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THE AVERAGE INTERACTION OF A THREE-BODY SYSTEM

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Abstract: The first-order interaction of a three-body system has been considered. An exact solution of the three-body problem has been used for the three-body wave function. The average interaction has been expressed through the evaluation of one-dimensional integrals. Particular interactions have been discussed. Characteristics of surface interactions have been deduced for spin-orbit and tensor-force average interactions.

1. Introduction

We consider the interaction between a particle with mass m and a bound three-particle system with masses of its constituents m_1, m_2 and m_3. The three particles in the bound state are supposed to interact with each other by the force of the isotropic harmonic oscillator[1]. The intensities of the three interactions involved are left unspecified, which means that the three free parameters characterize the bound state of the system. Since we consider only the ground state, we do not find it necessary to assign quantum number labels to the wave function in our considerations. According to Ref.[1] the ground state wave function of the described three-body system is given by

$$\Phi = N \cdot \mathrm{e}^{-\frac{1}{2}(\beta_1 r_1^2 + \beta_2 r_2^2 + \beta_3 r_3^2)}, \tag{1}$$

where $\vec{r}_i$, $i = 1, 2, 3$, represent three linearly dependent relative vectors for the three particles involved: $\vec{r}_1 = \vec{r}_{23}$, $\vec{r}_2 = \vec{r}_{31}$ and $\vec{r}_3 = \vec{r}_{12}$. Consequently, the relation

$$\vec{r}_1 + \vec{r}_2 + \vec{r}_3 = 0$$

holds.

The normalizing constant is given by

$$N^2 = \frac{1}{\pi^3} (\beta_1 \beta_2 + \beta_2 \beta_3 + \beta_3 \beta_1)^{3/2}. \tag{2}$$

Let the symbols $v_i(\vec{r_{0i}})$, $i = 1, 2, 3$, denote three potentials representing interactions between particle 0 and three particles 1, 2 and 3, constituting the bound state.

The total Hamiltonian for the four particles can be written in the centre-of-mass system as

$$H = H_{in} + T_{\vec{r}} + \sum_{i=1}^{3} v_i(\vec{r_{oi}}). \tag{3}$$

Here, H_{in} is the internal Hamiltonian of the three-body system of particles 1, 2 and 3; the wave function Φ, Equ. (1), is the ground state eigenvector of H_{in}, i. e.,

$$(H_{in} - \varepsilon)\, \Phi = 0. \tag{4}$$

The kinetic operator

$$T_{\vec{r}} = -\frac{\hbar^2}{2\mu} \Delta_{\vec{r}} \tag{5}$$

corresponds to the relative kinetic energy of motion of particle 0 with respect to the centre of mass of the three-body system. The relative coordinates are represented by the vector $\vec{r}$ and the reduced mass is given by

$$\frac{1}{\mu} = \frac{1}{m_0} + \frac{1}{m_1 + m_2 + m_3}$$

A particular solution ψ_0 of the Schrödinger equation

$$H\psi = E\psi \tag{6}$$

can easily be obtained in the separated form

$$\psi_0 = \phi(\vec{r})\, \Phi(\vec{r_1}, \vec{r_2}), \tag{7}$$

where the unknown function $\phi(\vec{r})$ satisfies the one-particle Schrödinger equation

$$(T_{\vec{r}} + V(\vec{r}) - E')\Phi(\vec{r}) = 0. \tag{8}$$

The potential $V(\vec{r})$ given by

$$V(\vec{r}) = (\Phi \mid \sum_{i=1}^{3} v_i \mid \Phi) \tag{9}$$

represents an effective average interaction between particle 0 and the bound three-particle system. The relative energy is $E' = E - \varepsilon$.

The solution (7) can only approximately describe the actual physical situation, because it does not include the excitation effects of the bound three-particle system, i. e., the inelastic scattering of particle 0 on the bound system. The general solution ψ, which takes into account all effects is given by the expansion

$$\psi = \sum_n f_n(\vec{r})\, \Phi_n(\vec{r}_1 \vec{r}_2), \tag{10}$$

where Φ_n represents a complete set of wave functions for the three-particle system; all these wave functions satisfy Equ. (4) with the corresponding energies ε_n. The functions $f_n(\vec{r})$ satisfy a complicated coupled system of differential equations which can be solved only approximately by the perturbation method, for example. In this case, the solution (7) serves as a good starting point as far as elastic scattering is concerned. The difference $\sum v_i - V(\vec{r})$ acts as the perturbation interaction, which is more suitable than the interaction $\sum v_i$ alone. Better convergence of the Born expansion can be expected for the elastic scattering amplitude in which the first order term vanishes identically.

In this paper we consider the problem of finding the analytic expressions for the potential $V(\vec{r})$ corresponding to several well-known shapes of the potentials v_i.

2. *The averaging formalism*

The mathematical problem we are facing consists in evaluating the integral which defines the function $V(\vec{r})$ in closed analytic form. Three integrals are involved

$$V_i(\vec{r}) = \int\int v_i(\vec{r}_{0i}) \mid \Phi(\vec{r}_1, \vec{r}_2) \mid^2 \mathrm{d}\vec{r}_1\, \mathrm{d}\vec{r}_2 \qquad i = 1,\ 2,\ 3, \tag{11}$$

so that

$$V(\vec{r}) = \sum_{i=1}^{3} V_i(\vec{r}). \tag{12}$$

The three vectors $\vec{r}_{0i}$, $i = 1, 2, 3$, can be expressed linearly through the vectors $\vec{r}$, $\vec{r}_1$ and $\vec{r}_2$ as follows

$$\begin{aligned}
\vec{r}_{01} &= \vec{r} + \frac{m_2}{m}\vec{r}_1 + (1 - \frac{m_1}{m})\vec{r}_2, \\
\vec{r}_{02} &= \vec{r} - (1 - \frac{m_2}{m})\vec{r}_1 - \frac{m_1}{m}\vec{r}_2, \\
\vec{r}_{03} &= \vec{r} + \frac{m_2}{m}\vec{r}_1 - \frac{m_1}{m}\vec{r}_2,
\end{aligned} \tag{13}$$

where $m = m_1 + m_2 + m_3$.

In compact notation we can write

$$\vec{r}_{0i} = \vec{r} + a_i\vec{r}_1 + b_i\vec{r}_2 \quad i = 1, 2, 3, \tag{14}$$

where a_i and b_i follow from eqs. (13). Owing to relation (14), the particular variable dependence under the sign of integration suggests that it is easier to evaluate the Fourier transform of $V_i(\vec{r})$ than $V_i(\vec{r})$ itself.

We define the Fourier transform of the potential $V_i(\vec{r})$

$$\tilde{V}_i(\vec{k}) = \int e^{i\vec{k}\vec{r}} V_i(\vec{r})\, d\vec{r}. \tag{15}$$

After a simple variable transformation we obtain the factorized form for $\tilde{V}_i(\vec{k})$

$$\tilde{V}_i(\vec{k}) = \tilde{v}_i(\vec{k})\,\tilde{\Delta}_i(\vec{k}), \tag{16}$$

where

$$\tilde{v}_i(\vec{k}) = \int e^{i\vec{k}\vec{x}} v_i(\vec{x})\, d\vec{x}, \tag{17}$$

$$\tilde{\Delta}_i(\vec{k}) = \int\int e^{-i\vec{k}(a_i\vec{r}_1 + b_i\vec{r}_2)} |\Phi(\vec{r}_1, \vec{r}_2)|^2\, d\vec{r}_1\, d\vec{r}_2. \tag{18}$$

Now, the functions $\tilde{\Delta}_i(\vec{k})$ are fully determined by the three-body bound state and can readily be evaluated

$$\tilde{\Delta}_i(\vec{k}) = \mathrm{e}^{-\alpha_i^2 k^2}, \tag{19}$$

where α_i are real and equal

$$\alpha_i^2 = \frac{1}{4m^2} \frac{(m_j + m_k)^2 \beta_i + m_j^2 \beta_j + m_k^2 \beta_k}{\beta_1 \beta_2 + \beta_2 \beta_3 + \beta_3 \beta_1}, \quad i, j, k \text{ cyclic}. \tag{20}$$

The constants α_i are proportional to the root-mean-square values $\sqrt{\langle R_i^2 \rangle}$ of the distance R_i of the i-th particle from the centre of mass of the bound three-particle system. The connection between the two quantities (Ref. [1]) is given by

$$\alpha_i^2 = \frac{1}{6} \langle R_i^2 \rangle. \tag{21}$$

We see that the internal structure of the three-body system enters the definition of the effective interaction $\tilde{V}_i(\vec{k})$ only through the quantity $\tilde{\Delta}_i$ appearing as a multiplicative factor to the interaction $\tilde{v}_i(\vec{k})$. The factor $\tilde{\Delta}_i$ modifies the original interaction between the outside particle, particle 0, and the constituents of the three-body system. This modification has a smoothing effect in the coordinate representation of the potential $V_i(\vec{r})$. For example, the extremely short-range and singular interaction $\delta(\vec{r}_{0i})$ for $v_i(\vec{r}_{0i})$ leads to $\tilde{v}_i(\vec{k}) \equiv 1$ and $\tilde{V}_i(\vec{k}) = \tilde{\Delta}_i(\vec{k}) = \mathrm{e}^{-\alpha_i^2 k^2}$. Using a transformation which is inverse to relation (15)

$$V_i(\vec{r}) = \frac{1}{(2\pi)^3} \int \mathrm{e}^{-i\vec{k}\vec{r}} \tilde{V}_i(\vec{k}) \, \mathrm{d}\vec{k}, \tag{22}$$

we obtain

$$V_i(\vec{r}) = \frac{1}{(4\pi\alpha_i^2)^{3/2}} \mathrm{e}^{-\frac{r^2}{4\alpha_i^2}}.$$

This is a nonsingular and smooth function resembling the mass density distribution of the three-body system.

In the general case of an arbitrary function $v(\vec{r})$ (to simplify the notation, we shall omit the unessential subscripts) the averaging procedure would involve the evaluation of two three-dimensional integrations, Eqs. (17) and

(22). It is, however, possible to reduce these integrations to one-dimensional integrations if the multipole expansion of the interaction $v(\vec{r})$ is employed. We can write the expansion

$$v(\vec{r}) = \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} w_{\lambda}^{\mu}(r)\, Y_{\lambda}^{\mu}(\hat{r}) - \sum_{\lambda\mu} v_{\lambda}^{\mu}(\vec{r}), \tag{23}$$

where $Y_{\lambda}^{\mu}(\hat{r})$ are spherical harmonics and $w_{\lambda}^{\mu}(r)$ are functions independent of angles. In order to ensure the rotational invariance of the interaction, the functions $w_{\lambda}^{\mu}(r)$ should possess appropriate tensorial properties, which, however, have no influence on our considerations.

Owing to the linearity of the averaging procedure, we can average the expansion (23) term by term. For the term $v_{\lambda}^{\mu}(\vec{r})$ we obtain the Fourier transform, equ. (17),

$$\tilde{v}_{\lambda}^{\mu}(\vec{k}) = 4\pi i^{\lambda}\, \tilde{w}_{\lambda}^{\mu}(k)\, Y_{\lambda}^{\mu}(\hat{k}). \tag{24}$$

where

$$\tilde{w}_{\lambda}^{\mu}(k) = \int_0^{\infty} w_{\lambda}^{\mu}(r)\, j_{\lambda}(kr)\, r^2\, dr. \tag{25}$$

Here, $j_{\lambda}(x)$ denotes the spherical Bessel function. Forming $\tilde{V}_{\lambda}^{\mu}(\vec{k})$ according to Equ. (16) and using the inverse transformation (22), we obtain

$$V_{\lambda}^{\mu}(\vec{r}) = W_{\lambda}^{\mu}(r)\, Y_{\lambda}^{\mu}(\hat{r}). \tag{26}$$

where

$$W_{\lambda}^{\mu}(r) = \frac{2}{\pi} \int_0^{\infty} e^{-\alpha^2 k^2}\, j_{\lambda}(kr)\, \tilde{w}_{\lambda}^{\mu}\, k^2\, dk = \tag{27}$$

$$= \frac{2}{\pi} \int_0^{\infty} \left\{ \int_0^{\infty} e^{-\alpha^2 k^2}\, j_{\lambda}(kr)\, j_{\lambda}(kr')\, k^2\, dk \right\} w_{\lambda}^{\mu}(r')\, r'^2 dr'.$$

Here, the integral over k can be expressed through the modified spherical Bessel function (formula 6.633/2 of Ref. [2]), and the results is

$$W_\lambda^\mu(r) = \frac{1}{2\sqrt{\pi}} \frac{1}{\alpha^3} \int_0^\infty e^{-\frac{r^2 + r'^2}{4\alpha^2}} y_\lambda\left(\frac{rr'}{2\alpha^2}\right) w_\lambda^\mu(r') r'^2 \, dr', \tag{28}$$

where $y_\lambda(x)$ denotes the modified spherical Bessel function. This function is related to the spherical Bessel and modified Bessel functions in the following manner

$$y_\lambda(x) = (-i)^\lambda j_\lambda(ix) = \left(\frac{\pi}{2x}\right)^{1/2} I_{\lambda + \frac{1}{2}}(x).$$

The average interaction $V(\vec{r})$ corresponding to the interaction $v(\vec{r})$, Equ. (23), is finally given by the expansion

$$V(\vec{r}) = \sum_{\lambda\mu} W_\lambda^\mu(r) \, Y_\lambda^\mu(\hat{r}) \quad . \tag{29}$$

In the particular case of a spherically symmetric interaction $v(r)$, only the first term with $\lambda = 0$ is present in the expansion (23). Then, since $y_0(x) = \operatorname{sh} x/x$, we can write

$$V(r) = \frac{1}{2\sqrt{\pi}} \frac{1}{\alpha r} \int_0^\infty v(r') \left\{ e^{-\left(\frac{r - r'}{2\alpha}\right)^2} - e^{-\left(\frac{r + r'}{2\alpha}\right)^2} \right\} r' \, dr'. \tag{30}$$

To give an example of the case when the terms $\lambda \neq 0$ are present, let us consider the so-called tensor-force interaction. The standard form of this interaction is

$$v(\vec{r}_{0i}) = w(r_{0i}) \left\{ 3 \frac{(\vec{\sigma}_0 \vec{r}_{0i})(\vec{\sigma}_i \vec{r}_{0i})}{r_{0i}^2} - \vec{\sigma}_0 \vec{\sigma}_i \right\}, \tag{31}$$

where a somewhat more explicit labelling has been used to ensure clarity. The operators $\vec{\sigma}_0$ and $\vec{\sigma}_i$ represent the Pauli spin operator for particles 0 and i, respectively. This interaction can be put in the form (23), since we can write

$$v(\vec{r}_{oi}) = w(r_{oi}) \sum_\mu T_2^{\mu\dagger} \, Y_2^\mu(\hat{r}_{oi}), \tag{32}$$

where the spin tensor operator components T_2^μ are defined through the spin operators $\vec{\sigma}$. The explicit form of these operators is here irrelevant, for it

remains unchanged by averaging over the coordinates. According to Equ. (31) only the radial functions in the expansion (32) will be changed, and we can write

$$V(\vec{r}) = W(r)\left\{3\frac{(\vec{\sigma}_0\vec{r})\,(\vec{\sigma}_i\vec{r})}{r^2} - \vec{\sigma}_0\vec{\sigma}_i\right\} \tag{33}$$

with $W(r)$ given by formula (28) for $\lambda = 2$.

The well-known type of interaction is the so-called spin-orbit interaction

$$v_i(\vec{r}_{0i}) = \frac{w_i(r_{0i})}{r_{0i}}\vec{l}_{0i}\cdot\vec{s}_{0i}, \tag{34}$$

where $\vec{s}$ represents the operator of the total spin of the two particles in question, $\vec{s}_{0i} = \vec{s}_0 + \vec{s}_i$. The orbital angular momentum operator is given by $\vec{l}_{0i} = \vec{r}_{0i} \times \vec{p}_{0i}$, where $\vec{p}_{0i}$ is the linear momentum operator. The interaction (34) includes derivatives and cannot be treated by formula (28). To include derivatives in our considerations of the averaging procedure, it is necessary to express the gradient operators ∇_{0i}, $i = 1, 2, 3$ in terms of the gradient operators ∇, ∇_1 and ∇_2. From Equs. (13) it follows

$$\begin{aligned}
\nabla_{01} &= \frac{m_1}{m}\nabla + \nabla_2, \\
\nabla_{02} &= \frac{m_2}{m}\nabla - \nabla_1, \\
\nabla_{03} &= \frac{m_3}{m}\nabla + \nabla_1 - \nabla_2.
\end{aligned} \tag{35}$$

When the derivatives enter linearly into the interaction as in (34), the interaction splits into two or three terms depending on the number of terms on the right-hand side of Equs. (35). The first term includes no derivatives with respect to the averaging variables $\vec{r}_1$ and $\vec{r}_2$, and consequently it can be treated by formula (28). The terms involving the operators ∇_1 or ∇_2 can readily be put in the form (16). The function $v_i(\vec{x})$ in formula (17) should include the whole spacial dependence and $\tilde{\Delta}_i^{(1)}(\vec{k})$ and $\tilde{\Delta}_i^{(2)}(\vec{k})$ should replace $\tilde{\Delta}_i(k)$, depending on whether the operator ∇_1 or the operator ∇_2 is involved.

The two functions $\tilde{\Delta}_i^{(s)}(\vec{k})$ $s = 1, 2$, are defined by

$$\tilde{\Delta}_i^{(s)}(\vec{k}) = \int\int e^{-i\vec{k}(a_i\vec{r}_1 + b_i\vec{r}_2)} \Phi^*(\vec{r}_1, \vec{r}_2) \nabla_s \Phi(\vec{r}_1, \vec{r}_2) \, d\vec{r}_1 \, d\vec{r}_2 \qquad s = 1, 2. \tag{36}$$

The integration by parts gives

$$\begin{aligned} \tilde{\Delta}_i^{(1)}(\vec{k}) &= \frac{1}{2} i a_i \vec{k} \tilde{\Delta}_i(k), \\ \tilde{\Delta}_i^{(2)}(\vec{k}) &= \frac{1}{2} i b_i \vec{k} \tilde{\Delta}_i(k). \end{aligned} \tag{37}$$

Applying this procedure to the interaction (34), we readily find that only the first term in the right-hand side of the substitution (35) contributes to the average interaction. The other terms vanish identically. The spacial average of the interaction (34) is equal to

$$V_i(r) = -\frac{m_i}{m} \frac{W_i(r)}{r} \vec{l} \cdot \vec{s}_{0i}, \tag{38}$$

with $W_i(r)$ given by formula (28) for $\lambda = 1$.

We note that in examples (33) and (38) the averaging over the spacial coordinates represents only part of the complete averaging procedure over the intrinsic degrees of freedom of the three-body system. The averaging over the spin »variables« will also be required if the internal structure of the three-body system is to be completely avoided. The result of the latter procedure leads to the definition of the operator $\vec{s}$ which represents the spin

Table 1

	$s = \frac{1}{2}$		$s = \frac{3}{2}$
	$k = 0$	$k = 1$	
g_1	0	$\frac{2}{3}$	$\frac{1}{3}$
g_2	0	$\frac{2}{3}$	$\frac{1}{3}$
g_3	1	$-\frac{1}{3}$	$\frac{1}{3}$

of the three-body system and replaces the operators $\vec{s}_i$ in the spacial averaged interactions. The proportionality factors g_i are defined by the relation

$$\vec{s}_i = g_i \vec{s}. \tag{39}$$

It should be noted that both sides of relation (39) give the same value when averaged over the spin state coordinates of the three-body system. For the most interesting case, namely for particles of spin $\frac{1}{2}$ with the coupling scheme

$$\begin{aligned} \vec{s}_1 + \vec{s}_2 &= \vec{k} \\ \vec{k} + \vec{s}_3 &= \vec{s}, \end{aligned} \tag{40}$$

we find the values of g_i listed in Table 1.

The total average interaction of the *l-s* type resulting from Equs. (38) and (39) is

$$V = \frac{1}{r}\{f(r)\vec{l}\cdot\vec{s}_0 + g(r)\vec{l}\cdot\vec{s}\}, \tag{41}$$

where

$$\begin{aligned} f(r) &= \frac{1}{m}\sum_{i=1}^{3} m_i W_i(r), \\ g(r) &= \frac{1}{m}\sum_{i=1}^{3} m_i g_i W_i(r). \end{aligned} \tag{42}$$

This example clearly shows that the character of the interaction with a composite particle is affected by its structure. Spherically symmetric interactions and tensor-force interactions remain unaltered after the averaging procedure. Only the shape of the radial dependence is changed. The role of the composite particle is the same as the role of the fourth particle (supposed to be an elementary particle). This kind of symmetry is lost after the averaging of the *l-s* interaction (34). The balance of the two terms obtained, Equ. (41), is strongly affected by the values of masses and spin states of particles forming the three-body composite particle.

The averaging procedure described here can easily be extended to more complicated types of interactions, such as the quadratic *l-s* interaction. The averaging of this interaction results in a number of terms including invariants (scalars) formed by the vectors $\vec{r}$, $\vec{p}$, $\vec{s}$ and $\vec{s}_0$, each multiplied by an appropriate radial function obtained by formula (28). We are not going to write down the result for the average potential of the quadratic *l-s* interaction. We only want to point out that the average interactions easily assume forms much more complex than the initial interparticle interactions involved, par-

ticularly in the case when higher-order derivatives are included. However, these more complex interactions are of less importance than the examples given here.

3. *Discussion*

It was shown in the preceding chapter that the averaging of the interaction between one of the particles forming the three-body system, and the fourth particle is simply reduced to the multiplication of that interaction by the function $\tilde{\Delta}_i(\vec{k})$, Equ. (16), if the momentum representation of the interactions is considered. The structure of the three-body system is incorporated in the three functions $\tilde{\Delta}_i(\vec{k})$, $i = 1, 2, 3$, each of them containing a single parameter α_i, Equ. (19), closely related to the radius root-mean-square value of the corresponding particle, Equ. (21). In this way the average interaction apparently corresponds to completely uncorrelated particles in the three-body system in spite of the fact that the groundstate wave function (1) represents the state of a correlated motion of the three particles. This is an interesting feature of the first-order interaction (9) with the three-body system. This feature can be expected to be also present in the interaction with the many--body system. The average interaction, therefore, gives some insight into the independent particle model for the many-body system. The average interaction involves three independent contributions to the total interaction, in

Table 2

$v(\vec{r})$	$V(\vec{r})$
δ-function $\delta(\vec{r})$	$\frac{1}{(4\pi\alpha^2)^{3/2}} e^{-(\frac{r}{2\alpha})^2}$
Gaussian $e^{-(\frac{r}{2a})^2}$	$(\frac{a^2}{\alpha^2+a^2})^{3/2} e^{-\frac{r^2}{4(\alpha^2+a^2)}}$
Yukawa $\frac{1}{r} e^{-\mu r}$	$\frac{1}{2r} e^{-(\frac{r}{2\alpha})^2} \{f(\mu\alpha - \frac{r}{2\alpha}) - f(\mu\alpha + \frac{r}{2\alpha})\}$
Coulomb $\frac{1}{r}$	$\frac{1}{r}\,\mathrm{erf}(\frac{r}{2\alpha})$

close analogy to the system of three independent particles moving in the potential well. By adjusting the parameters of three independent wave functions, we can obtain the average interaction equal to that obtained from the function (1). The lowest-order approximation of the interaction with a composite system cannot, therefore, be used to test the validity of the independent particle model of the system in spite of the fact that the three-body system, according to this model, contains nine internal degrees of freedom in the configuration space instead of only six degrees available. It is not surprising, therefore, that a theoretically, not strictly founded, independent particle description of a physical system should work in practice.

We derives formula (28), which reduces the problem of finding the average interaction to one-dimensional integration. In the case of spherically symmetric interactions this formula simplifies to expression (30). In the case of very well-known shapes for two-particle interactions, the integral in formula (30) can be resolved or expressed through the standard functions. We give several examples in Table 2.

Because of the smoothing effect of the averaging process, the singularities in the initial potentials disappear in the average potential. The behaviour of the averaged Yukawa potential at the origin is given by

$$V_Y(r) \cong \frac{1}{\alpha}\left[\frac{1}{\sqrt{\pi}} - \alpha\mu f(\mu\alpha) + \right.$$

$$\left. + \frac{1}{12}\left\{\frac{2\mu^2\alpha^2 - 1}{\sqrt{\pi}} - 2\alpha^3\mu^3 f(\mu\alpha)\right\}\frac{r^2}{\alpha^2}\right],$$

where

$$f(x) = e^{x^2}\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}e^{x^2}\int_x^\infty e^{-t^2}\,dt = e^{x^2}\{1 - \operatorname{erf}(x)\},$$

and the behaviour of the Coulomb interaction is

$$V_C(r) \cong \frac{1}{\sqrt{\pi}\,\alpha}\left(1 - \frac{1}{12}\frac{r^2}{\alpha^2}\right),$$

with the finite values $V(0)$.

The asymptotic behaviour of the average potential depends on the decrements of the initial potential and the wave function for $r \to \infty$, and it belongs to the function of smaller decrement. The cut-off type of potentials exhibits the asymptotic behaviour of the wave function (see the δ function). In this

case the potential has a stronger decrement than the wave function (squared). The exponential function has a smaller decrement than the Gaussian function so that the asymptotic behaviour belongs to the initial potential in the case of the Yukawa and the Coulomb potentials

$$V_{\mathrm{Y}} \simeq \frac{1}{r} \mathrm{e}^{-\mu(r-\alpha^2\mu)},$$

$$V_{\mathrm{C}} \simeq \frac{1}{r}.$$

The behaviour at $r \simeq 0$ of the average interaction in the general case of formula (28) can be obtained by the behaviour of the modified spherical Bessel function

$$y_\lambda(x) \simeq \frac{x^\lambda}{(2\lambda+1)!!}; \qquad x \simeq 0.$$

This gives

$$W_\lambda(r) \simeq A_\lambda r^\lambda$$

with

$$A_\lambda = \frac{1}{\sqrt{\pi}} \frac{1}{2^{\lambda+1}} \frac{1}{(2\lambda+1)!!} \frac{1}{\alpha^{2\lambda+3}} \int_0^\infty \mathrm{e}^{-\left(\frac{x}{2\alpha}\right)} w(x)\, x^{\lambda+2}\, \mathrm{d}x.$$

The consequence of this behaviour at the origin is that the average spin-orbit and tensor-force interactions vanish linearly and quadratically, respectively, at the origin (note that $\vec{l}/r$ is finite for $r \to 0$). Both interactions behave as surface interactions when composite particles are concerned. This property is in agreement with the well-known assumption that the radial dependence of the spin-orbit interaction is proportional to the derivative of the spherically symmetric volume interaction[3]. We note that this property is not assumed in the case of interactions between elementary particles[4].

Finally, it should be mentioned that particle 0, which interacts with the three-body system, is not necessarily an elementary particle similar to those constituting a bound three-body system. Particle 0 can be assumed to be a composite particle, such as a heavy nucleus. Then the initial three interactions v_i assume the meaning of optical potentials for three particles and the average interaction becomes the optical potential of the nucleus for the

three-body composite particle system, such as the H^3 and the He^3 nucleus. This interpretation is in close analogy to similar considerations concerning the deuteron optical potential[5].

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PROSJEČNO MEĐUDJELOVANJE TROČESTIČNOG SISTEMA

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Sadržaj

Razmatraju se svojstva prosječnog međudjelovanja čestice s tročestičnim sistemom. Usrednjavanje je definirano izrazima (9) i (11) u kojima nastupa valna funkcija tročestičnog sistema (1).

Provedena je redukcija šesterodimenzionalne integracije na jednodimenzionalnu, radijalnu integraciju s rezultatom prikazanim formulom (28).

Diskutirani su primjeri s interakcijom skalarnog, vektorskog i tenzorskog karaktera. Ukazuje se na osobine tenzorske sile i sile *l-s* vezanja koje nakon usrednjavanja pokazuju svojstva površinske sile.

Eksplicitni rezultati usrednjavanja dobiveni su za osnovne oblike skalarnih potencijala: δ funkcije, Gaussovog, Yukawinog i Coulombovog potencijala (Tablica 2).