escape time algorithm, etc. ([1, 9, 14]), many of them, such as the fractals mentioned above, are created as the attractors of iterated function systems (IFS) ([5, 9, 11, 12, 13]).

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This approach allows us to define the dynamical systems through code representations of the points associated with contraction mappings. This way makes it easier to investigate chaos conditions and compute periodic points. In the present paper, our main objective is to establish a dynamical system family on the discrete pentagon fractal. It is known that dynamical systems can be defined in various ways, including those associated with iterated function systems, which emerge naturally [9]. On the other hand, special transformations such as expanding, folding, translation, rotation, etc. [3, 4, 15], with the help of the symmetries of polygons [6, 16], are another approach to define dynamical systems. For instance, in [6], a family of dynamical systems is defined on the Cantor dust using a combination of elements of the 4th Dihedral group and the shift map, via code representations of the points. It is sure that special shift maps enable to define different types of dynamical systems which are also chaotic on a code space (see [2, 7]).

Our aim ise to use elements of the symmetry group of an equilateral pentagon D_5 to define a dynamical system family on P^d. As any convex n-gon has $2n$ symmetries, the symmetry group of the pentagon comprises ten elements, five of which are translations and five reflections. To define the dynamical systems using these symmetries, we first represent the points on P^d via quinary numbers, meaning that each point is represented by a sequence of terms from $\{0,1,2,3,4\}$. Then we obtain the expressions of the symmetries of pentagon in Proposition 1 and define the family of the dynamical system $\{P^d;F\}$ via code representations of the points in Definition 1. Furthermore, we give the forms of some periodic points and verify Devaney's chaos conditions (see detail [10]) for this family in Lemma 1 and Theorem 1, respectively. Finally, we examine the topological conjugacy of the dynamical systems through Theorem 3.

2. A dynamical system family on P^d via the elements of symmetry group D_5

In this part, we first introduce a discrete pentagon fractal and representations of its points via quinary numbers. To construct pentagon fractal, we begin with an equilateral pentagon scaling it down by a certain scale factor and then translating it into five copies. If the scale factor is $r=\frac{3-\sqrt{5}}{2}$, the classical pentagon fractal (pentagasket or Sierpinski pentagon) is obtained as the attractor of the iterated function systems $\{\mathbb{R}^2; f_0,f_1,f_2,f_3,f_4\}$, where

$$\begin{aligned}
f_0(x,y)&=(rx,ry)\\
f_1(x,y)&=(rx+0.618,ry)\\
f_2(x,y)&=(rx+0.809,ry+0.588)\\
f_3(x,y)&=(rx+0.309,ry+0.951)\\
f_4(x,y)&=(rx-0.191,ry+0.588)\,.
\end{aligned}$$

Using smaller scale factors than r, different discrete pentagon fractals can be obtained; one of them is illustrated in Figure 1.

We now define the code sets of the pentagon fractal, $P_0^d,P_1^d,P_2^d,P_3^d,P_4^d$ such as $f_i(P^d)=P_i^d$, where $i=0,1,2,3,4$. For any $\alpha=\alpha_1\alpha_2\alpha_3\ldots\alpha_k\in\{0,1,2,3,4\}^k$,

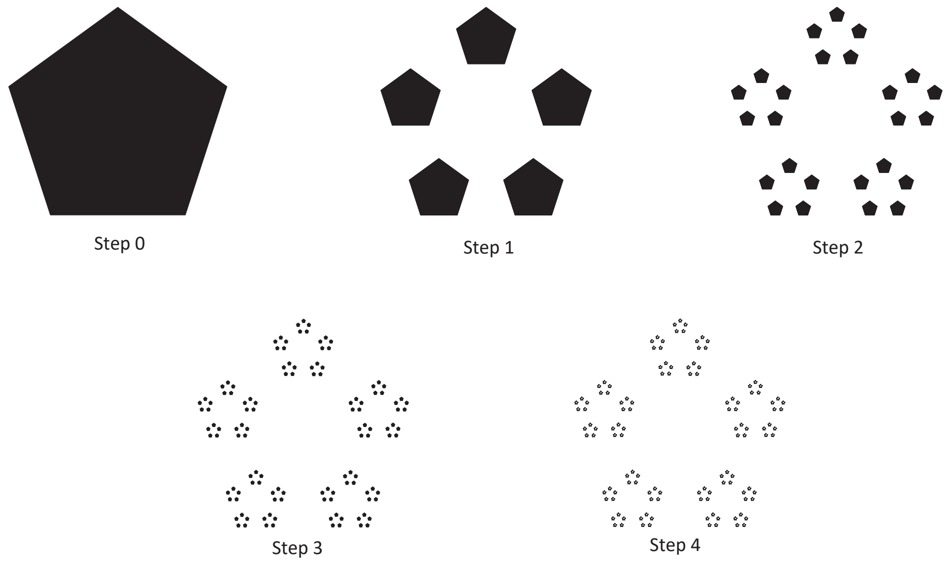

Figure 1: *The first four steps of the construction of a discrete pentagon fractal.*

$P^d_\alpha = f_\alpha(P^d)$, where $f_\alpha = f_{\alpha_1} \circ f_{\alpha_2} \circ f_{\alpha_3} \circ \cdots \circ f_{\alpha_k}$. The code set of P^d_α refers to the kth level sub-pentagon. Unquestionably, $P^d_{\alpha_1}, P^d_{\alpha_1\alpha_2}, P^d_{\alpha_1\alpha_2\alpha_3}, \ldots, P^d_{\alpha_1\alpha_2\alpha_3\ldots\alpha_k} \ldots$ are nested sets respectively. Thus, these sets of sequences intersect at a unique point $A \in P^d$. This point is redefined as $\alpha_1\alpha_2\alpha_3\ldots$, which is called the code representation of A. Since P^d is totally disconnected, every point has a unique code representation on P^d (for details, see Figure 3 and Figure 2).

In this section, we first represent the elements of D_5, which is a symmetry group of the equilateral pentagon, via code representations. It is well known that half of the elements of D_5, ρ_0, ρ_1, ρ_2, ρ_3, ρ_4, are counterclockwise rotational of $0°, 72°$, $144°$, $216°$, $288°$, and the others, μ_1, μ_2, μ_3, μ_4, and μ_5 are axial symmetries with respect to the lines in Figure 3, respectively.

Proposition 1. *Let the code representations of the points X and Y on P^d be $x_1x_2x_3\ldots$ and $y_1y_2y_3\ldots$ for $x_i, y_i \in \{0,1,2,3,4\}$, respectively. Then, the elements of D_5 are redefined via the code representations as follows:*

$$\begin{aligned}
\rho_0(X) = Y &\Leftrightarrow y_i \equiv x_i \pmod 5\\
\rho_1(X) = Y &\Leftrightarrow y_i \equiv x_i + 1 \pmod 5\\
\rho_2(X) = Y &\Leftrightarrow y_i \equiv x_i + 2 \pmod 5\\
\rho_3(X) = Y &\Leftrightarrow y_i \equiv x_i + 3 \pmod 5\\
\rho_4(X) = Y &\Leftrightarrow y_i \equiv x_i + 4 \pmod 5\\
\mu_1(X) = Y &\Leftrightarrow y_i \equiv 4x_i + 1 \pmod 5\\
\mu_2(X) = Y &\Leftrightarrow y_i \equiv 4x_i + 2 \pmod 5\\
\mu_3(X) = Y &\Leftrightarrow y_i \equiv 4x_i + 3 \pmod 5
\end{aligned}$$

$$\mu_4(X) = Y \Leftrightarrow y_i \equiv 4x_i + 4 \pmod 5$$
$$\mu_5(X) = Y \Leftrightarrow y_i \equiv 4x_i \pmod 5.$$

Proof. By using the code representations of the points and the geometrical illustrations for the pentagon fractal (Figure 2, Figure 3), one can easily verify the above expressions of the symmetries. The images of $P_0^d, P_1^d, P_2^d, P_3^d, P_4^d$ under the mapping ρ_3 indicating the rotation of 144° become $P_3^d, P_4^d, P_0^d, P_1^d, P_2^d$, respectively. This means that other cases for rotational symmetries can be shown by observing the paths of green, blue, yellow, colorful and dark blue parts in Figure 2, while the cases for axial symmetries can be observed in Figure 3. □

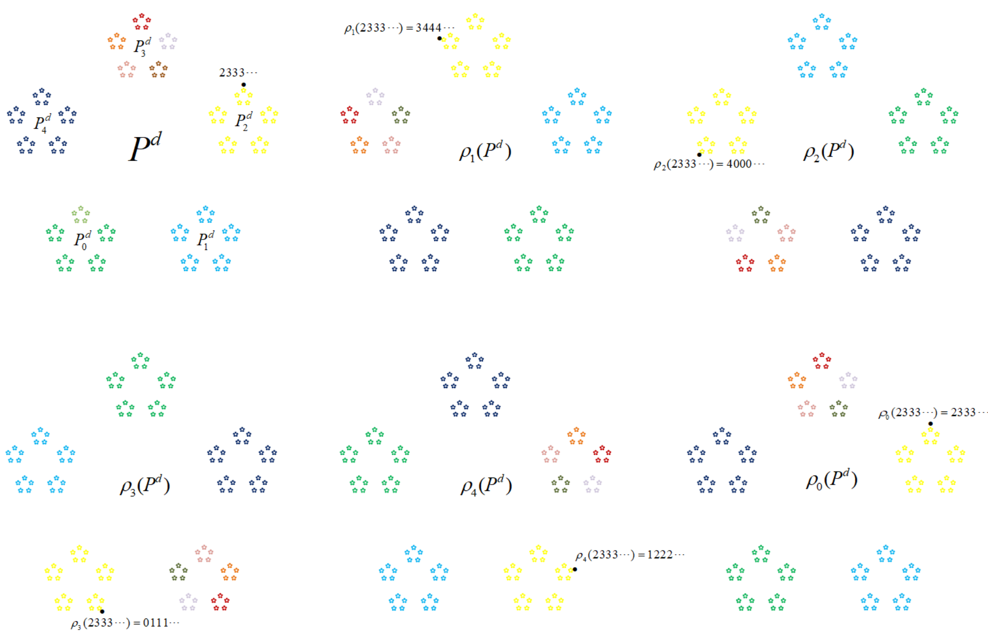

Figure 2: *Rotational symmetries on P^d and the image of some points under these symmetries.*

In order to construct a dynamical system family $\{P^d; F\}$ on the pentagon fractal, the function F is defined by using the composition of the elements of D_5 with the shift map. Then $\{P^d; F\}$ is expressed via quinary numbers as follows:

Definition 1. *Let $f_{i\alpha_i^j}$ be the elements of D_5 and α_i^j an array with j terms of quinary numbers for $i = 0, 1, 2, 3, 4$ and $j = 0, 1, 2, \ldots$. Then $\{P^d; F\}$ is a family of dynamical systems, where $F : P^d \to P^d$ is expressed as*

$$F(X) = \begin{cases} \sigma^{k+1} f_{0\alpha_0^k}(X) & , X \in P^d_{0\alpha_0^k} \\ \sigma^{l+1} f_{1\alpha_1^l}(X) & , X \in P^d_{1\alpha_1^l} \\ \sigma^{m+1} f_{2\alpha_2^m}(X) & , X \in P^d_{2\alpha_2^m} \\ \sigma^{r+1} f_{3\alpha_3^r}(X) & , X \in P^d_{3\alpha_3^r} \\ \sigma^{s+1} f_{4\alpha_4^s}(X) & , X \in P^d_{4\alpha_4^s} \end{cases}.$$

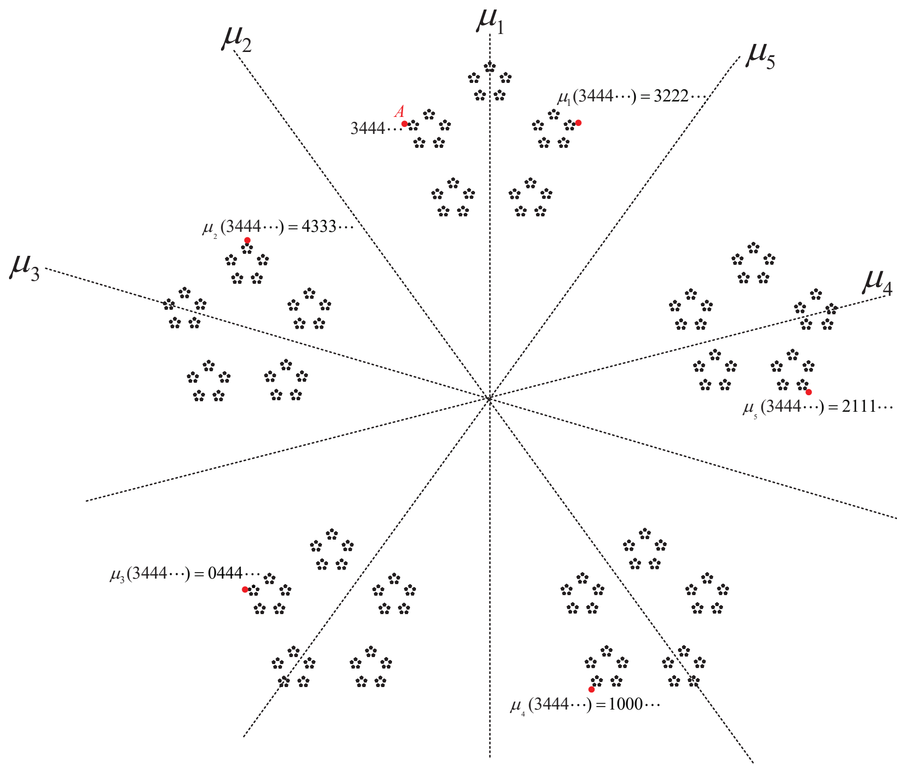

Figure 3: *Axial symmetries on P^d and the image of some points under these symmetries.*

Although there are $10^{5^k+5^l+5^m+5^r+5^s}$ number of different dynamical systems, some of them can be topologically equivalent. Thus, we investigate these systems in terms of topological conjugacy in the next section.

For the case $k=0$, $l=0$, $m=0$, $r=0$, $s=0$, there are 10^5 dynamical systems and $F: P^d \to P^d$ is defined as

$$F(x) = \begin{cases} \sigma f_0(X) \ , \ X \in P_0^d \\ \sigma f_1(X) \ , \ X \in P_1^d \\ \sigma f_2(X) \ , \ X \in P_2^d \\ \sigma f_3(X) \ , \ X \in P_3^d \\ \sigma f_4(X) \ , \ X \in P_4^d \end{cases} .$$

In the case $k=0$, $l=0$, $m=0$, $r=1$, $s=0$, the number of different dynamical systems is 10^9. The dynamical system family is defined as $F : P^d \to P^d$ such that

$$F(X) = \begin{cases} \sigma f_0(X) & , X \in P_0^d \\ \sigma f_1(X) & , X \in P_1^d \\ \sigma f_2(X) & , X \in P_2^d \\ \sigma^2 f_{3\alpha_3^1}(X) & , X \in P_{3\alpha_3^1}^d \\ \sigma f_4(X) & , X \in P_4^d \end{cases} = \begin{cases} \sigma f_0(X) & , X \in P_0^d \\ \sigma f_1(X) & , X \in P_1^d \\ \sigma f_2(X) & , X \in P_2^d \\ \sigma^2 f_{30}(X) & , X \in P_{30}^d \\ \sigma^2 f_{31}(X) & , X \in P_{31}^d \\ \sigma^2 f_{32}(X) & , X \in P_{32}^d \\ \sigma^2 f_{33}(X) & , X \in P_{33}^d \\ \sigma^2 f_{34}(X) & , X \in P_{34}^d \\ \sigma f_4(X) & , X \in P_4^d \end{cases}. \tag{1}$$

According to a specified dynamical system for F given in (1), the movement of some code sets can be observed in Figure 4 and Figure 5.

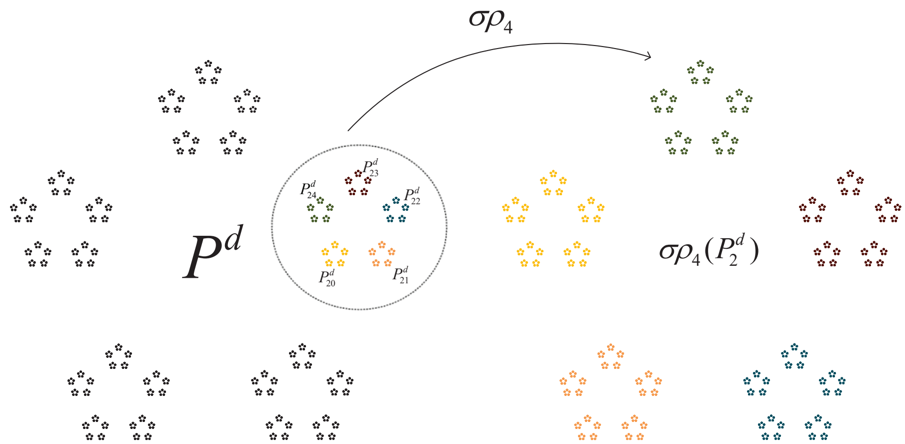

Figure 4: *The image of P_2^d, where $f_2 = \rho_4$ for the dynamical system given in (1).*

In the following lemma, we introduce the form of some periodic points of $\{P^d; F\}$. Thanks to the given forms, we can find a periodic point close enough to any arbitrary points of F.

Lemma 1. *Let the code representation of a point X on the discrete pentagon fractal P^d be denoted by $x_1x_2x_3\ldots$, where each $x_i \in \{0,1,2,3,4\}$ and $\{P^d; F\}$ is the family of dynamical systems such that*

$$F(X) = \begin{cases} \sigma^{k+1} f_{0\alpha_0^k}(X) & , X \in P_{0\alpha_0^k}^d \\ \sigma^{l+1} f_{1\alpha_1^l}(X) & , X \in P_{1\alpha_1^l}^d \\ \sigma^{m+1} f_{2\alpha_2^m}(X) & , X \in P_{2\alpha_2^m}^d \\ \sigma^{r+1} f_{3\alpha_1^r}(X) & , X \in P_{1\alpha_1^r}^d \\ \sigma^{s+1} f_{4\alpha_1^s}(X) & , X \in P_{1\alpha_1^s}^d \end{cases},$$

where $f_{i\alpha_i^j} \in D_5$, $\alpha_i^j \in \{0,1,2,3,4\}^j$ for $i = 0,1,2,3,4$ and $j = 0,1,2,\ldots$. Here, $F^n(X) = (\sigma^p \circ f_i)(X)$ represents the n-times composition and p depends on t and

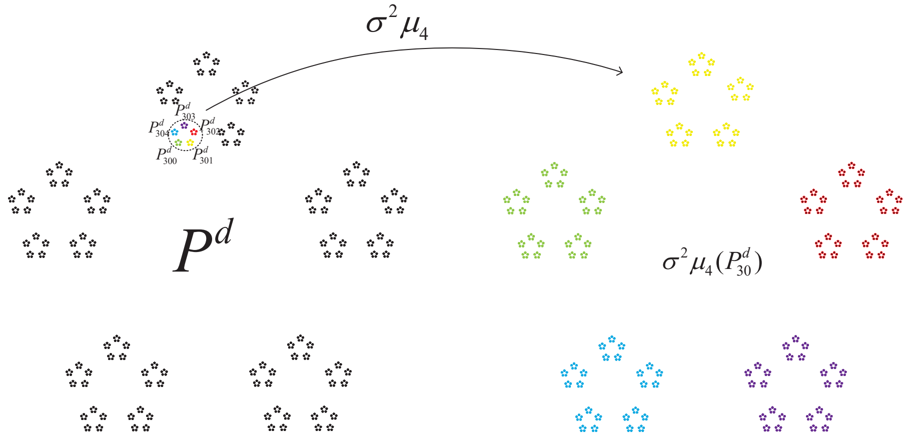

Figure 5: *The image of P_{30}^d, where $f_{30} = \mu_4$ for the dynamical system given in (1).*

T, where $t = \min\{k, l, m, r, s\}$ and $T = \max\{k, l, m, r, s\}$ such that $n(t+1) \leq p \leq n(T+1)$. Thus, some periodic points can be expressed in the following forms:

Case 1: *If $F^n(X) = (\sigma^p \circ \rho_0)(X)$, then one of the n-periodic points takes the form:*

$$\overline{x_1 x_2 x_3 \ldots x_n \ldots x_p}.$$

Case 2: *If $F^n(X) = (\sigma^p \circ \rho_1)(X)$, then one of the n-periodic points takes the form:*

$$\overline{x_1x_2x_3\ldots x_p(x_1+4)(x_2+4)\ldots(x_p+4)(x_1+3)(x_2+3)}$$
$$\overline{\ldots(x_p+3)(x_1+2)(x_2+2)\ldots(x_p+2)(x_1+1)(x_2+1)\ldots(x_p+1)}.$$

Case 3: *If $F^n(X) = (\sigma^p \circ \rho_2)(X)$, then one of the n-periodic points takes the form:*

$$\overline{x_1x_2x_3\ldots x_p(x_1+3)(x_2+3)\ldots(x_p+3)(x_1+1)(x_2+1)}$$
$$\overline{\ldots(x_p+1)(x_1+4)(x_2+4)\ldots(x_p+4)(x_1+2)(x_2+2)\ldots(x_p+2)}.$$

Case 4: *If $F^n(X) = (\sigma^p \circ \rho_3)(X)$, then one of the n-periodic points takes the form:*

$$\overline{x_1x_2x_3\ldots x_p(x_1+2)(x_2+2)\ldots(x_p+2)(x_1+4)(x_2+4)}$$
$$\overline{\ldots(x_p+4)(x_1+1)(x_2+1)\ldots(x_p+1)(x_1+3)(x_2+3)\ldots(x_p+3)}.$$

Case 5: *If $F^n(X) = (\sigma^p \circ \rho_4)(X)$, then one of the n-periodic points takes the form:*

$$\overline{x_1x_2x_3\ldots x_p(x_1+1)(x_2+1)\ldots(x_p+1)(x_1+2)(x_2+2)}$$
$$\overline{\ldots(x_p+2)(x_1+3)(x_2+3)\ldots(x_p+3)(x_1+4)(x_2+4)\ldots(x_p+4)}.$$

Case 6: *If $F^n(X) = (\sigma^p \circ \mu_1)(X)$, then one of the n-periodic points takes the form:*

$$\overline{x_1x_2x_3\ldots x_p(4x_1+1)(4x_2+1)(4x_3+1)\ldots(4x_p+1)}.$$

Case 7: *If $F^n(X) = (\sigma^p \circ \mu_2)(X)$, then one of the n-periodic points takes the form:*

$$\overline{x_1x_2x_3 \dots x_p(4x_1+2)(4x_2+2)(4x_3+2)\dots(4x_p+2)}.$$

Case 8: *If $F^n(X) = (\sigma^p \circ \mu_3)(X)$, then one of the n-periodic points takes the form:*

$$\overline{x_1x_2x_3 \dots x_p(4x_1+3)(4x_2+3)(4x_3+3)\dots(4x_p+3)}.$$

Case 9: *If $F^n(X) = (\sigma^p \circ \mu_4)(X)$, then one of the n-periodic points takes the form:*

$$\overline{x_1x_2x_3 \dots x_p(4x_1+4)(4x_2+4)(4x_3+4)\dots(4x_p+4)}.$$

Case 10: *If $F^n(X) = (\sigma^p \circ \mu_5)(X)$, then one of the n-periodic points takes the form:*

$$\overline{x_1x_2x_3 \dots x_p(4x_1)(4x_2)(4x_3)\dots(4x_p)}.$$

Proof. Since the other cases can be proven in a similar manner, we only show Case 5:

$$F^n(x_1x_2x_3\dots) = (\sigma^p \circ \rho_4)(x_1x_2x_3\dots.).$$

This means

$$\begin{aligned} F^n(x_1x_2x_3\dots) &= (\sigma^p \circ \rho_4)(x_1x_2x_3\dots) \\ &= (x_{p+1}+4)(x_{p+2}+4)(x_{p+3}+4)\dots. \end{aligned}$$

To find n periodic points, we solve the following equality: $F^n(x_1x_2x_3\dots) = x_1x_2x_3\dots$. We thus get $x_i \equiv x_{p+i}+4 \pmod 5$, equivalently $x_{p+i} \equiv x_i + 1 \pmod 5$ for $i = 1,2,3,\dots,p$.

Similarly, we compute $F^{2n}(x_1x_2x_3\dots), F^{3n}(x_1x_2x_3\dots)$, $F^{4n}(x_1x_2x_3\dots)$ and $F^{5n}(x_1x_2x_3\dots)$ to follow repeating terms. By solving $F^{2n}(x_1x_2x_3\dots) = x_1x_2x_3\dots$,

$$\begin{aligned} F^{2n}(x_1x_2x_3\dots) &= (\sigma^{2p} \circ \rho_4^2)(x_1x_2x_3\dots) \\ &= \sigma^{2p}((x_1+3)(x_2+3)(x_3+3)\dots) \\ &= (x_{2p+1}+3)(x_{2p+2}+3)(x_{2p+3}+3)\dots, \end{aligned}$$

we obtain $x_i \equiv x_{2p+i}+3 \pmod 5$, which means $x_{2p+i} \equiv x_i + 2 \pmod 5$ for $i = 1,2,3,\dots,p$.

Analogously, from the equalities $F^{3n}(x_1x_2x_3\dots) = x_1x_2x_3\dots$, $F^{4n}(x_1x_2x_3\dots) = x_1x_2x_3\dots$ and $F^{5n}(x_1x_2x_3\dots) = x_1x_2x_3\dots$, we have $x_{3p+i} \equiv x_i + 3 \pmod 5$, $x_{4p+i} \equiv x_i + 4 \pmod 5$ and $x_{5p+i} \equiv x_i \pmod 5$ for $i = 1,2,3,\dots,p$, respectively.

Consequently, we conclude that the terms are repeated after the first $5p$ blocks and $n-$periodic points are expressed in the following form:

$$\overline{x_1x_2\dots x_p(x_1+1)(x_2+1)\dots(x_p+1)(x_1+2)(x_2+2)}$$
$$\overline{\dots(x_p+2)(x_1+3)(x_2+3)\dots(x_p+3)(x_1+4)(x_2+4)\dots(x_p+4)}.$$

□

Theorem 1. *In a family of dynamical systems defined in (1), $\{P^d; F\}$ is chaotic in the sense of Devaney.*

Proof. Based on the statement about chaos conditions in [8], it is enough to show that $\{P^d; F\}$ is topologically transitive and the sets of periodic points are dense. If F is continuous, then the above properties imply sensitivity dependence on initial conditions. Furthermore, density of periodic points is automatically satisfied thanks to the cases given in Lemma 1. One can easily find a periodic point close enough to any arbitrary points.

We only need to prove that $\{P^d; F\}$ is topologically transitive. Let us choose two non-empty open sets U and $V \in P^d$. Since P^d is a self-similar set, any ball for every point on P^d consists of a small copy of itself. Thus, there is a sub-pentagon fractal at level p:

$$P^d_{a_1a_2a_3\ldots a_p} = \{a_1a_2a_3\ldots a_p x_{p+1}x_{p+2}x_{p+3}\ldots | x_i \in \{0,1,2,3,4\} \text{ are arbitrary}\}$$

such that $P^d_{a_1a_2a_3\ldots a_p} \subseteq U$. There also exists a number n, where $n(t+1) \leq p \leq n(T+1)$, satisfying $t = \min\{k,l,m,r,s\}$ and $T = \max\{k,l,m,r,s\}$, such that

$$F^n(P^d_{a_1a_2a_3\ldots a_p}) = \{x'_{p+1}x'_{p+2}x'_{p+3}\ldots | x'_i \in \{0,1,2,3,4\} \text{ are arbitrary}\}.$$

Then we conclude that

$$P^d = F^n(P^d_{a_1a_2a_3\ldots a_p}) \subseteq F^n(U)$$

which implies $F^n(U) = P^d$. This means that it is possible to find a natural number n such that $F^n(U) \cap V \neq \emptyset$. This completes the proof. □

3. Classification of topologically conjugate dynamical systems on P^d

In order to classify topologically conjugate dynamical systems in Definition 1, we firstly specify the conjugacies between the elements of D_5 and the conjugate maps. We first recall the definition of conjugate map and the theorem expressing the relation of topologically conjugate dynamical systems.

Definition 2. *Let (X, d_1) and (Y, d_2) be two metric spaces and $f : X \to X$ and $g : Y \to Y$ two functions. If there is a homeomorphism h such that $h \circ f = g \circ h$, then f and g are topologically conjugate maps denoted by $f \approx_h g$; here h is called conjugacy ([10, 11]).*

Definition 3. *If $f : X \to X$ and $g : Y \to Y$ are topologically conjugate, $f \approx_h g$, then $\{X; f\}$ and $\{Y; g\}$ are topologically conjugate dynamical systems via conjugacy h.*

Theorem 2. *([10, 11]) Let $f : X \to X$ and $g : Y \to Y$ be topologically conjugate via h. Then the following statements hold:*

- ***i.*** *If $\{x_1, x_2, \ldots, x_n\}$ is an n-periodic cycle for f, then $\{h(x_1), h(x_2), \ldots, h(x_n)\}$ is an n-periodic cycle for g;*

ii. *If and only if f is topologically transitive, g is topologically transitive;*

iii. *If and only if the set of periodic points of f are dense in X, the set of periodic points of g are dense in Y.*

We determine the conjugate maps of D_5 with the following lemma.

Lemma 2. *Some elements of D_5 are topologically equivalent through the following homeomorphisms h:*

i. $\mu_1 \approx_{\rho_3,\mu_4} \mu_2,\ \ \mu_1 \approx_{\rho_1,\mu_2} \mu_3,\ \ \mu_1 \approx_{\rho_4,\mu_5} \mu_4,\ \ \mu_1 \approx_{\rho_2,\mu_3} \mu_5,$

ii. $\mu_2 \approx_{\rho_2,\mu_4} \mu_1,\ \ \mu_2 \approx_{\rho_3,\mu_5} \mu_3,\ \ \mu_2 \approx_{\rho_1,\mu_3} \mu_4,\ \ \mu_2 \approx_{\rho_4,\mu_1} \mu_5,$

iii. $\mu_3 \approx_{\rho_4,\mu_2} \mu_1,\ \ \mu_3 \approx_{\rho_2,\mu_5} \mu_2,\ \ \mu_3 \approx_{\rho_3,\mu_1} \mu_4,\ \ \mu_3 \approx_{\rho_1,\mu_4} \mu_5,$

iv. $\mu_4 \approx_{\rho_1,\mu_5} \mu_1,\ \ \mu_4 \approx_{\rho_3,\mu_3} \mu_2,\ \ \mu_4 \approx_{\rho_2,\mu_1} \mu_3,\ \ \mu_4 \approx_{\rho_3,\mu_2} \mu_5,$

v. $\mu_5 \approx_{\rho_3,\mu_3} \mu_1,\ \ \mu_5 \approx_{\rho_1,\mu_1} \mu_2,\ \ \mu_5 \approx_{\rho_4,\mu_4} \mu_3,\ \ \mu_5 \approx_{\rho_2,\mu_2} \mu_4,$

vi. $\rho_1 \approx_{\mu_i} \rho_4,\ \ \rho_2 \approx_{\mu_i} \rho_3,\ \ \rho_3 \approx_{\mu_i} \rho_2,\ \ \rho_4 \approx_{\mu_i} \rho_1$ *for* $i = 1, 2, 3, 4, 5.$

Proof. We give the proof for $\mu_1 \approx_{\rho_3} \mu_2$ in Case (i).

Let the code representation of $X \in P^d$ be $x_1x_2x_3\dots$, where $x_i \in \{0, 1, , 2, 3, 4\}$ for $i = 1, 2, 3, \dots$. If μ_1 and μ_2 are topologically conjugate maps via conjugacy ρ_3, then the following equality holds:

$$(\rho_3 \circ \mu_1)(x) = (\mu_2 \circ \rho_3)(X).$$

On the other hand, by using the symmetries

$$\begin{aligned}\mu_1(X) =& y \Leftrightarrow y_i \equiv 4x_i + 1 \pmod 5,\\ \mu_2(X) =& y \Leftrightarrow y_i \equiv 4x_i + 2 \pmod 5\end{aligned}$$

and

$$\rho_3(X) = y \Leftrightarrow y_i \equiv x_i + 3 \pmod 5,$$

we obtain

$$\begin{aligned}(\rho_3 \circ \mu_1)(X) &= \rho_3((4x_1 + 1)(4x_2 + 1)(4x_3 + 1)\dots)\\ &= (4x_1 + 4)(4x_2 + 4)(4x_3 + 4)\dots\end{aligned}$$

$$\begin{aligned}(\mu_2 \circ \rho_3)(X) &= \mu_2((x_1 + 3)(x_2 + 3)(x_3 + 3)\dots)\\ &= (4x_1 + 4)(4x_2 + 4)(4x_3 + 4)\dots.\end{aligned}$$

Therefore, $(\rho_3 \circ \mu_1)(X) = (\mu_2 \circ \rho_3)(X)$ holds, which means that μ_1 and μ_2 are topologically equivalent via conjugacy ρ_3. □

Throughout this study, we denote the conjugate maps of f by $f_{\sim}$. For instance, according to Lemma 2, we thus note that the conjugate maps of μ_1 are denoted by $\mu_{1\sim}$, which are μ_2, μ_3, μ_4 and μ_5. The relationships of conjugate maps of ρ_i are $\rho_{1\sim} = \rho_4,\ \rho_{4\sim} = \rho_1,\ \rho_{2\sim} = \rho_3,\ \rho_{3\sim} = \rho_2$.

We now give a practicable theorem to classify topologically conjugate dynamical systems.

Theorem 3. *Let f_i, g_i be elements of the symmetry group D_5 for $i = 0, 1, 2, 3, 4$. $\{P^d; f\}$ and $\{P^d; g\}$ are topologically conjugate dynamical systems via homeomorphism h, where*

$$f : P^d \to P^d, f(x) = \begin{cases} f_0(X)\ ,\ X \in P_0^d \\ f_1(X)\ ,\ X \in P_1^d \\ f_2(X)\ ,\ X \in P_2^d \\ f_3(X)\ ,\ X \in P_3^d \\ f_4(X)\ ,\ X \in P_4^d \end{cases},$$

$$g : P^d \to P^d, g(x) = \begin{cases} g_0(X)\ ,\ X \in P_{h(0)}^d \\ g_1(X)\ ,\ X \in P_{h(1)}^d \\ g_2(X)\ ,\ X \in P_{h(2)}^d \\ g_3(X)\ ,\ X \in P_{h(3)}^d \\ g_4(X)\ ,\ X \in P_{h(4)}^d \end{cases}$$

if the conditions below are satisfied:

i. *$g_i = \rho_0$ if $f_i = \rho_0$ for $i = 1, 2, 3, 4$*

ii. *Otherwise, $g_i = f_i$ or $g_i = f_{i\sim}$ for $i = 1, 2, 3, 4$.*

Proof. **i.** Since ρ_0 is an identity element, the proof is clear.

ii. Let $f_i \neq \rho_0$ and $f_i \in \{\rho_1, \rho_2, \rho_3, \rho_4\}$. Then either $g_i = f_i$ according to the commutative property such that $\rho_m \circ \rho_n = \rho_n \circ \rho_m$, where $\rho_m, \rho_n \neq \rho_0$ for $m \neq n$ or $g_i = f_{i\sim}$ from Lemma 2.

If $f_i \in \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$, then g_i must only be $f_{i\sim}$ from Lemma 2 that satisfies $(h \circ f_i)(x) = (g_i \circ h)(x)$ for any $x \in P^d$. □

Corollary 1. *Let f and g be topologically equivalent in terms of Theorem 3. Then the composition of f and g with the shift map given as $F = \sigma \circ f$ and $G = \sigma \circ g$ is also topologically conjugate.*

We discuss the following examples in terms of topologically conjugacy and periodic points.

Example 1. *Suppose that the functions are defined as*

$$f : P^d \to P^d, f(X) = \begin{cases} \mu_2(X)\ ,\ X \in P_0^d \\ \rho_1(X)\ ,\ X \in P_1^d \\ \rho_3(X)\ ,\ X \in P_2^d \\ \rho_0(X)\ ,\ X \in P_3^d \\ \mu_5(X)\ ,\ X \in P_4^d \end{cases},$$

$$g : P^d \to P^d, g(X) = \begin{cases} \mu_3(X) \ , X \in P_0^d \\ \rho_0(X) \ , X \in P_1^d \\ \rho_2(X) \ , X \in P_2^d \\ \rho_4(X) \ , X \in P_3^d \\ \mu_1(X) \ , X \in P_4^d \end{cases} .$$

Then, the dynamical systems $F = \sigma \circ f$ and $G = \sigma \circ g$ are topologically equivalent via $h = \mu_4$.

One can compute the fixed points of F as $\overline{02}$, $\overline{10432}$, $\overline{24130}$, $\overline{3}$, $\overline{41}$, by using the definition of F. On the other hand, the fixed points of G can be calculated in an easier way via homeomorphism μ_4 such that

$$\mu_4(\overline{02}) = \overline{42}, \quad \mu_4(\overline{10432}) = \overline{34012}, \quad \mu_4(\overline{24130}) = \overline{20314}, \quad \mu_4(\overline{3}) = \overline{1}, \quad \mu_4(\overline{41}) = \overline{03}.$$

On the other hand, the dynamical system

$$T(X) = \begin{cases} \sigma\mu_4(X) \ , X \in P_0^d \\ \sigma^2\mu_1(X) \ , X \in P_{10}^d \\ \sigma^2\mu_5(X) \ , X \in P_{11}^d \\ \sigma^2\rho_2(X) \ , X \in P_{12}^d \\ \sigma^2\rho_0(X) \ , X \in P_{13}^d \\ \sigma^2\mu_2(X) \ , X \in P_{14}^d \\ \sigma\rho_3(X) \ , X \in P_2^d \\ \sigma\mu_3(X) \ , X \in P_3^d \\ \sigma\rho_4(X) \ , X \in P_4^d \end{cases}$$

on P^d is not topologically conjugate with both F and G since $\{P^d; T\}$ has a different number of fixed points than $\{P^d; F\}$ and $\{P^d; G\}$. The fixed points of T are $\overline{04}$, $\overline{1001}$, $\overline{1144}$, $\overline{1240230134}$, $\overline{13}$, $\overline{1413}$, $\overline{24130}$, $\overline{30}$, $\overline{40123}$.

In addition, you can observe that the fixed points $\overline{24130}$ and $\overline{40123}$ are compatible with the forms given in Case 4 and Case 5 of Lemma 1, respectively. One can compute and check the other periodic points according to Lemma 1.

4. Conclusion

In this paper, we define a new family of chaotic dynamical systems on the discrete pentagon fractal P^d via the elements of D_5 and a shift map. We also find useful forms to compute the periodic points of $\{P^d; F\}$ in Lemma 1. Considering this study, the method can be generalized for the discrete n-flake fractal via symmetry groups of any $n-$gon.

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