FIZIKA, 4 (1972) 1-11

COULOMB ENERGY OF ^{3}He DEDUCED BY SYMMETRY RELATIONS FROM THE CHARGE FORM FACTORS*

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Received 25 June 1971

Abstract: In the frame work of the hyperspherical formalism, a relation, independant of the shape of the trinucleons wave function, giving the Coulomb energy of ^{3}He in terms of the charge form factors of the nucleon ^{3}He and Tritium, has been obtained. The utilisation of the experimental data leads to an energy of about 0.67 MeV, smaller than the 0.76 MeV experimental value. This result may be interpreted as a proof of a small charge assymetry of the nuclear forces.

1. Introduction

The energy difference between the ground states of ^{3}H and ^{3}He is one of the most precise tests of the charge symmetry of nuclear forces, it is for this reason that the calculation of the Coulomb energy of these mirror nuclei is so important.

Until now the method used to evaluate this energy, was to fit the form factors with a trial wave function, taking or not into account the existence of a hard core in nuclear forces, and to calculate with this function the average of the Coulomb energy[1].

This method is sensitive to the choice of the trial function consequently bearing an uncertainty on the Coulomb energy. The aim of this work is to show that it is possible to deduce the Coulomb energy from the knowledge of the ^{3}H and ^{3}He charge form factors avoiding the intermediate step of defining a wave function for the ground state of the 3 nucleon system.

* An abstract of this paper has been published in »The Three Body Problem« (North-Holland Publishing Co., 1970)

2. *Theoretical development using the method of hyperspherical functions*

The tri nucleons form factor is given by

$$\frac{3+2T_z}{2} F_{T_z}(k) = \langle \psi_{T_z} | \sum_{i=1}^{3} \left\{ G_{ES}(k) + G_{EV}(k)\, \tau_z(i) \right\} e^{i\vec{k}(\vec{x}_i - \vec{X})} | \psi_{T_z} \rangle , \tag{1}$$

where the $\vec{x}_i$ are the coordinates of the nucleons, $\vec{X}$ the centre of mass and $\tau_z(i)$ the isospin operator acting on the nucleon i. ψ_{T_z} is the trinucleous ground state wave function including the spin-isospin variables which takes the form

$$\psi_{T_z} = A_{T_z} \Phi_s + A'_{T_z} \Phi_+ + S'_{T_z} \Phi_- + \text{non central contributions.}$$

A, A', and S' being respectively the completely symmetrical and mixed symmetrical spin-isospin function for spin and isospin $S = T = \frac{1}{2}$.

The relations

$$G_{ES}(k) = \frac{1}{2} [F_p(k) + F_n(k)] \quad \text{and}$$

$$G_{EV}(k) = \frac{1}{2} [F_p(k) - F_n(k)]$$

are the scalar and vector nucleon charge form factors given in terms of the proton and neutron charge form factors F_p and F_n and T_z the third component of the isospin of the 3 nucleon system ($T_z = 1/2$ for ^{3}He).

The nuclear matrix element can be separated in two terms[2], the first, independent of the isospin, is the Fourier transform of the nucleons distribution

$$\langle \psi | \frac{1}{3} \sum_{i=1}^{3} e^{i\vec{k}(\vec{x}_i - \vec{X})} | \psi \rangle = \frac{2F_{^3\mathrm{He}}(k) + F_{^3\mathrm{H}}(k)}{6\, G_{ES}(k)} ; \tag{2}$$

the second which is responsible for the difference between the charge form factors $F_{^3\mathrm{H}}$ and $F_{^3\mathrm{He}}$ of Tritium and ^{3}He consists of the cross term of the mixed symmetry states and the completely symmetric state

$$\langle \Psi_{T_z} | \left(\tau_+ \sum_+ + \tau_- \sum_- \right) e^{i\vec{k}(\vec{x}_i - \vec{X})} | \Psi_{T_z} \rangle =$$

$$= 4\, T_z \left\{ \langle \Phi_+ | \sum_+ e^{i\vec{k}(\vec{x}_i - \vec{X})} | \Phi_s \rangle + \langle \Phi_- | \sum_- e^{i\vec{k}(\vec{x}_i - \vec{X})} | \Phi_s \rangle \right\},$$

where the operators are defined as follows

$$\tau_+ = \tau_z(3) - \frac{1}{2}[\tau_z(1) + \tau_z(2)],$$

$$\tau_- = \frac{\sqrt{3}}{2}[\tau_z(1) - \tau_z(2)],$$

$$\sum_+ f(x_i) = \frac{1}{3}[2 f(x_3) - f(x_1) - f(x_2)],$$

$$\sum_- f(x_i) = \frac{1}{\sqrt{3}}[f(x_1) - f(x_2)].$$

φ_+ and φ_- being the symmetric and antisymmetric part in the x_1, x_2 exchange of the mixed symmetry state and φ_S the completely symmetrical wave function. From the difference between the ^{3}H and ^{3}He charge form factors one obtains the nuclear matrix element in terms of $F_{^3H}$, $F_{^3He}$, G_{ES} and G_{EV}

$$\langle \Phi_+ | \sum_+ e^{i\vec{k}(\vec{x}_i - \vec{X})} | \Phi_s \rangle + \langle \Phi_- | \sum_- e^{i\vec{k}(\vec{x}_i - \vec{X})} | \Phi_s \rangle = \tag{3}$$

$$= \frac{1}{3\, G_{EV}(k)} \left\{ F_{^3H}(k) - F_{^3He}(k) - \frac{1}{4} \frac{G_{ES}(k) - G_{EV}(k)}{G_{ES}} \left(2 F_{^3He}(k) + F_{^3H}(k) \right) \right\}.$$

Assuming that the difference between the ground state wave function of ^{3}H and ^{3}He is negligible (charge independence of nuclear forces) it is possible to obtain the Coulomb energy just by taking the average over the Coulomb interaction

$$\langle \psi_{T_z} | \sum_{i,\, j>i} V_c(\vec{x}_i - \vec{x}_j) | \psi_{T_z} \rangle .$$

The Coulomb interaction between nucleons depends on the charge distribution around the nucleons and is given in terms of the nucleon form factors by[3]:

$$V_c(\vec{x}_i - \vec{x}_j) = \frac{e^2}{2\pi^2} \int [G_{ES}(q) + G_{EV}(q)\, \tau_z(i)]\, [G_{ES}(q) +$$

$$+ G_{EV}(q)\, \tau_z(j)] \cdot e^{i\vec{q}\cdot(\vec{x}_i - \vec{x}_j)} \frac{d^3 q}{q^2} .$$

Taking into account the relations

$$\tau_z(i)\, \tau_z(j)\, \tau_z(k) = -2\, T_z,$$

$$\tau_z(i) + \tau_z(j) + \tau_z(k) = 2\, T_z,$$

the energy difference between the ^3He and ^3H ground states takes the form:

$$E_c = \frac{e^2}{2\pi^2} \int 4\, G_{ES}(q)\, G_{EV}(q) \langle \psi_{\frac{1}{2}} | \sum_c \frac{1 - \tau_z(k)}{2} e^{i\vec{q}\cdot(\vec{x}_i - \vec{x}_j)} | \psi_{\frac{1}{2}} \rangle \frac{d^3 q}{q^2} .$$

The summation Σ being taken over the cyclic permutations of 1, 2, 3 with i, j and k different.

A procedure similar to the one used for the calculation of the form factors gives the nuclear matrix element

$$\langle \psi_{\frac{1}{2}} | \sum_c \frac{1 - \tau_z(k)}{2} e^{i\vec{q}\cdot(\vec{x}_i - \vec{x}_j)} | \psi_{\frac{1}{2}} \rangle = \frac{1}{3} \langle \psi | \sum_c e^{i\vec{q}\cdot(\vec{x}_i - \vec{x}_j)} | \psi \rangle +$$

$$+ \langle \Phi_+ | \sum_+ e^{i\vec{q}\cdot(\vec{x}_i - \vec{x}_j)} | \Phi_s \rangle + \langle \Phi_- | \sum_- e^{i\vec{q}\cdot(\vec{x}_i - \vec{x}_j)} | \Phi_s \rangle , \tag{5}$$

where

$$\sum_+ f(\vec{x}_i - \vec{x}_j) = \frac{1}{3} [2 f(\vec{x}_1 - \vec{x}_2) - f(\vec{x}_2 - \vec{x}_3) - f(\vec{x}_3 - \vec{x}_1)],$$

$$\sum f(\vec{x}_i - \vec{x}_j) = \frac{1}{\sqrt{3}} [f(\vec{x}_3 - \vec{x}_1) - f(\vec{x}_1 - \vec{x}_2)].$$

This expression contains matrix elements similar to Equs. (2) and (3) when $e^{i\vec{k}\cdot(\vec{x}_i - \vec{X})}$ is replaced by $e^{i\vec{q}\cdot(\vec{x}_i - \vec{x}_j)}$. The problem now reduces to find a relation enabling us to deduce the matrix elements contained in (5) and from those calculated in (2) and (3). The key to the solution of this problem lies in the utilization of the »hyperspherical formalism«[4, 5].

This method consists of deriving the fundamental equations from the expansion of the space functions into the complete set of the angular eigen functions at the surface of a hypersphere whose dimension is the number of the free variables but one. The linear coordinates of the particles after the elimination of the mass center are the free variables of a 3 (A — 1) dimensional space. They are transformed into polar coordinates in this 3 (A — 1) dimensional space and the partial differential many-body Shrödinger equation is reduced by this procedure to an infinite set of one dimensional coupled differential equations.

For a trinucleon system one eliminates the mass center by the standard change of variables

$$\vec{x}_1 - \vec{x}_2 = \vec{\xi},$$

$$\vec{x}_3 - \vec{X} = \frac{\vec{\zeta}}{\sqrt{3}},$$

$$\vec{X} = \frac{1}{3}(\vec{x}_1 + \vec{x}_2 + \vec{x}_3).$$

The polar coordinates for the six dimensional space of the free variables $\vec{\xi}, \vec{\zeta}$ are:

— the angular coordinates ω_ξ and ω_ζ of the vector $\vec{\xi}$ and $\vec{\zeta}$,

— the angle φ related to the length of $\vec{\xi}$ and $\vec{\zeta}$ by $\mathrm{tg}\,\varphi = \frac{\zeta}{\xi}$,

— the length $r = \sqrt{\xi^2 + \zeta^2}$.

The introduction of a φ angular parameter dependant kinematic rotation vector[6] $\vec{z}(\varphi) = \vec{\xi}\cos\varphi + \vec{\zeta}\sin\varphi$ enables one to express the positions of the particles in terms of $\vec{z}(\varphi)$ by giving particular values to φ

$$\vec{z}(0) = \vec{x}_1 - \vec{x}_2, \qquad \vec{z}\left(\frac{\pi}{2} - \frac{2\pi}{3}\right) = \sqrt{3}\,(\vec{x}_1 - \vec{X}),$$

$$\vec{z}\left(-\frac{2\pi}{3}\right) = \vec{x}_2 - \vec{x}_3, \qquad \vec{z}\left(\frac{\pi}{2} + \frac{2\pi}{3}\right) = \sqrt{3}\,(\vec{x}_2 - \vec{X}).$$

$$\vec{z}\left(\frac{2\pi}{3}\right) = \vec{x}_3 - \vec{x}_1, \qquad \vec{z}\left(\frac{\pi}{2}\right) = \sqrt{3}\,(\vec{x}_3 - \vec{X}), \tag{6}$$

A $\frac{\pi}{2}$ rotation of the φ parameter transform $\vec{x}_i - \vec{x}_j$ into $\sqrt{3}\,(\vec{x}_k - \vec{X})$ where $k \neq i, j$. The expansion of either $\sum_c e^{i\vec{k}(\vec{x}_i - \vec{x}_j)}$ or $\sum_c e^{i\vec{k}\sqrt{3}(\vec{x}_k - \vec{X})}$ into

the angular eigen functions is obtained in starting from the one of $e^{i\vec{k}\cdot\vec{z}(\varphi)}$. Especially the part of this expansion which has an angular momentum zero is given by

$$\frac{1}{4\pi}\int e^{i\vec{k}\cdot\vec{z}(\varphi)}\, d\omega_k = \sum_{K=0}^{+\infty} (-1)^K a'_K(\varphi)\, {}^{(2)}P_{2K}(\Omega,\varphi)\, \frac{J_{2K+2}(kr)}{(kr)^2}, \tag{7}$$

where the ${}^{(2)}P_{2k}(\Omega,\varphi)$ constitute a set of orthognale angular eigen functions whose parity is $(-1)^K$ when φ is changed in $\varphi \pm \frac{\pi}{2}$. Ω stands for the angles ω_ξ, ω_ζ, Φ and J_ν is a Bessel function.

In applying the symmetrization operator $\sum_0$ introduced in Ref.[5] to this expansion, one defines a new set of symmetrized angular functions ${}^{(2)}P_{2k}^{(0)}(\Omega,\varphi)$ which are invariant for any exchange of the position of the particles

$$\sum_0 \frac{1}{4\pi}\int e^{i\vec{k}\cdot\vec{z}(\varphi)}\, d\omega_k = \sum_{K=0}^{+\infty} (-1)^K a_K(\varphi)\, {}^{(2)}P_{2K}^{(0)}(\Omega,\varphi)\, \frac{J_{2K+2}(kr)}{(kr)^2}$$

The $a_k(0)$ are the following constants[7]:

$$a_k^2(0) = \begin{cases} \frac{1}{3}(K+1)(K-1) & K = 3n+1, \\ \frac{1}{3}(K+1)(K+1) & K = 3n+2, \\ \frac{1}{3}(K+1)(K+3) & K = 3n. \end{cases}$$

The ${}^{(2)}P_{2k}^{(0)}(\Omega, \varphi)$ have, also the parity $(-1)^K$ for a change of φ into $\varphi \pm \frac{\pi}{2}$, there is no ${}^{(2)}P_2^{(0)}$ function because the symmetrization operation cancels the contribution for which $K = 1$. The symmetrical wave function has to be expanded on the ${}^{(2)}P_{2k}^{(0)}(\Omega, 0)$ basis[5, 8]

$$\Phi_s = \sum_{K=0}^{+\infty} {}^{(2)}P_{2K}^{(0)}(\Omega, 0)\, \Phi_{2K}(r), \tag{8}$$

and the numerical calculations[7, 8] have shown that the two first terms ($K = 0,2$) contribute more than 99 % to the symmetrical wave function, the contribution for which $K = 2$ alone being less than 1 %.

The parameter dependent matrix element

$$\langle \psi \,|\sum_{0} e^{\,i\vec{k}\cdot\vec{z}(\varphi)}\,|\psi\rangle = \langle \psi | \sum_{0} \frac{1}{4\pi} \int e^{i\vec{k}\cdot\vec{z}(\varphi)}\, d\omega_h \,|\psi\rangle \tag{9}$$

(for a total angular momentum $e = 0$) does not contain any cross term between the completely symmetric state Φ_s and the mixed symmetry or noncentral terms of the wave function. In cutting the expansion $\sum_{0} e^{i\vec{k}\cdot\vec{z}(\varphi)}$ to its third term one ignores in the calculation of the matrix element (9) a part of the wave function whose weight is less than 1 % of the total contribution only.

For $\varphi = 0$ the matrix element (9) is the correlation form factor

$$J_s(k) = \langle \psi \,|\, \frac{1}{3} \sum_{i,j>i} e^{i\vec{q}\cdot(\vec{x}_i - \vec{x}_j)} \,|\, \psi \rangle.$$

For $\varphi = \frac{\pi}{2}$ it is the nuclear form factor related to the part of the wave function which is completely symmetrical for any exchange of the position of the particles

$$F_s(\sqrt{3}\ k) = \langle \psi \left| \frac{1}{3} \sum_{i=1}^{3} e^{i\vec{k}\sqrt{3}(\vec{x}_i - \vec{X})} \right| \psi \rangle .$$

When φ is rotated by $\frac{\pi}{2}$ the ${}^{(2)}P_{2K}^{(0)}(\Omega, \varphi)$ angular function of the expansion basis is transformed into $(-1)^K\, {}^{(2)}P_{2K}^{(0)}(\Omega, \varphi)$. Therefore the two first terms of the expansion of $\int e^{i\vec{k}\cdot\vec{z}(\varphi)}\, d\omega_k$, for which $K = 0,2$ are invariant for a $\frac{\pi}{2}$ rotation of φ.

This proves that the accuracy of the relation (9)

$$f_s(k) = F_S(\sqrt{3}\, k) \tag{10}$$

is very good, because it ignores less than 1 % of the wave function. In the same way as it has been done for the completely symmetrical state, the mixed symmetry wave function has to be constructed on the angular basis ${}^{(2)}P_{2K}^{\pm}(\Omega, 0)$ obtained by applying to (7) the $\sum_{+}$ or $\sum_{-}$ operator[5]

$$\frac{1}{4\pi} \sum_{(\pm)} \int e^{i\vec{k}\cdot\vec{z}(\varphi)}\, d\omega_k = \sum_{K=1}^{+\infty} (-1)^k\, a_{2k}^{(\pm)}(\varphi)\, {}^{(2)}P_{2k}^{(\pm)}(\Omega, \varphi)\, \frac{J_{2K+2}(kr)}{(kr)^2} .$$

It takes the form

$$A'\,\Phi_+ + S'\,\Phi_- = \sum_{K=1}^{+\infty} \{A'\,{}^{(2)}P_{2K}^{(+)}(\Omega, 0) + S'\,{}^{(2)}P_{2K}^{(-)}(\Omega, 0)\}\, \Phi_{2K}(r). \tag{11}$$

The weight of the mixed symmetry state into the total ground state wave function of the trinucleon being of the order of 1 % it is practically sufficient to retain the first term of the expansion (10) only. To this approximation the matrix element

$$\langle \Phi_+ | \sum_+ e^{i\vec{k}\cdot\vec{z}(\varphi)} | \rangle + \langle \Phi_- | \sum_- e^{i\vec{k}\cdot\vec{z}(\varphi)} | \Phi_S \rangle$$

changes its sign when φ is rotated by $\frac{\pi}{2}$. In defining the mixed symmetry correlation form factor by

$$f_M(k) = \rangle \Phi_+ | \sum_+ e^{i\vec{k}\cdot(\vec{x}_i - \vec{x}_j)} | \Phi_S \rangle + \langle \Phi_- | \sum_- e^{i\vec{k}\cdot(\vec{x}_i - \vec{x}_j)} | \Phi_S \rangle$$

and the mixed simmetry nuclear form factor by

$$F_M(k) \langle = \Phi_+ | \sum_+ e^{i\vec{k}\cdot(\vec{x}_i - \vec{X})} | \Phi_S \rangle + \langle \Phi_- | \sum_- e^{i\vec{k}\cdot(\vec{x}_i - \vec{X})} | \Phi_S \rangle,$$

one has

$$f_M(k) = -F_M(\sqrt{3}\,k).$$

Replacing in (5) the matrix elements between two nucleons by those deduced by symmetry from the form factors one obtains

$$\langle \psi_{\frac{1}{2}} | \sum_c \frac{1 - \tau_z(k)}{2} e^{i\vec{q}\cdot(\vec{x}_i - \vec{x}_j)} | \psi_{\frac{1}{2}} \rangle = F_S(\sqrt{3}\,q) - F_M(\sqrt{3}\,q)\cdot$$

Putting $q = k/\sqrt{3}$ in (4) and taking into account the relation (2) and (3) one gets the following expression for the Coulomb energy given in terms of the charge form factors G_{ES}, G_{EV}, $F_{^3H}$ and $F_{^3He}$:

$$E_c = \frac{2e^2}{\pi\sqrt{3}} \int_0^{+\infty} 4\, G_{ES}\left(\frac{k}{\sqrt{3}}\right) G_{EV}\left(\frac{k}{\sqrt{3}}\right) \left\{ \frac{2 F_{^3He}(k) + F_{^3H}(k)}{6\, G_{ES}(k)} - \right.$$
$$\left. - \frac{1}{3\, G_{EV}(k)} \left[F_{^3H}(k) - F_{^3He}(k) - \frac{G_{ES}(k) - G_{EV}(k)}{4\, G_{ES}(k)} \left(2 F_{^3He}(k) + \right.\right.\right.$$
$$\left.\left.\left. + F_{^3H}(k) \right) \right] \right\} dk.$$

An interesting property of this relation can be obtained in using for the nucleon form factors a Gaussian approximation. The Coulomb energy E_c increases with the size of the nucleon charge distribution.

3. *Numerical estimation and conclusion*

For numerical calculation the three poles fit the approximate analytical expression given in Ref. [10], has been utilized for G_{ES} and G_{EV} and the experimental values given in Refs. [11, 12] for $F_{^3H}$ and $F_{^3He}$. Integrating over the range of experimental knowledge, one obtains

$$E_c = 0.65\,\mathrm{MeV},$$

in which the contribution of the additional term arising from the mixed symmetry wave functions is —0.03 MeV. only (i.e. 5% of E_c).

The small contribution due to the proton-neutron mass difference which makes the kinetic energy to be charge dependent has been calculated to raise the Coulomb energy by about 0.02 MeV. Therefore all together the Coulomb energy is not expected to exceed about 0.67 or 0.68 MeV. This corroborates the original remark of K. Okamoto[1] that the nucleon potential is slightly charge dependent and is responsible for about 10% of the energy difference between the Triton and ^{3}He. Anyway if it holds for other nuclei, the nuclear radii deduced from the Coulomb energy difference between analogue states may be raised by about 10% to take into account the existence of charge dependent forces.

The relations (10) and (11) between the correlation and the nuclear form factors make any calculation of the Coulomb energy of ^{3}He with wave functions which does not fit the charge form factors of Tritium and ^{3}He completely meanigless.

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COULOMBSKA ENERGIJA ^3He IZVEDENA IZ RELACIJA SIMETRIJE ZA NABOJNE FORM-FAKTORE

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Sadržaj

U radu je — koristeći hipersferni formalizam — izvedena relacija između Coulombske energije ^3He i nabojnih form-faktora nukleona, ^3He i tricija.

Primjenom eksperimentalnih podataka, račun veličine energije dao je iznos od oko 0.67 MeV a što je manje od eksperimentalne vrijednosti, koja iznosi 0.76 MeV.

Dobiveni rezultat se može smatrati kao indikacija za malo narušenje nabojne nezavisnosti nuklearnih sila.