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ON THE CLOCK PROBLEM IN THE SPECIAL THEORY OF RELATIVITY

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Abstract: In connection with the clock problem in special relativity a new thought experiment is discussed leading to the traditional solution. The accelerated frame is replaced by a family of inertial ones. In this case the round trip times can be calculated from the viewpoint of any of the inertial observers within the special theory of relativity. No additional fundamental assumptions are introduced besides the supposition that ideal clocks exist. The solution of the problem in this case reflects the structure of four-dimensional space-time.

1. Introduction

The clock (or twin) problem is a familiar topic in special relativity. Many relativists agree that no paradox arises in solving this problem and accept the solution found at the advent of special relativity[1,2]. A variety of derivations of this solution has been put forward[3-12]. However, some relativists defend the contrary view that a real paradox exists and even dispute the traditional solution[13-17]. Their arguments differ appreciably and the proposed solutions range from denying any difference of the round trip times to accepting the traditional result as one of two possibilities.

Derivations of the traditional solution and doubts expressed were extensively reviewed by numerous authors[6] and the problem has been considered thoroughly from the observational point of view[5]. In this contribution introducing a new thought experiment further arguments are exposed in favour of the traditional solution.

2. *The usual thought experiment*

In this experiment two reference frames S_0 and S_0' participate with the origins O_0 and O_0' and mutually parallel coordinate axes. The first frame is inertial throughout the experiment, whereas the second frame is not. Identical clocks are situated at the origins together with observers called S and S_0', respectively. The clock in S_0' should be ideal, i.e. it should not change its internal state on account of the acceleration.

Let v be the velocity of O_0' in the x-direction at the event (x_0, t_0) as observed in S_0. Initially, at the event 0, both observers coincide and the velocity v is zero. At this event they synchronize their clocks: $t_{00} = t_{00}' = 0$. (The first index refers to the frame, the second to the event). Then the observer S_0' is accelerated in the x-direction. Later on he is accelerated in the opposite direction and finally, he is accelerated once more. At the event N he coincides with O_0 again having zero velocity v.

Each observer describes the trip from his viewpoint. From the viewpoint of the observer S_0 the round trip time is T measured by his own clock and T' measured by the clock in S_0'. From the viewpoint of the observer S_0' the round trip time is T_0' measured by his own clock and T_0 measured by the clock in S_0.

At the events 0 and N both observers coincide and are at relative rest. Since the frame S_0 is inertial throughout, the frame S_0' is inertial at the beginning and at the end of the round trip. (It is even not necessary to demand that at the events 0 and N the observers are in relative rest). On the basis of the time homogeneity in inertial frames it can be claimed that

$$T = T_0, \quad T' = T_0', \tag{1}$$

i.e. that the round trip times measured by the clock in S_0 have to be equal for both observers and that the same is true for the round trip times measured by the clock in S_0'. Any other result would contradict the fundamental assumptions of special relativity or would reveal a serious internal inconsistency of the theory.

A further step[13]: $T = T'$, however, would imply that the observer S_0' was inertial throughout between the events 0 and N and would prohibit the round trip.

To complete the solution of the problem a relation between T and T' has to be given.

Equations (1) cannot be proven directly from the viewpoint of the uninertial observer S_0' within the special theory of relativity. But with the use of the equations of general relativity or at least the equivalence principle the validity of (1) can be ascertained[18-21].

3. *The new thought experiment*

A new thought experiment corresponding to the usual one can be introduced for which Eqs. (1) can be deduced within the special theory of relativity.* Select a sequence of events $1, 2, \ldots n, \ldots N-1$ on the world line of the origin O_a and replace the accelerated frame S_a' with a family of inertial frames S_n $(1 \leqq n \leqq N)$ with origins O_n. Their coordinate axes are mutually parallel to the coordinate axes of S_0. At the origins identical clocks are situated together with observers. The event n corresponds to the encounter of O_n and O_{n+1}; the event N coresponds to the encounter of O_N and O_0. The events $n-1$ and n line on the world line of O_n, (Fig. 1). The observer S_n synchronizes his clock at the event $n-1$ with the clock of the observer S_{n-1}.

In the frame S_0 the event n is defined with 0_n (x_{0n}, t_{0n}). The velocity of O_n in S_0 is $v(n,0) = (x_{0n} - x_{0n-1})/(t_{0n} - t_{0n-1})$ in the x_0 direction. Further, we have $x_{00} = 0$, $t_{00} = 0$ and

$$x_{0N} = \sum_{n=1}^{N} (x_{0n} - x_{0n-1}) = \sum_{n=1}^{N} v(n,0)(t_{0n} - t_{0n-1}) = 0 . \tag{2}$$

The round trip time for the observer S_0 according to his own clock is

$$T = t_{0N} - t_{00} = t_{0N} = \sum_{n=1}^{N} (t_{0n} - t_{0n-1}) = \sum_{n=1}^{N} (x_{0n} - x_{0n-1})/v(n,0) . \tag{3}$$

To obtain the round trip time T' the Lorentz transformation, connecting the frames S_0 and S_n, has to be exploited. The observer S_n has not synchronized his clock directly with the clock of observer S_0. Nevertheless, owing to the homogeneity of space and time the Lorentz transformation applies to space and time intervals (c is put equal to 1):

$$\begin{aligned} \Delta x_n &= (1 - v^2(n,0))^{-1/2} (\Delta x_0 - v(n,0)\Delta t_0) \\ \Delta t_n &= (1 - v^2(n,0))^{-1/2} (\Delta t_0 - v(n,0)\Delta x_0) . \end{aligned} \tag{4}$$

* In the usual thought experiment often an approximation is considered. The time intervals of acceleration of S_a' measured in S_0 are taken to be negligible as compared with T and the velocity is taken $v = $ const. during the first half and $v = -$const. during the second half of the trip. This approximation is useful but if the experiment is described from the standpoint of the observer S_a' it must be taken into account that the acceleration becomes infinite. From the viewpoint of the observer S_0 in this approximation the accelerated observer can be replaced by two inertial ones. This was first noted by Halsbury[13] and exploited by others[3, 4, 7]. It enables us to get Eqs. (1) for this case within special relativity[12]. A generalization of this procedure leads to the new thought experiment.

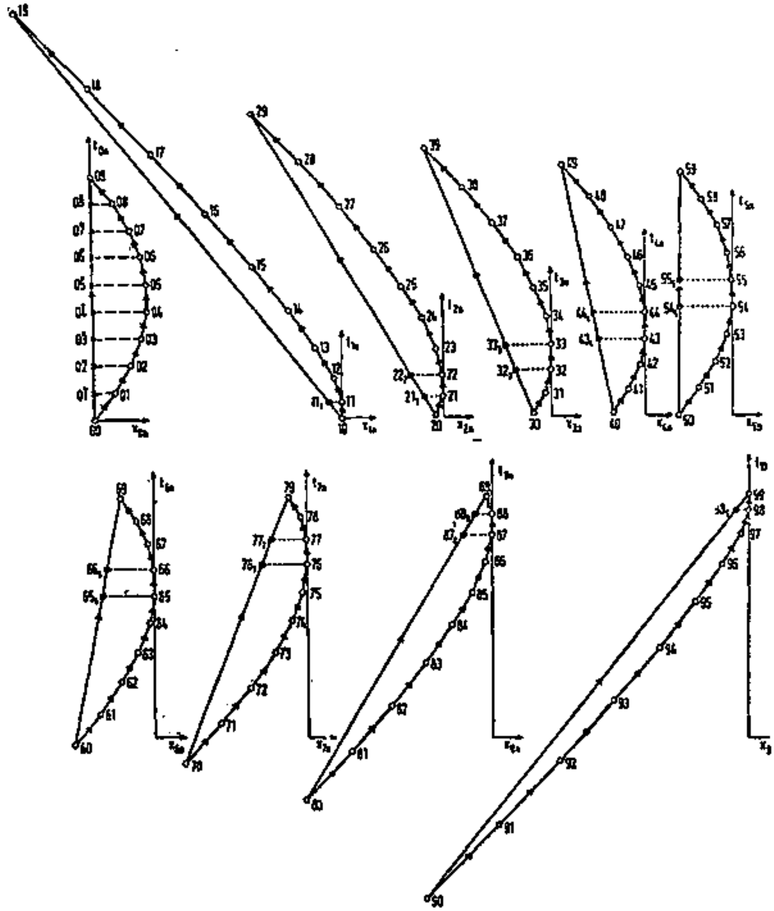

Fig. 1. World lines of the origins O_0 and O_1 to O_9 in the inertial frames S_0 and S_1 to S_9 for an example of the alternative thought experiment: the accelerated frame was replaced by a family of nine inertial ones S_1 to S_9. S_0 is the frame of the »proper« inertial observer and S_1 to S_9 are the frames of »tangential« inertial observers. The equation of the world line of O_n in S_0 is $x_0 = v(n, 0)\,(t_0 - t_{0n-1}) + x_{0n-1}$. — At an event the first number specifies the frame and the second the event. — The events n_n and $(n-1)_n$ happen at the origin O_0 for the observer S_n simultaneously with the events n and $n-1$, respectively.

The time interval between events n and n—1 according to the clock in the frame S_n from the viewpoint of the observer S_0 is

$$t_{nn} - t_{nn-1} = (1 - v^2(n,0))^{-1/2}\,(t_{0n} - t_{0n-1} - v(n,0)\,(x_{0n} - x_{0n-1}))$$
$$= (1 - v^2(n,0))^{1/2}\,(t_{0n} - t_{0n-1})\,.$$

Here we have used the relation $x_{0n} - x_{0n-1} = v(n,0)\,(t_{0n} - t_{0n-1})$. This result can be interpreted in the traditional way: the proper time interval $t_{nn} - t_{nn-1}$ in the frame S_n is calculated by multiplying with $(1 - v^2(n,0))^{1/2}$ the corresponding unproper time interval in the frame S_0 in which the origin O_n is moving with the velocity $v(n, 0)$. The round trip time T' in our thought experiment according to the clocks in the frames $S_1, S_2 \ldots S_N$ from the viewpoint of the observer S_0 is equal to the sum

$$T' = \sum_{n=1}^{N} (t_{nn} - t_{nn-1}) = \sum_{n=1}^{N} (1 - v^2(n,0))^{1/2} (t_{0n} - t_{0n-1}) =$$

$$= \sum_{n=1}^{N} (1 - v^2(n,0))^{1/2} (x_{0n} - x_{0n-1})/v(n,0) . \tag{5}$$

Now we turn to the description from the viewpoint of one of the observers S_n, $1 \leqq n \leqq N$. For this observer the time interval between events 0 and N according to the clock in S_0 is

$$T_n = (1 - v^2(n,0))^{1/2} (t_{nN} - t_{n0}) .$$

The first factor is due to the fact the events 0 and N happen at the same space point in the frame S_0. The proper time is calculated from the viewpoint of the observer in S_n for whom the origin O_0 is moving with the velocity $v(0, n) = -v(n, 0)$. The corresponding unproper time interval in the frame S_n is $t_{nN} - t_{n0}$. Using the Lorentz transformation (4) and taking into account that $x_{0N} - x_{00}$ equals zero we obtain*

$$T_n = t_{0N} - t_{00} = T . \tag{6}$$

This result is independent of the particular value of n.

It remains to calculate the round trip time T_n' from the viewpoint of the observer S_n according to the clocks in the frames $S_1, S_2, \ldots S_N$. The time interval between the events m—1 and m according to the clock in S_m from the viewpoint of the observer S_n is

$$(1 - v^2(m,n))^{1/2} (t_{nm} - t_{nm-1}) .$$

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* Also, $\sum_{n=1}^{N} (t_{nm} - t_{nm-1}) = T$ with the means of Eqs. (7) taking into account that $\sum_{n=1}^{N} v(m,0) (t_{0m} - t_{0m-1}) = 0$.

The events $m-1$ and m happen at the same space point in the frame S_m. The proper time is calculated from the viewpoint of the observer in S_n for whom the origin O_m is moving with the velocity $v\,(m,n)$. Here $t_{nm} - t_{nm-1}$ is the corresponding unproper time interval in the frame S_n $(n \neq m)$. Of course, $v(m,m) = 0$, and $t_{mm} - t_{mm-1}$ is a proper time.

The velocity of the origin O_m in the frame S_n is

$$v\,(m,n) = -v\,(n,m) = (v\,(m,0) - v\,(n,0))/(1 - v\,(m,0)\,v\,(n,0))\,.$$

The frames S_m and S_n are connected through the Lorentz transformation

$$\begin{aligned} \Delta x_n &= (1 - v^2\,(m,n))^{-1/2}\,(\Delta x_m - v\,(m,n)\,\Delta t_m) \\ \Delta t_n &= (1 - v^2\,(m,n))^{-1/2}\,(\Delta t_m - v\,(m,n)\,\Delta x_m)\,. \end{aligned} \tag{7}$$

Here $(1 - v^2(m,n))^{-1/2} = (1 - v^2(n,0))^{-1/2}\,(1 - v^2(m,0))^{-1/2}\,(1 - v(m,0)\,v(n,0))$.

Utilizing Eqs. (7) we get after a short calculation the round trip time from the viewpoint of the observer S_n according to the clocks in $S_1, S_2 \dots S_N$ (Fig. 2)

$$\begin{aligned} T'_n &= \sum_{n=1}^{N} (1 - v^2\,(m,n))^{1/2}\,(t_{nm} - t_{nm-1}) = \\ &= \sum_{n=1}^{N} (1 - v^2\,(m,0))^{1/2}\,(t_{0m} - t_{0m-1}) = T'\,. \end{aligned} \tag{8}$$

In the above calculation we have made use of the relation $x_{0m} - x_{0m-1} = v\,(m,0)\,(t_{0m} - t_{0m-1})$. Again, this result does not depend upon the particular value of n.

It is worthwhile to introduce two additional sequences of events. An event of the first sequence (m) happens at the origin O_0 for the observer S_0 simultaneously with the event m. In the frame S_0 these events are defined through $0(m)$ $(x_{0(m)} = 0,\ t_{0(m)} = t_{0m})$. Using the Lorentz transformation (4) we get in the frame S_n

$$\begin{aligned} x_{n(m)} - x_{n(m-1)} &= -v\,(n,0)\,(1 - v^2\,(n,0))^{-1/2}\,(t_{0m} - t_{0m-1}) \\ t_{n(m)} - t_{n(m-1)} &= (1 - v^2\,(n,0))^{-1/2}\,(t_{0m} - t_{0m-1})\,. \end{aligned}$$

For $m = n$ we have

$$x_{n(n)} - x_{m(n-1)} = -(1 - v^2\,(n,0))^{1/2}\,(x_{0n} - x_{0n-1}).$$

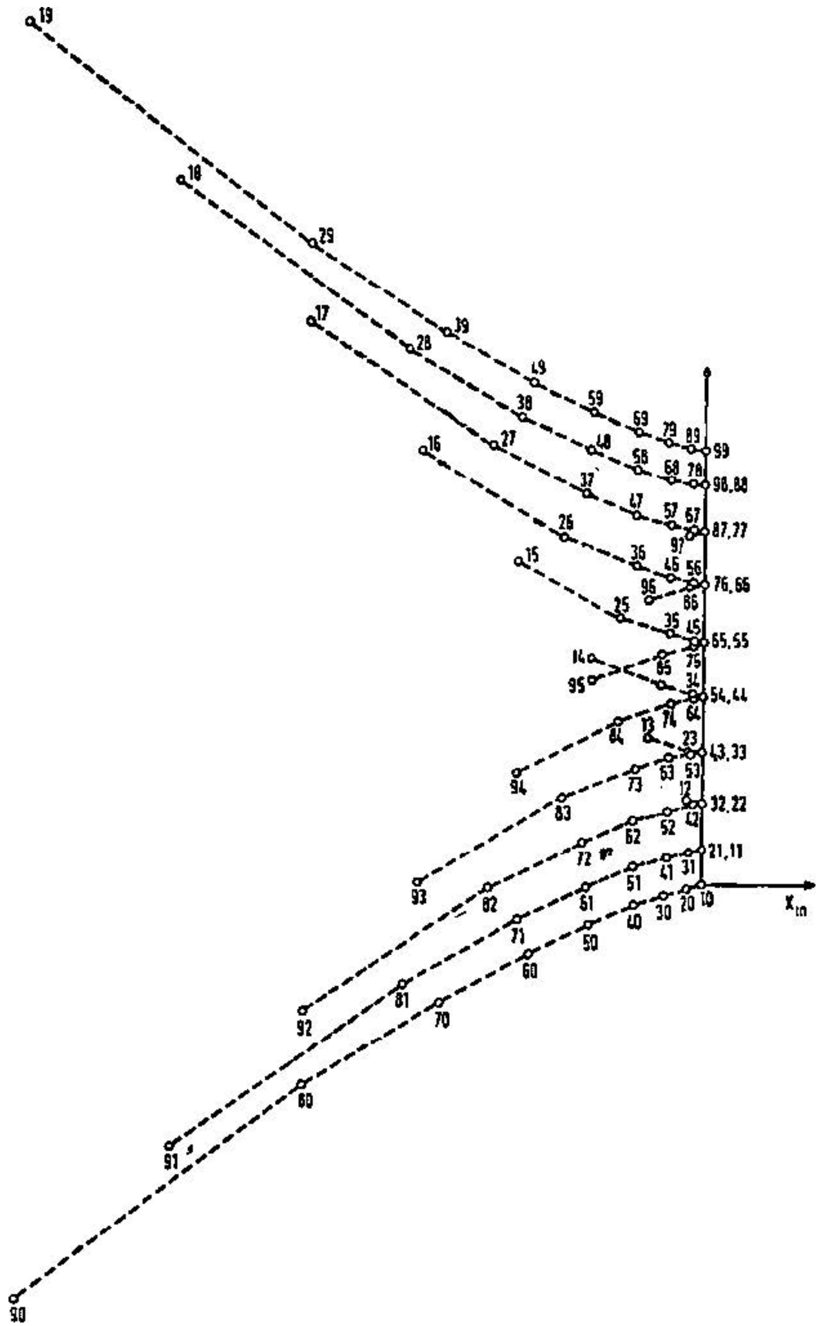

Fig. 2. Parts of the diagrams of the world lines in S_1 to S_9 from Fig. 1, put together to show on the ordinate axis the sum of time intervals $\sum_{n=1}^{N} (t_{nn} - t_{nn-1})$. The shifts of events because of the transitions from frames S_1 to frame $S_2, \ldots S_9$ show the »rotation« of the lines of simultaneity and of the lines of equal distance.

This can be interpreted as the Lorentz contraction of the corresponding proper space intervals. For the observer S_n the events m and (m) are not simultaneous. But if the succesive events m, and so (m), happen for the observer S_0 at equal time intervals so do the events (m) for the observer S_n. This does not hold for the events m from the viewpoint of the observer S_n.

An event of the second sequence m_n happens at the origin O_n simultaneously for the observer S_n with the event m. In the frame S_0 an event of this sequence is defined through $0m_n$ $(x_{0m_n} = 0,\ t_{m_n})$ and in the frame S_n through nm_n $(x_{nm_n},\ t_{nm_n} = t_{nm})$. After applying the Lorentz transformation (4) we get finally

$$t_{0m_n} - t_{0(m-1)_n} = t_{0m} - t_{0m-1} = v(n,0)\,(x_{0m} - x_{0m-1})$$

$$x_{nm_n} - x_{n(m-1)_n} = -v(n,0)\,(1 - v^2(n,0))\,(t_{0m} - t_{0m-1} - v(n,0)\,(x_{0m} - x_{0m-1})).$$

For $m = n$ we get

$$t_{0n_n} - t_{0(n-1)_n} = (1 - v^2(n,0))\,(t_{0n} - t_{0n-1}),$$

and

$$x_{nn_n} - x_{n(n-1)_n} = -(1 - v^2(n,0))\,(x_{0n} - x_{0n-1}).$$

The events m and m_n are not simultaneous for the observer S_0. The successive events m_n do not happen for the observer S_0 at equal time intervals in the case the events m for this observer do.

4. *Discussion*

In Chapter 3 it was shown that the time interval between events 0 and N measured by the clock in S, i.e. along the world line of O_0, from the viewpoint of any of the inertial observers S_k, $0 \leqq k \leqq N$, is equal to T. The time interval between the same events measured by the clock in S_n, $1 \leqq n \leqq N$, as a sum of time intervals between events n—1 and n on the world lines of O_n from the viewpoint of any of the inertial observers S_k equals $T' < T$. Hereby only equations of the special theory of relativity have been used and the results are within this theory.

Now let us compare the alternative thought experiment with the corresponding usual thought experiment. In the second one the two clocks that coincide at the event 0 coincide also at the event N whereas in the first experiment this is not the case. If in the first case the same pair of clocks should meet at the event N the clock from S_1 at the event 1 should be transferred to S_2 and from this at the event 2 to S_3 etc. This procedure shows that the transfer of a clock from an inertial frame to another inertial frame — this cannot be accomplished without acceleration — corresponds to a rotation of the hyperplane of simultaneity. In the alternative thought experiment only the information is transferred from clock to clock at a given space-time point. Thus the supposition of the synchronizations of clocks in the alternative experiment amounts to the supposition of the existence of ideal clocks in the usual thought experiment. The existence of ideal clocks can be based on experimental data,* with the condition that the acceleration does not exceed an upper limit[3,5]. Atoms and subatomic particles can be considered as ideal clocks up to high accelerations.

It can be conjectured that Eq. (5) leading to $T > T'$ for the new thought experiment is a consequence of the structure of four-dimensional space-time. The same equation for the usual thought experiment cannot be a consequence of dynamical effects on the accelerated clock. The acceleration cannot be held directly responsible for the difference of T and T'. Neither there is any reason to base this result on the interactions of the clock with the environment[16].

It should be remarked that the result (5) is not reversible in the sense that T could be replaced by T' and v by $-v$ and vice versa. However, the Eqs. (5), (6) and (8) are derived on an equal footing. One has only to bear in mind that both events 0 and N lie on the world line of the observer S_0, whereas these events do not lie on the world line of a single inertial observer S_n, $1 \leqq n \leqq N$. This causes the asymmetry of the final results. In the interpretation of the alternative thought experiment no ambiguity arises. So no ambiguity can arise in the interpretation of the usual thought experiment[16]. In the limit $N \to \infty$ and $t_{0n} - t_{0n-1} \to 0$ from Eq. (5) directly follows the equation

$$T' = \int_0^T (1 - v^2(t_0))^{1/2} dt_0 = \oint (1 - v^2(x_0))^{1/2} \, dx_0/v(x_0) .$$

This equation can be derived with the assumptions of special relativity and the suppositions that ideal clocks exist and that the accelerated clock is ideal. There is no need to introduce additional fundamental assumptions, as is done in Refs.[22,23]

5. Conclusion

The analysis of an alternative thought experiment affirms the traditional solution of the clock problem. The time between the second and the first encounter measured by the accelerated clock is less than the corresponding time measured by the clock at rest in the inertial frame. An essential supposition is that the accelerated clock is ideal, i.e. that its internal state is unaffected by the acceleration.

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* Some details are given in Ref. [7].

References

1) A. Einstein, Ann. Phys. **17** (1905) 891;
2) P. Langevin, Scientia **10** (1911) 31, 231;
3) H. Bondi, Discovery **18** (1957) 505;
4) D. Scott. Am. J. Phys. **27** (1959) 580;
5) J. Terrell, Nuovo Cim. **16** (1960) 457;
6) G. Holton, Am. J. Phys. **30** (1962) 472;
7) E. S. Lowry, Am. J. Phys. **31** (1963) 59;
8) H. Lass, Am. J. Phys. **31** (1963) 274;
9) L. Levi, Am. J. Phys. **35** (1967) 968;
10) W. H. McCrea, Nature **216** (1967) 122;
11) J. Lang, A. Ruh, Phys. Lett. **29A** (1969) 324;
12) J. Strnad, Phys. Lett. **29A** (1969) 325;
13) H. Dingle, Nature **216** (1967) 119, and **217** (1968) 19;
14) L. Essen, Nature **180** (1957) 1061;
15) L. Essen, Nature **202** (1964) 787, **217** (1968) 19 and Proc. Phys. Soc. **86** (1965) 671;
16) R. Schlegel, Time and the Physical World, Dover, New York, 1968 and priv. comm.
17) M. Pardy, Phys. Lett. **28A** (1969) 766;
18) A. Einstein, Naturwiss. **6** (1918) 697;
19) R. G. Tolman, Relativity, Thermodynamics and Cosmology, Clarendon Press, Cambridge, 1934;
20) C. Møller, The Theory of Relativity, Oxford University Press, London, 1952;
21) H. L. Helfer, Am. J. Phys. **38** (1970) 916;
22) H. M. Schwartz, Introduction to Special Relativity, McGraw-Hill, New York, 1968;
23) S. J. Prokhovnik, The Logic of Special Relativity, University Press, Cambridge, 1967;
24) J. Strnad, Institute »J. Stefan« Report R-583, 1970 (unpubl.).

K PROBLEMU UR V SPECIALNI TEORIJI RELATIVNOSTI

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Vsebina

Sestavek uvaja nov miselni poskus v zvezi s problemom ur v specialni teoriji relativnosti. Ta poskus privede do ustaljene rešitve problema. Pri poskusu nadomestimo pospešeni opazovalni sistem z nizom inercialnih. Čase za krožno pot lahko v tem primeru izračunamo za katerega koli inercialnega opazovalca z enačbami specialne teorije relativnosti. Pri tem ni treba sprejeti nobene dodatne temeljne predpostavke, če le privzamemo, da obstajajo idealne ure. O obstoju idealnih ur pa pričajo poskusi. Po tem lahko sklenemo, da je v našem primeru rešitev problema posledica zgradbe štiridimenzionalnega prostora — časa.