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DECOMPOSITION OF A THREE-BODY WAVE FUNCTION

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Abstract: The decomposition of a three-body wave function into a series of separable functions has been considered in the case of nonseparable interactions between particles. A particular complete set of functions has been singled out for which the best convergence of the expansion series has been obtained. The effective two-body force strength in a three-body system has been determined by the condition of strongest convergence. It has been shown that the effective force strength differs essentially from the strength of the two-body system. The expansion series assumes a simplified and symmetric form if effective forces are used. The interpretation in terms of the independent particle shell model is given for the residual interaction and the finiteness of the mass of the nucleus.

1. Introduction

It has been shown[1] that the harmonic force of the isotropic harmonic oscillator offers the possibility of obtaining an exact analytical solution of the three-body problem in the most general case of three arbitrary masses and force intensities. This solution is equivalent to the quantum mechanical state of two independent isotropic oscillators involving variables of no direct physical meaning. These variables are represented by two vectors which are not vectors of two interparticle distances. Consequently a three-body wave function is not a simple combination of two-body wave functions. However, a three-body wave function can be interpreted as a linear superposition of independent two-body wave functions. The question remains to choose a suitable basis to obtain a rapidly convergent expansion series .In addition to the convergence problem, we meet the problem of expanding a three-body wave function into a series of two-body wave functions. The knowledge of such an expansion is equivalent to the knowledge of the solution of the three-body problem, and we can expect that the exact solution of the three-body problem not involving harmonic forces will be obtained analytically or numerically in the form of series expansion. It is desirable to study expansions of this kind in a particular case, in order to gain a general insight into the three-body problem. The knowledge of such an expansion in a particular case is important for application. Methods for numerical calculations with two-body wave functions are already well known, so the decomposition of a three-body wave function into a series of two-body wave functions is a suitable tool in dealing with three-body wave functions.

On the other hand, a three-body quantum mechanical system appears to be the simplest possible system that possesses a subsystem, i. e., a two-body system in our case. The expansion of a wave function over a complete set of subsystem wave functions is known to be a very useful formal procedure for handling complex quantum mechanical systems, since it leads to various approximate solutions of physical problems. Such an epxansion is the starting point for several well-known approximations: plane wave Born approximation, distorted wave Born approximation[2)] and coupled channel approximation[3)]. We should point out that formal expansions have been used but have not been written down explicitly. However the present paper is an attempt to give an example of the exact treatment of the whole problem.

2. *Parametrization of three-body wave functions*

We shall use the same parametrization of a three-body wave function as in Ref. 1). Accordingly, we may write the intrinsic ground state three-body wave function in the form

$$\psi(\vec{r}_1, \vec{r}_2) = N\, e^{-\frac{1}{2}(\beta_1 r_1^2 + \beta_2 r_2^2 + \beta_3 r_3^2)}, \tag{1}$$

where $\vec{r}_i$, $i = 1, 2, 3$ are vectors of three interparticle distances in cyclic notation, i. e., $\vec{r}_1 = \vec{r}_{23}$, $\vec{r}_2 = \vec{r}_{31}$ and $\vec{r}_3 = \vec{r}_{12}$. Obviously, the three vectors are linearly dependent because of the relation

$$\vec{r}_1 + \vec{r}_2 + \vec{r}_3 = 0. \tag{2}$$

The normalization constant is given by

$$N^2 = \pi^{-3}(\beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1)^{3/2} \equiv \frac{\Upsilon^{3/2}}{\pi^3}. \tag{3}$$

To insure the integrability of the wave function (1), we have to impose some restrictions on the parameters β_i. These are as follows: a) Only one of the three parameters β_i can be negative, b) $\gamma \equiv \beta_1 \beta_2 + \beta_2 \beta_3 + \beta_3 \beta_1 > 0$. It follows from these conditions that $\beta_i + \beta_j > 0$ for every i and j, $i \neq j$.

In our previous paper[1] we determined the parameters β_i in terms of frequencies ω_i appearing in the Hamiltonian. By contrast, here we state inverse relations for the determination of frequencies ω_i given by the parameters β_i

$$\frac{\mu_i \omega_i^2}{\hbar^2} = \frac{\beta_i^2}{\mu_i} - \frac{\beta_j \beta_k}{m_i} + \frac{\beta_k \beta_i}{m_j} + \frac{\beta_i \beta_j}{m_k} . \tag{4}$$

Here, i, j and k should be taken in a cyclic order of (1, 2, 3) and the reduced masses μ_i are designated by $1/\mu_i = 1/m_j + 1/m_k$. Expressed in terms of the parameters β_i, the energy eigenvalue is given by

$$E = \frac{3}{2} \hbar^2 \left(\frac{\beta_1}{\mu_1} + \frac{\beta_2}{\mu_2} + \frac{\beta_3}{\mu_3} \right) . \tag{5}$$

Since we have thus obtained the correspondence between the force intensities and the parameters β_i, we are able to write down the Hamiltonian for a three-body system described by the three-body wave function (1).

The parametrization of the wave function given by Equ. (1) is suitable for a three-body system in which one of the three parameters β_i is small. For example, $\beta_3 \to 0$ is equivalent to $m_3 \to \infty$, $\omega_3 \to 0$, which represents the motion of two independent particles in two central fields with the centre in the infinitely heavy particle (denoted by label 3). In this case two interactions are strong and one is weak. When one of the interactions is stronger than the other two, a new parametrization would be more convenient. This parametrization corresponds to the variables in the configuration space represented by two vectors $\vec{r}$ and $\vec{\rho}$. The symbol $\vec{r}$ is the vector of the relative distance between two strongly interacting particles, and $\vec{\rho}$ denotes the position of the third particle with respect to the centre of mass of the other two particles. We shall assume that the strongly interacting particles are labelled by 1 and 2. Then

$$\begin{aligned} \vec{r} &= \vec{r}_3 = -(\vec{r}_1 + \vec{r}_2), \\ \vec{\rho} &= \frac{\mu_3}{m_2} \vec{r}_2 - \frac{\mu_3}{m_1} \vec{r}_1 , \end{aligned} \tag{6}$$

and the wave function (1) takes the form

$$\psi(\vec{r}, \vec{\rho}) = N \, \mathrm{e}^{-\frac{1}{2} \{(b_1 + b_3) r^2 + (b_2 + b_3) \rho^2 + 2 b_3 \vec{r} \vec{\rho}\}} , \tag{7}$$

where the quantities b_i are completely analogous to the parameters β_i in expression (1). Interrelations between the two sets of parameters are

$$b_1 + b_3 = \frac{\mu_3^2}{m_2^2}\beta_1 + \frac{\mu_3^2}{m_1^2}\beta_2 + \beta_3,$$

$$b_2 + b_3 = \beta_1 + \beta_2, \tag{8}$$

$$b_3 = \frac{\mu_3}{m_2}\beta_1 - \frac{\mu_3}{m_1}\beta_2.$$

We readily obtain the relation $b_1 b_2 + b_2 b_3 + b_3 b_1 = \beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1$, so that the normalization constant is the same in both representations. The equality $d\vec{r}_1\, d\vec{r}_2 = d\vec{r}\, d\vec{\rho}$ is valid for the volume elements of the configuration space. Now we see that $b_3 = 0$ corresponds to a particular balance of the two forces and two masses labelled by 1 and 2. Generally, b_3 is small if the interactions involving the third particle are weak.

In the following considerations we shall use the first representation for three-body wave functions. However, the results will also be valid for the second representation if the parameters β_i are read as b_i and the variables $(\vec{r}_1, \vec{r}_2)$ as $(\vec{r}, \vec{\rho})$.

3. The two-body wave function

The natural choice for a complete set of two-body wave functions is obviously an orthonormal set of isotropic harmonic oscillator wave functions satisfying the two-body Schrödinger equation of relative motion

$$\left(\frac{p^2}{2\mu} + \frac{\mu\omega^2}{2} r^2 - E_{nl}\right)\phi_{nlm}(\vec{r}) = 0. \tag{9}$$

The solution has the form

$$\phi_{nlm}(\alpha, \vec{r}) = R_{nl}(\alpha, r)\, Y_l^m(\hat{r}), \tag{10}$$

where we have introduced the characteristic parameter $\alpha = \mu\omega/\hbar$. The radial wave function[4] is given by

$$R_{nl}(\alpha, r) = |\alpha|^{\frac{l}{2}+\frac{3}{4}} \left\{\frac{2\cdot n!}{\Gamma\left(n+l+\frac{3}{2}\right)}\right\}^{1/2} r^l e^{-\frac{\alpha r^2}{2}} L_n^{\left(l+\frac{1}{2}\right)}(\alpha r^2). \tag{11}$$

The functions Y_l^m and $L_n^{(\kappa)}$ are spherical harmonics and Laguerre polynomials, respectively, defined as in[5, 6]. The energy eigenvalues are

$$E_{nl} = \hbar\,\omega\,(2\,n + l + \frac{3}{2}). \tag{12}$$

We should point out that no particular choice of the parameter α appearing in the two-body wave function, has been made. This means that the two-body interaction strength remains unspecified. The two-body interaction is not necessarily taken to be equal to the original two-body interaction in the three-body Hamiltonian. Because of the other two interactions, an effective two-body interaction will be formed inside the three-body system. We can expect that this interaction will give a more suitable set of two-body wave functions in terms of which the expansion of a three-body wave function should be carried out if we wish to obtain a better convergence of the expansion series.

4. *Expansion*

The expression for the three-body wave function (1) over a complete set of two-body wave functions (10) has the form

$$\psi\,(\vec{r}_1, \vec{r}_2) = \sum_{nlm} f_{nlm}\,(\vec{r}_2)\,\phi_{nlm}\,(\alpha_1, \vec{r}_1). \tag{13}$$

Here, the functions

$$f_{nlm}\,(\vec{r}_2) = \int \overset{*}{\phi}_{nlm}\,(\alpha_1, \vec{r}_1)\,\psi\,(\vec{r}_1, \vec{r}_2)\,\mathrm{d}\,\vec{r}_1, \tag{14}$$

besides being depending on $\vec{r}_2$, also depend on the free parameter $\alpha_1 > 0$. Because of the normalization of the function ψ, the positive quantities

$$W_{nlm} = \int \left| f_{nlm}\,(\vec{r}_2) \right|^2 \mathrm{d}\,\vec{r}_2 \tag{15}$$

satisfy the summation rule

$$\sum_{nlm} W_{nlm} = 1\,, \tag{16}$$

in accordance with the probabilistic interpretation of the integral W_{nlm}. Here arises the question of the uniform convergence of the series (13) and the convergence of the series (16). The uniform convergence of the expansion

(13) is predominantly a mathematical problem, while the convergence of the series (16) may also be given a physical interpretation. Each term W_{nlm} represents the probability of finding the two-body configuration ϕ_{nlm} in the three-body state ψ.

We consider the convergence of the series (16), mainly its dependence on the parameters involved. First, we consider the dependence on the free parameter α_1. This parameter measures the strength of the two-body interaction, giving the bound states ϕ_{nlm}. Second, we study the dependence on the three-body parameters β_i.

We can obtain an explicit analytical expression for the functions $f_{nlm}(\vec{r}_2)$ by expanding the wave function ψ over spherical harmonics (the modified plane wave expansion with an imaginary wave number) and applying formula 7.421/4 of Ref. 5) for the radial integral over r_2. The result is

$$f_{nlm}(\vec{r}_2) = a_{nl}\, e^{-\frac{\delta}{2} r_2^2}\, \overset{*}{\phi}_{nlm}(\alpha_2, \vec{r}_2), \tag{17}$$

where $\phi_{nlm}(\alpha_2, \vec{r}_2)$ are functions defined by Equ. (9). We have introduced a new free parameter δ for the sake of convenience. We have the following expressions for the δ-dependent parameters entering the expansion (13)

$$\alpha_1 = (\beta_1 + \beta_3)\left(\frac{\delta_0 - \delta}{\delta_\infty - \delta}\right)^{1/2} \tag{18}$$

$$\alpha_2 = \alpha_1 \cdot \frac{\delta_\infty - \delta}{\beta_1 + \beta_3} = \pm\sqrt{(\delta_0 - \delta)(\delta_\infty - \delta)} \qquad \delta \left[\begin{matrix} < \delta_0 \\ > \delta_\infty \end{matrix}\right. \tag{19}$$

$$a_{nl} = (\text{sign}\,\alpha_2)^n\,(-\text{sign}\,\beta_3)^l \left(\frac{2\sqrt{\gamma}}{|\beta_3|}\right)^{\frac{3}{2}} t^{2n+l+\frac{3}{2}}, \tag{20}$$

$$t = \frac{\sqrt{\delta_\infty - \delta_0}}{\sqrt{|\delta_\infty - \delta|} + \sqrt{|\delta_0 - \delta|}}\,; \quad 0 < t < 1. \tag{21}$$

The two critical values of the parameter δ are

$$\delta_0 = \frac{\gamma}{\beta_1 + \beta_3}, \tag{22}$$

$$\delta_\infty = \beta_2 + \beta_3,$$

with the properties

$$\delta_\infty - \delta_o = \frac{\beta_3^2}{\beta_1 + \beta_3} > 0\,, \tag{23}$$

$$\alpha_1(\delta_o) = 0\,, \qquad \alpha_1(\delta_\infty) = \infty\,. \tag{24}$$

To insure the reality of the parameter α_1, ($\phi_{nlm}(\alpha_1, \vec{r}_2)$ is the bound state wave function) we should exclude the interval $[\delta_o, \delta_\infty]$ for the values of the parameters δ. In the case $\delta < \delta_0$, $\alpha_2 > 0$, the function $\phi_{nlm}(\alpha_2, \vec{r}_2)$ represents the bound state wave function, while in the case $\delta > \delta_\infty$, $\alpha_2 < 0$ it represents the unphysical solution of the Schrödinger equation (9) with negative frequency ω (or negative E_{nl}). In both cases we achieve the square integrability of the functions $f_{nlm}(\vec{r}_2)$, since the inequalities

$$\delta_0 < \alpha_2(\delta) + \delta < \delta_\infty\,, \quad \text{for} \quad \delta < \delta_0 \quad \text{or} \quad \delta > \delta_\infty \tag{25}$$

are satisfied. The condition (25) insures the decreasing exponential in the expression for $|f_{nlm}|^2$ and, consequently, the convergence of the integral over $d\vec{r}_2$. This integral, which represents the value W_{nlm}, Equ. (15), can be evaluated in closed form by applying the integration formula 7.414/4 of Ref. 5) and relation 22.543 of Ref. 6). The result is

$$W_{nlm} = |a_{nl}|^2 \int e^{-\delta r^2} |\Phi_{nlm}(\alpha_2, \vec{r})|^2\, d\vec{r} =$$

$$= W_{nl} = |a_{nl}|^2 \left(\frac{|\alpha_2|}{\alpha_2 + \delta}\right)^{2n + l + \frac{3}{2}} \left(\frac{\alpha_2^2 - \delta^2}{|\alpha_2|^2}\right)^n P_n^{\left(0, l + \frac{1}{2}\right)} \left(\frac{\alpha_2^2 + \delta^2}{\alpha_2^2 - \delta^2}\right), \tag{26}$$

where $P_n^{(\alpha,\beta)}(x)$ represents the Jacobi polynomial defined as in Refs. 5, 6). The two critical values δ_0 and δ_∞ of the parameter δ belong to the two extreme values 0 and ∞ of the parameter α_1, eqs. (24). For these values of δ the expansion (13) becomes meaningless. Then we also have $\alpha_2 = 0$ and $t = 1$. We can single out the particular value of δ, i. e., $\delta = 0$. For this value the expansion (13) assumes the form

$$\psi(\vec{r}_1, \vec{r}_2) = \sum_{nlm} a_{nl}^{(0)}\, \phi^*_{nlm}(\alpha_2^{(0)}, \vec{r}_2)\, \phi_{nlm}(\alpha_1^{(0)}, \vec{r}_1). \tag{27}$$

Here, only the bound states of the two-body wave functions are present. The coefficients $a_{nl}^{(0)}$, parameters $\alpha_{1,2}^{(0)}$ and probabilities $W_{nl}^{(0)}$ are now equal to

$$a_{nl}^{(0)} = (-\operatorname{sign}\beta_3)^l \left(\frac{2\sqrt{\gamma}}{|\beta_3|}\right)^{\frac{3}{2}} t_0^{2n+l+\frac{3}{2}} = \tag{28}$$
$$= (-\operatorname{sign}\beta_3)^l \left(\frac{1}{t_0} - t_0\right)^{\frac{3}{2}} t_0^{2n+l+\frac{3}{2}},$$

where

$$t_0 = \frac{1}{|\beta_3|}\{\sqrt{\gamma+\beta_3^2} - \sqrt{\gamma}\}, \quad 0 < t_0 < 1, \tag{29}$$

$$\alpha_1^{(0)} = \left(\frac{\beta_1+\beta_3}{\beta_2+\beta_3}\gamma\right)^{\frac{1}{2}},$$
$$\alpha_2^{(0)} = \left(\frac{\beta_2+\beta_3}{\beta_1+\beta_3}\gamma\right)^{\frac{1}{2}}, \tag{30}$$

$$W_{nl}^{(0)} = (a_{nl}^{(0)})^2. \tag{31}$$

We may write two interesting relations for the parameters $\alpha_{1,2}^{(0)}$

$$\alpha_1^{(0)} \cdot \alpha_2^{(0)} = \gamma, \quad \frac{\alpha_1^{(0)}}{\alpha_2^{(0)}} = \frac{\langle r_2^2\rangle_{III}}{\langle r_1^2\rangle_{III}}, \tag{32}$$

where the mean-square values $\langle r_1^2\rangle_{III}$ in the three-body state (1) are defined in Ref. 1). The mean-square value $\langle r^2\rangle_{II}$ in the two-body state (10) is $\langle r^2\rangle_{II} = (2n+l+\frac{3}{2})/\alpha$, so that by relation (32) we state that the ratio $\langle r_2^2\rangle_{II}/\langle r_1^2\rangle_{II}$ is constant and equal to the same ratio for a three-body state, for each term of the expansion (27). We may take this fact as a hint that the expansion (27), i. e., the case $\delta = 0$, must play a particular role in the expansion of the type (13), especially in the discussion of convergence. This will prove true in the following considerations.

5. Convergence

In the preceding section we obtained the expansion of a three-body state over a complete set of two-body states with the free parameter δ chosen outside the interval $[\delta_0, \delta_\infty]$. The variation of the parameter δ leads to the variation of the parameter α_1 and, consequently, to the variation of the strength of the two-body force between particles 2 and 3. We expect by physical intuition that a certain strength of the two-body force will be particularly favourable to the expansion (13) and will produce the best convergence of the series. To find the corresponding value of the parameter δ, we shall examine the convergence of the probability series (16) in its dependence on the parameter δ.

Let us first assess the upper limit of the positive quantity W_{nl}. Obviously, for $\delta > 0$ and $\alpha_2 > 0$ (i. e., $\delta < \delta_0$) the integral

$$I_{nlm}(\delta, \alpha) = \int e^{-\delta r^2} |\Phi_{nlm}(\alpha, \vec{r})|^2 d\vec{r} \tag{33}$$

satisfies the inequality

$$I_{nlm}(\delta, \alpha) \leq 1\,, \qquad \delta > 0\,. \tag{34}$$

To extend this inequality to the region $\delta < 0$, we write

$$I_{nlm}(\delta, \alpha) = \left(\frac{|\alpha|}{\alpha + \delta}\right)^{2n+l+\frac{3}{2}} G\left(\frac{\delta}{\alpha}\right), \tag{35}$$

where $G(x)$ is an even function of x defined by Equ. (26). It follows from Equ. (34) that

$$G\left(\frac{\delta}{\alpha}\right) \leq \left(\frac{\alpha + \delta}{|\alpha|}\right)^{2n+l+\frac{3}{2}}, \ \delta \geq 0,\ \alpha > 0.$$

Replacing δ by $-\delta$ with $\delta < 0$, we obtain

$$G\left(-\frac{\delta}{\alpha}\right) = G\left(\frac{\delta}{\alpha}\right) \leq \left(\frac{\alpha - \delta}{|\alpha|}\right)^{2n+l+\frac{3}{2}}, \ \delta \leq 0, \quad \alpha > 0,$$

$$G\left(\frac{\delta}{\alpha}\right) \leq \left(\frac{\alpha + |\delta|}{|\alpha|}\right)^{2n+l+\frac{3}{2}}, \quad \alpha > 0,$$

where δ is positive or negative. Replacing α by $-\alpha$ with $\alpha < 0$, we obtain

$$G\left(\frac{\delta}{\alpha}\right) \leq \left(\frac{|\alpha|+|\delta|}{|\alpha|}\right)^{2n+l+\frac{3}{2}}$$

or

$$I_{nlm}(\delta, \alpha) \leq \left(\frac{|\alpha|+|\delta|}{\alpha+\delta}\right)^{2n+l+\frac{3}{2}} \tag{36}$$

and finally

$$W_{nl} \leq \left(\frac{1}{t_0} - t_0\right)^3 \left\{t^2 \frac{|\alpha_2|+|\delta|}{\alpha_2+\delta}\right\}^{2n+l+\frac{3}{2}} \tag{37}$$

where the equality is valid for $\delta = 0$.

The right-hand side of Equ. (37) can considerably differ from the left-hand side, particularly for $\delta \to \delta_0$ (or δ_∞) when $\alpha_2 \to 0$ and $t \to 1$. In that case $W_{nl} \to 0$, while the right-hand side remains > 0. However, what counts in convergence considerations is the quotient of two consecutive terms and not their magnitude. In our case the quantity

$$q(\delta) = t^2(\delta) \frac{|\alpha_2(\delta)|+|\delta|}{\alpha_2(\delta)+\delta} \tag{38}$$

is responsible for the convergence of the series (16). The behaviour of this series is approximately the same as that of the geometrical series.

The function $q(\delta)$ is a smooth function in the intervals $(-\infty, 0)$, $(0, \delta_0)$ and (δ_∞, ∞). The values at the ends of these intervals are given in Table 1.

The relation between q_0 and q_∞ is

$$2q_0 = (1+q_0^2)\, q_\infty \simeq q_\infty . \tag{39}$$

Here, the approximation is valid, since both quantities are small ($q_\infty \leq \frac{1}{7}$; equality for $\beta_1 = \beta_2 = \beta_3$; otherwise β_3 is taken to be the least of β_1, β_2 and β_3).

From the qualitative display for the quotient q as a function of δ we conclude that the minimal value of q is $q_0 = q(0)$ obtained for the value $\delta = 0$.

Table 1

δ	$q(\delta)$	$q'(\delta)$	$q''(\delta)$
$-\infty$	$q_\infty = \frac{\beta_3^2}{2\gamma + \beta_3^2}$		
		< 0	< 0
0	$q_0 = t_0^2 \simeq \frac{q_\infty}{2}$		
		> 0	> 0
δ_0	1		
		forbidden region	
δ_∞	1		
		< 0	> 0
$+\infty$	$\frac{\beta_3^2}{2\gamma + \beta_3^2}$		

Consequently, for this value of δ we obtain the best convergence of the series (16) and also of the expansion (13). In that case the series (16) behaves as a geometric series with the quotient $q_0 \lesssim \frac{1}{14}$, with the power index $2n + l$ denoting the energy levels of the two-body states over which the expansions have been carried out. In the case $\delta \neq 0$ such a simple picture no longer exists, since the dependence on n and l becomes complicated. The best way of getting a certain feeling of what happens in the case $\delta \neq 0$ is to examine the neighbourhood of the point $\delta = 0$ and expand the function $W_{nl}(\delta)$ in power series at the point $\delta = 0$. It can be shown that the function $W_{nl}(\delta)$ has no linear term in δ and therefore it reaches the extreme value at the point $\delta = 0$. It has a maximum for $n = 0$ and a minimum for $n \geq 1$. The larger n, the more pronounced the minimum.

After discussing the question of convergence, we should say a few words about the β dependence. Since $\delta = 0$ is the best choice for the parameter δ, there is no need to consider the β dependence in the case $\delta \neq 0$. The coefficients $a_{nl}^{(0)}$ of the expansion (27) in dependence on n, l are simply powers of the quantity $t_0 > 0$. As $1 - t_0^2 = \frac{2\sqrt{\gamma}}{|\beta_3|} t_0$, it follows $t_0 < 1$, and the convergence is insured. Actually, $t_0 \leq 2 - \sqrt{3} \simeq 0.268$, the equality being obtained for the least favourable case when $\beta_1 = \beta_2 = \beta_3$. If β_3 is small with respect to β_1 and β_2, we approximately have $t_0 \simeq \frac{|\beta_3|}{2\sqrt{\gamma}} << 1$, and we find that the expansion series (27) converges very rapidly thus reducing to a single term in the limit $\beta_3 = 0$. This term is just the function (1) for $\beta_3 = 0$.

6. Conclusion

From our considerations we can draw general conclusions on the expansion of quantum mechanical state wave functions over a complete set of substate wave functions. A suitable choice of substate wave functions will not only simplify the explicit form of expansion »coefficients« (functions $f_{nlm}(\vec{r}_2)$ in our considerations) but will also improve the convergence of the series. The strength of the interaction in a subsystem plays the most important role in the choice of subsystem wave functions. There is a value of the interaction strength with stationary properties that yields the best convergence of the series. It should be pointed out that the strength thus obtained is not equal to the strength of the force in a free subsystem. The effective force can be defined as a force the subsystem feels while incorporated in the whole system. In our example, in which the two-body system is a subsystem of the three-body system, the addition of a third particle to the two-body system results in a stronger two-body effective interaction.

Table 2

Quantity \ Case	I	II	III
δ	0	1.2	0
α_1^2	3	1.5	$\sqrt{2}$
α_2^2	3	0.24	$\sqrt{2}$
W_{00}	0.799704	0.764827	0.97775797
W_{01}	0.057416	0.053304	0.00730353
W_{02}	0.004122	0.003715	0.00005455
W_{10}	0.004122	0.037149	0.00005455
W_{03}	0.000296	0.000259	0.00000041
W_{11}	0.000296	0.004143	0.00000041
$\sum_{nl}^{2n+l=3} (2l+1) W_{nl}$	0.999644	0.994705	0.99999995

For example, let $m_1 = m_2 = m_3$ and $\beta_1 = \beta_2 = \beta_3 = 1$; then from eq. (4) it follows $\alpha^2 = \frac{3}{2}$, while the effective value obtained from eqs. (30) is $\alpha^2_{\text{eff}} = 3$. As the force strength is proportional to α^2, we find that the effective force is twice as strong as the free force. For illustration, in Table 2 we give several W_{nl} values (for the first three energy levels) for the effective case and the free case of our simple example (cases II and III, respectively). Table 2 shows the effect of the change of the parameter δ on the sequence of probabilities.

In addition to the already mentioned particular properties of the series (27), we may also emphasize its symmetrical appearance. The set of relevant formulas (27), (28), (29) and (30) transforms into itself by an interchange of labels 1 and 2. (Note that the position of the asterisk in formula (27) is irrelevant, because the sum over m is real). The expansion can be understood as an expansion over subsystem 2 as well as an expansion over subsystem 1. In fact, the expansion represents the decomposition of the wave function of the whole system in terms of two-particle states, each state being coupled into the total angular momentum $L = 0$. The correspondence with the nuclear shell model states is complete if we take $m_3 \rightarrow \infty$ (the mass of the heavy nucleus). However, $\beta_3 \neq 0$ indicates that the two particles 1 and 2 interact with each other through the so-called residual interaction .The considered ground-state wave function can only represent two innermost particles immersed in nuclear matter. In the simple shell-model picture these two particles are treated as noninteracting particles with the wave functions $\phi_{000}(\vec{r}_1)$ and $\phi_{000}(\vec{r}_2)$, respectively. The interaction between the particles gives rise to higher components with probabilities W_{nl}. The presence of higher components is also caused by the finiteness of the mass m_3. To be more quantitative, we write down the relevant expressions.

For

$$m_1 = m_2 = m\,, \qquad \omega_1 = \omega_2 = \omega\,, \tag{40}$$

we have

$$\beta_1 = \beta_2 = \beta = \frac{m\,\omega}{\hbar}\,\{(1 + \frac{m}{m_3})\,(1 + 2\,\frac{m}{m_3})\}^{-1/2}\,, \qquad \beta_3 = \frac{\beta}{2}\,(\lambda - 1)\,, \tag{41}$$

where

$$\lambda = \{(1 + 2\,\frac{m}{m_3})\,[1 + (1 + \frac{m}{m_3})\,\frac{\omega_3^2}{\omega^2}]\}^{1/2} = \frac{\gamma}{\beta^2}\,.$$

Hence we derive

$$t = \frac{\sqrt{\lambda} - 1}{\sqrt{\lambda} + 1}\,. \tag{42}$$

For the already quoted example (case I, Table 2) we have $\lambda = 3$ and $t_0 = 0.26795$, which corresponds to the conditions $m = m_3$ and $\omega = \omega_3$. If we take $m_3 = \infty$ and $\omega = \omega_3$, we obtain $\lambda = \sqrt{2}$ and $t_0 = 0.086427$, which yields a considerably better convergence than the previous example, as shown in Table 2 (case III). The same result is obtained by taking $\omega_3 = 0$ and $m_3 = 2m$. These results can be put in the following words: In the decomposition of a three-body state into a series of two independent two-body states the first term, or the ground state, enters with a probability not smaller than 80 %, leaving 20 % to all excited states. These values are achieved in the least favourable case of three equal masses and three equal frequencies. If one of the particles is a heavy particle with three frequencies of comparable magnitude, the probability for the ground state is 98 % with only a 2 % probability for higher states. The same result is obtained in the case of three comparable masses with the vanishing interaction between the two particles. If neither all the three masses nor all the three frequencies are of comparable size, i. e., in the case $m_3 >> m$ and $\omega_3 >> \omega_2$, we have a rapidly convergent series with $t_0 \simeq \frac{1}{4}\left(\frac{m}{m_3} + \frac{1}{2}\frac{\omega_3^2}{\omega^2}\right)$. This gives a quadratically small quantity in the ratios $\frac{m}{m_3}$ and $\frac{\omega_3^2}{\omega^2}$ for the probability quotient of two consecutive levels.

Finally we note that in the symmetric case considered under the conditions (40), the wave function (1) assumes a separable form in the representation (7), since the relation (8) yield $b_1 = \frac{\beta\lambda}{2}$, $b_2 = 2\beta$ and $b_3 = 0$. The exact zero value for the parameter b_3 causes the reduction of the expansion series to a single term in the $\vec{r}, \vec{\rho}$ representation. This is a very particular property of the two-body force used in our considerations and it cannot be applied with major significance to a general case. It can be expected that the convergence in the $\vec{r}, \vec{\rho}$ representation of the expansion series will be good if the conditions analogous to those in (40) are met in the case of the general force law.

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DEKOMPOZICIJA TROČESTIČNE VALNE FUNKCIJE

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Sadržaj

Razmatra se rastav tročestične valne funkcije (1) u obliku reda separabilnih funkcija (13) na primjeru neseparabilnog međudjelovanja među česticama. Izdvojen je takav kompletni skup funkcija s kojim se postizava najbolja konvergencija rada. Uslov najbolje konvergencije određuje intenzitet efektivnog dvočestičnog međudjelovanja za koji se pokazuje da se bitno razlikuje od intenziteta međudjelovanja u slobodnom dvočestičnom sistemu (tablica 2).

Upotrebom efektivnog međudjelovanja postizava se pojednostavnjen i simetričan oblik razvoja (27) koji u tom obliku dopušta interpretaciju u okviru modela nezavisnih čestica za efekte konačnosti mase jezgre te rezidualnog međudjelovanja. Oba se efekta manifstiraju u višim članovima razvoja (27) pri čemu vodećem članu pripada dominantna uloga.