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# THE SYMMETRY OF AN EXACT THREE-BODY SOLUTION

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*Abstract:* The symmetry of the solution of an exactly solvable three-body problem has been considered. A three-body wave function invariant under permutations of the particle numeration has been constructed. The uniqueness of the solution has been proved.

## *1. Introduction*

In a previously published article[1] the author described the way of solving the three-body problem for harmonic oscillator forces acting between pairs of three nonidentical particles. It was shown that the separation of the intrinsic Hamiltonian could be carried out in terms of two linearly independent vectors denoted by $\vec{\xi}_1$ and $\vec{\xi}_2$. These vectors were expressed by linear combinations of two independent vectors $\vec{r}_1$ and $\vec{r}_2$, the vectors of the relative distances between particles 2,3 and 3,1 respectively. The third relative vector $\vec{r}_3$, being linearly dependent on $\vec{r}_1$ and $\vec{r}_2$ (because $\vec{r}_1 + \vec{r}_2 + \vec{r}_3 = 0$) was eliminated from our considerations. The wave function was expressed by the vectors $\vec{\xi}_1$ and $\vec{\xi}_2$ in which three particle labels were not treated symmetrically. This might give the impression that the obtained solution is not complete or does not represent a complete set of functions. The question can be raised what kind of solution we should obtain if we eliminated the vector $\vec{r}_1$ or $\vec{r}_2$ instead of the vector $\vec{r}_3$ in the expressions for the vectors $\vec{\xi}_1$ and $\vec{\xi}_2$.

On the other hand, the numeration of the three particle is arbitrary and we should expect the invariance of the wave function under permutations of the particle indices. Such a behaviour of the wave function is not evident, since the expression for the wave function does not include the three indices symmetrically. It is, therefore, useful to examine the transformation properties of the three-body wave function under a group of permutations of the three indices. It will be shown that the three-body wave function, apart from the phase factor, remains unaltered under the permutation group. The phase factor can be chosen in such a way that the wave function becomes invariant under permutations of the particle numeration.

## *2. Transformation of the relevant quantities*

The solution of a three-body system includes several quantities with definite transformation properties under permutations of the three indices. Before examining these properties we should mention that[2] the whole permutation group (or symmetric group) of three elements 1, 2 and 3 consists of six group elements which can be expressed in cyclic notation as (1), (1,2), (2,3), (3,1), (1,2,3), (2,1,3). Here (1) represents identity. The other symbols denote permutations in which each label in the sequence is to be replaced by the next and the last by the first. The multiplication law table can easily be found by the reader himself. We only state that $(1,2,3)^2 = (2,1,3) = (1,2,3)^{-1}$, $(i, j) = (i, j)^{-1} = (i, j)^2$, $(2,3) = (1,2)\,(1,2,3) = (2,1,3)\,(1,2)$, $(3,1) = (1,2,3)\,(1,2) =$ $= (1,2)\,(2,1,3)$. This shows that it is sufficient to consider only two permutations, (1,2,3) and (1,2), for example; the others can be obtained by combining these two. To simplify the notation, we shall introduce the permutation operators $P$ for (1,2,3) and $P_{ij}$ for $(i, j)$.

The three relative vectors between the three pairs of particles are defined as $\vec{r}_i = \vec{r'}_j - \vec{r'}_k$, with $i, j, k$ in the cyclic order of 1, 2, 3, where $\vec{r'}_i$ represents the position vector of particle $i$. The three vectors under permutations behave as follows

$$
\begin{aligned}
&P\vec{r}_1 = \vec{r}_2,\ P\vec{r}_2 = \vec{r}_3,\ P\vec{r}_3 = \vec{r}_1, \\
&P_{12}\vec{r}_1 = -\vec{r}_2,\ P_{12}\vec{r}_2 = -\vec{r}_1,\ P_{12}\vec{r}_3 = -\vec{r}_3,
\end{aligned}
\tag{1}
$$

etc. cyclically.

For the reduced masses $\mu_i = m_j\, m_k/(m_j + m_k)$, where again $i, j, k$ is to be taken in the cyclic order of 1, 2, 3, we simply have

$$
\begin{aligned}
&P\,\mu_1 = \mu_2,\ P\,\mu_2 = \mu_3,\ P\,\mu_3 = \mu_1, \\
&P_{12}\,\mu_1 = \mu_2,\ P_{12}\,\mu_2 = \mu_1,\ P_{12}\,\mu_3 = \mu_3,
\end{aligned}
\tag{2}
$$

etc. cyclically.

The transformation of the frequencies $\omega_i$, $i = 1, 2, 3$, is equal to that of the reduced masses $\mu_i$, while the quantities $D_i = \frac{\mu_k \omega^2_k}{m_j} - \frac{\mu_j \omega^2_j}{m_k}$ behave like $\vec{r}_i$.

There are two quantities $x_1$ and $x_2$ which enter the definition of the vectors $\vec{\xi}_1$ and $\vec{\xi}_2$, respectively. These quantities are defined as

$$x_{1,2} = \frac{1}{2 D_2}(D_1 + D_2 - D_3 \pm \Delta), \tag{3}$$

with the upper sign for $x_1$ and the lower for $x_2$. Here $\Delta = \sqrt{\Delta^2}$, where $\Delta^2 = D_1^2 + D_2^2 + D_3^2 - 2 D_1 D_2 - 2 D_2 D_3 - 2 D_3 D_1$, so that $\Delta$ is invariant under permutations. Now we define four new quantities analogous to $x_{1,2}$, obtained from $x_{1,2}$ by applying the cyclic permutation $P$ to $x_{1,2}$

$$y_{1,2} = P\, x_{1,2},\ z_{1,2} = P\, y_{1,2} = P^2 x_{1,2}. \tag{4}$$

Obviously, $P z_{1,2} = x_{1,2}$, so that we have the cyclic notation with respect to letters $x, y$ and $z$. We can verify that the following interrelations between $x_{1,2}$, $y_{1,2}$ and $z_{1,2}$ hold (where the subscript 1 or 2 can be added)

$$(1 - x)\, y = (1 - y)\, z = (1 - z)\, x = 1. \tag{5}$$

From Eq. (5) we also have

$$x y z = -1. \tag{6}$$

In addition we have the following three relations

$$P_{12}\, x = \frac{1}{x},\ P_{23}\, x = \frac{1}{z},\ P_{31}\, x = \frac{1}{y}, \tag{7}$$

together with six relations which we obtain from these three by cyclic translations of numbers and letters (subscripts 1 or 2 of $x, y, z$ are not to be affected by this translation).

The last two quantities entering the definition for the vectors $\vec{\xi}_1$ and $\vec{\xi}_2$ are

$$A_{1,2} = \frac{1 - x_{1,2}}{\mu_1} - \frac{x_{1,2}(1 - x_{1,2})}{\mu_2} + \frac{x_{1,2}}{\mu_3}. \tag{8}$$

Analogous cyclic quantities are $B = P A$ and $C = P B = P^2 A$. For $A$, $B$ and $C$ we find the following relations

$$B = y^2 A, \; P_{12} A = C, \; P_{23} A = B, \; P_{31} A = A, \tag{9}$$

etc. cyclically in numbers and letters.

The six quantities $A_{1,2}$, $B_{1,2}$, $C_{1,2}$ are nonnegative quantities, so that we could write, for example, $\sqrt{B} = |y| \sqrt{A}$.

### 3. *Transformation of the vectors $\vec{\xi}_1$ and $\vec{\xi}_2$*

The sign in the definition for the vector $\vec{\xi}_1$ can be chosen arbitrarily. Once the sign of $\vec{\xi}_1$ is chosen, the sign of $\vec{\xi}_2$ is determined. The definitions are

$$\begin{aligned} \vec{\xi}_1 &= S \left(\frac{\Omega_1}{\hbar A_1}\right)^{1/2} (\vec{r}_1 + x_1 \vec{r}_2), \\ \vec{\xi}_2 &= -\operatorname{sign}(D_2)\, S \left(\frac{\Omega_2}{\hbar A_2}\right)^{1/2} (\vec{r}_1 + x_2 \vec{r}_2), \end{aligned} \tag{10}$$

where $S$ is the sign factor equal to $+1$ or $-1$.

In the previously published paper[1] the simplest choice was taken, $S = +1$. This is fully satisfactory for writing down a wave function which always includes an arbitrary phase factor. For permutation symmetry considerations another choice appears to be more suitable.

The two frequencies $\Omega_{1,2}$ are given by

$$\Omega_{1,2} = \frac{1}{2} (\omega_1^2 + \omega_2^2 + \omega_3^2 \mp \Delta)^{1/2}, \tag{11}$$

which shows the fully symmetric form invariant under permutations.

Now we can consider the behaviour of the vectors $\vec{\xi}_{1,2}$ under the permutation group. Applying the permutation $P$ to the vectors $\vec{\xi}_{1,2}$, we obtain two new vectors $\vec{\eta}_{1,2} = P \vec{\xi}_{1,2}$, and applying $P$ once more $\vec{\zeta}_{1,2} = P \vec{\eta}_{1,2} = P^2 \vec{\xi}_{1,2}$. Explicitly

$$\begin{aligned} \vec{\eta}_1 &= T \left(\frac{\Omega_1}{\hbar B_1}\right)^{1/2} (\vec{r}_2 + y_1 \vec{r}_3), \\ \vec{\zeta}_1 &= U \left(\frac{\Omega_1}{\hbar C_1}\right)^{1/2} (\vec{r}_3 + z_1 \vec{r}_1), \end{aligned} \tag{12}$$

with analogous expressions for $\vec{\eta}_2$ and $\vec{\zeta}_2$. The two new sign factors $T$ and $U$ are obtained cyclically from $S : T = P S,\ U = P T = P^2 S$.

Here we have reached the essential point of our considerations: It follows that all the three vectors $\vec{\xi}_1$, $\vec{\eta}_1$ and $\vec{\zeta}_1$ differ only by the sign factor. For example, after substituting $\sqrt{B_1} = |y_1| \sqrt{A_1}$, $\vec{r}_3 = -\vec{r}_1 - \vec{r}_2$ and noting that $y_1 = 1/(1 - x_1)$, we immediately obtain $\vec{\eta}_1$ expressed through $\vec{\xi}_1$. Altogether, we can find that

$$\begin{aligned} \vec{\eta}_{1,2} &= -\operatorname{sign} y_1\, T S\, \vec{\xi}_{1,2}, \\ \vec{\zeta}_{1,2} &= -\operatorname{sign} z_1\, U T\, \vec{\eta}_{1,2} = -\operatorname{sign} x_1\, U S\, \vec{\xi}_{1,2}, \end{aligned} \tag{13}$$

where Eq. (6) has been used in the second line.

A suitable choice for $S$ is now selfevident. By putting

$$S = \operatorname{sign} z_1, \tag{14}$$

i. e., $T = \operatorname{sign} x_1$ and $U = \operatorname{sign} y_1$, we obtain the equality

$$\vec{\xi}_{1,2} = \vec{\eta}_{1,2} = \vec{\zeta}_{1,2}. \tag{15}$$

The behaviour of $\vec{\xi}_1$ and $\vec{\xi}_2$ under the remaining elements $P_{ij}$ of the permutation group is

$$\begin{aligned} P_{12} \vec{\xi}_1 &= P_{23} \vec{\xi}_1 = P_{31} \vec{\xi}_1 = \vec{\xi}_1, \\ P_{12} \vec{\xi}_2 &= P_{23} \vec{\xi}_2 = P_{31} \vec{\xi}_2 = -\vec{\xi}_2, \end{aligned} \tag{16}$$

which is readily obtained using the formulae of Chapter 2.

The result can be expressed in the group theoretical language by saying that the vector $\vec{\xi}_1$ transforms according to the symmetric irreducible representation, while the vector $\vec{\xi}_2$ transforms according to the antisymmetric irreducible representation of the permutation group of three elements[2].

We note that the transformation character of $\vec{\xi}_1$ and $\vec{\xi}_2$ would be reversed if we took the sign factor to be equal to $S = \operatorname{sign}(D_1 D_2 D_3) \operatorname{sign} z_1 = \operatorname{sign}(D_2 z_2)$.

## 4. *The invariant wave function*

The intrinsic wave function of stationary states of the three-body system under consideration is

$$\psi = \varepsilon f_{n_1 l_1}(\xi_1) f_{n_2 l_2}(\xi_2) Y_{l_1}^{m_1}(\hat{\xi}_1) Y_{l_2}^{m_2}(\hat{\xi}_2), \tag{17}$$

where $f_{nl}(\xi)$ is the radial wave function of the isotropic harmonic oscillator. An arbitrary phase factor is denoted by $\varepsilon$. The moduli $\xi_1$ and $\xi_2$ of the two vectors $\vec{\xi}_1$ and $\vec{\xi}_2$ are invariant under permutations, so the radial wave functions are not affected by permutations. Since the vector $\vec{\xi}_1$ is an invariant vector and $\vec{\xi}_2$ may only change the sign, the transformation properties of the function $\psi$ depend only on the phase factor $\varepsilon$ and the spherical harmonic $Y_{l_2}^{m_2}(\hat{\xi}_2)$. We see that

$$P\psi = (P\varepsilon)\cdot\psi,$$

$$P_{ij}\psi = (P_{ij}\varepsilon)\cdot(-1)^{l_2}\psi.$$

To obtain the wave function $\psi$ invariant under the whole permutation group, the phase factor $\varepsilon$ can be chosen as

$$\varepsilon = [\operatorname{sign}(D_1 D_2 D_3)]^{l_2}, \tag{18}$$

with the properties $P\varepsilon = \varepsilon$ and $P_{ij}\varepsilon = (-1)^{l_2}\varepsilon$. Then

$$\mathbf{P}\psi = \psi, \tag{19}$$

where **P** can be any of the six group elements of the permutation group.

## 5. *Conclusion*

From the above considerations we can conclude that the solution of the three-body problem under consideration, although asymmetric in particle indices, is implicitly highly symmetric in them. The order of numeration of the particle affects neither the physical situation nor the wave function itself.

Our considerations also prove the uniqueness of the given three-body solution in the sense that no other solutions could be obtained by changing the numeration of the particles in the solving procedure. Formally different in appearance, through different representations of the variables, all these solutions coincide due to the interrelations (5), (6), (7) and (9) between the relevant parameters entering the solution.

Finally, it may be noticed that our symmetry considerations should not be confused with the widely discussed symmetry considerations including three identical particles, particularly three nucleons (see for example Refs.[3, 4]) We have been concerned with three nonidentical (spinless) particles, three arbitrary masses and three arbitrary force intensities.

## References

1) E. Coffou, Il Nuovo Cimento **61 B** (1969) 342;
2) E. P. Wigner, Group Theory, Academic Press, New York and London (1959);
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## SIMETRIJA EGZAKTNOG RJEŠENJA PROBLEMA TRIJU TIJELA

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### Sadržaj

Razmatrana su svojstva simetrije rješenja problema triju tijela u jednom egzaktno rješivom slučaju. Konstatirana je valna funkcija s invarijantnim svojstvima s obzirom na permutacije u numeraciji čestica. Dokazana je jednoznačnost rješenja.