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APPLICATION OF THE ABEL INTEGRAL EQUATION TO OPTICALLY THIN PLASMA SOURCES*

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Abstract: The paper describes the testing of five, most frequently used methods for computing the radial distribution of emitters or refractive indeces in cylindrical, optically thin plasma sources. These methods were tested for four analytically soluble distributions, by comparing the results of the numerical and analytical procedures. It is shown that for each distribution shape, depending on the accuracy of the input data, the most appropriate method can be selected.

1. Introduction

In plasma diagnostics a number of spectroscopic and interferometric measurements are related to a side-on investigation of circularly symmetric plasma sources such as arc columns, theta and z-pinches, jets from nozzles etc. If the source is optically thin it is possible using the Abel integral equation to obtain the radial distribution of emission coefficient or the refractive index distribution. Therefore a great effort has been expended in developping various methods for applying the Abel integral equation to the data gathered from rotationally symmetric source by side on observation.

The aim of this paper is to investigate the applicability and accuracy of various most frequently used methods of Abel inversion. The intention of this work is to facilitate best choice of the method for various radial distribution shapes.

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2. *Abel integral equation*

The Abel integral equation[1], which is a special case of Volterra equation of the first kind, may be written as

$$I(x) = 2 \int_x^R \frac{i(r)\, r\, \mathrm{d} r}{(r^2 - x^2)^{1/2}} \tag{1}$$

where $I(x)$ is for instance the radiance which is a function of the lateral coordinate x; r is the radius and $i(r)$ the radial distribution function, R is the over-all radius of the column. Since $i(r)$ is the unknown function it must be removed from the integral by a suitable assumption or by inverting Equ. (1)

$$i(r) = -\frac{1}{\pi} \int_r^R \frac{I'(x)\, \mathrm{d} x}{(x^2 - r^2)^{1/2}} \tag{2}$$

A number of different methods have been developed to solve either Equ. (1) or (2), but all these methods[2] may be divided in two general categories: numerical techniques and data approximation schemes utilizing curve fitting or other mathematical approximations.

3. *Numerical techniques*

A number of methods are available using Equ. (1). All these methods assume some sort of variation of $i(r)$ over a small interval. The simplest and earliest of these is given by Maecker[3], where $i(r)$ is assumed to be constant over a small interval so that the resultant series of integrals may be integrated. One elegant assumption is described in the paper by Frie[4]. It is assumed that normalized emission coefficients may be replaced over a small interval by a second degree interpolation formula

$$i(r) = \frac{i(r_k)(r^2_{k-1} - r^2) + i(r_{k-1})(r^2 - r^2_k)}{r^2_{k-1} - r^2_k}, \tag{3}$$

making again an integration possible.

Berge and Richter[5,6] expanded the integral in Equ. (1) into a set of linear equations given by

$$I(x_k) = \sum_{k=i}^{N} a_{ki}\, i(r_i). \tag{4}$$

Now it was possible to obtain a recursive formula

$$i(r_i) = \sum_{i=k}^{N} b_{ik}\, I(x_k), \tag{5}$$

convenient for numerical computations. The coefficients are given in Refs.[5, 6].

The numerical method reported by Edels, Hearne and Young[7] uses the Equ. (2). They devided the integration interval into equal subranges. Within these intervals, after the transformation of variables, the function $I(x)$ has been replaced by the Taylor expansion of the second order, so the integration may be performed.

4. *Data approximation techniques*

Two data approximation techniques are going to be considered, both using Equ. (2) and the method of fitting polinomials through the data points.

Bockasten[8] fitted third degres polynomials exactly to the data points and showed that the resultant errors were small. However this method requires prior smoothing of row data.

To avoid this Barr[9] included smoothing in the process of integration by changing the order of differentiation and integration in Equ. (2). He uses the Equ. (2) in the form

$$i(r) = \frac{1}{2\pi r}\frac{\mathrm{d}F(r)}{\mathrm{d}r}, \tag{6}$$

where

$$F(r) = 2\int_r^R \frac{I(x)\, x\, \mathrm{d}x}{(x^2 - r^2)^{1/2}}. \tag{7}$$

The integration interval was divided into equal increments Δ. Within each interval $I(x)$ was approximated by

$$I(x) = a_k + b_k x^2. \tag{8}$$

It is now possible to obtain a solution of the integral in Equ. (7) in the form

$$F_k = \Delta \sum_{n=k}^{N} \alpha_{kn} \cdot I_n. \tag{9}$$

At each point k, F_k was represented by the polinomial

$$F(k) = (A_k + B_k k^2 + c_k k^4)\, \Delta. \tag{10}$$

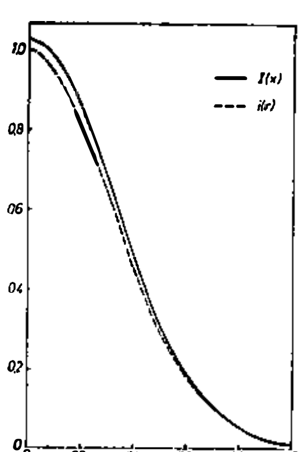
A least-squares method was than used to determine the coefficients giving best fit of $F(k)$ to the F_k at five points from $k = -2$ to $k = +2$. Finally the function $F(k)$ obtained in this way is introduced in Equ. (6) which is now convenient for numerical computations.

5. *Comparison of five methods*

Five methods were tested with $I(x)$ input data taken from analytically integrable functions. These four test curves are of two types. First two are for the usual bell shaped type distributions

$$i(r) = \exp\left(-\frac{r^2}{\sigma^2}\right),$$

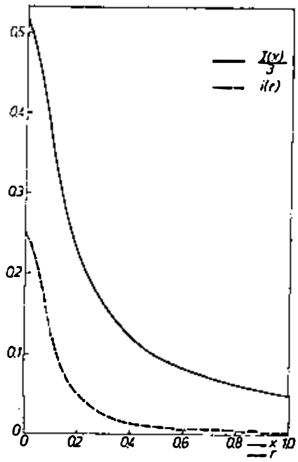
$$(\sigma = 0.58,\ R = 1.2); \tag{11}$$



and

$$i(r) = (\sigma^2 + r^2)^{-1},$$

$$(\sigma = 2,\ R = 20). \tag{12}$$



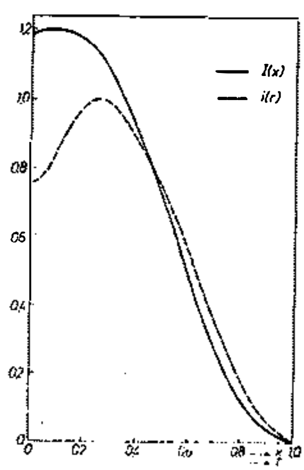
The third one is for the off-axis peak type of distribution typical for the radiances from high temperature arc columns Equ. (13) and for the shape of the radial distribution curves of refractive indices during temperature decay in a cylindrical plasma column Equ. (14).

$$i(r) = \frac{3}{4} + 12\,r^2 - 32\,r^3,$$

$$(0 \leq r \leq 0.25);$$

$$i(r) = \frac{16}{27}(1 + 6\,r - 15\,r^2 + 8\,r^3),$$

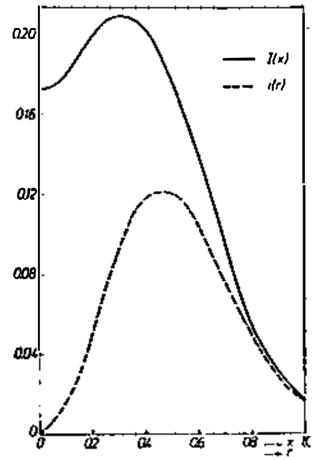
$$(0.25 < r \leq 1.0); \tag{13}$$



and

$$i(r) = r^2 \exp\left(-\frac{r^2}{\sigma^2}\right), \qquad (14)$$

$(\sigma = 0.58,\ R = 1.2)$.



The analytical functions $i(r)$ are taken from the papers by Bracewell[10] (Equs. 11, 12, 14) and Cremers and Birkebak[2] (Equ. 13). Relevant figures are given on the right side of the equations.

Table 1

Bell-shape distribution given by Equ. (11).

Test curve			Frie[4]	Edels et al.[7]	Berge et al.[5]	Barr[3]
k	$I(x)$	$i(r)$	$i - i_k$	$i - i_k$	$i - i_k$	$i - i_k$
0	1.028	1.000	—0.069	—0.406	—0.002	0.025
1	1.006	0.978	—0.071	—0.440	0.009	0.021
2	0.942	0.916	—0.064	—0.410	—0.013	0.014
3	0.845	0.822	—0.058	—0.301	0.025	0.006
4	0.725	0.705	—0.050	—0.427	—0.003	—0.017
5	0.595	0.579	—0.038	—0.261	0.000	—0.027
6	0.468	0.456	—0.028	—0.204	0.003	—0.019
7	0.353	0.343	—0.023	—0.148	0.000	—0.018
8	0.254	0.247	—0.016	—0.107	0.000	—0.018
9	0.175	0.170	—0.010	—0.074	0.000	—0.017
10	0.116	0.113	—0.007	—0.049	0.000	—0.013
11	0.073	0.071	—0.004	—0.032	—0.003	—0.011
12	0.044	0.043	—0.002	—0.018	+0.003	—0.009
13	0.026	0.025	—0.001	—0.014	—0.019	—0.007
14	0.014	0.014	—0.010	—0.019	—0.009	—0.004

Since the number of data points used is a parameter for this test comparisons are made for eleven fifteen and twenty one data intervals (the only exception is the computations based on Bockasten's method[8] where data intervals are limited to ten and twenty). The approximate $i(r)$ are calculated

Table 2

Bell-shape distribution given by Equ. (12).

Test curve			Frie[4]	Edels et al.[7]	Berge et al.[5]	Barr[9]
k	$I(x)$	$i(r)$	$i - i_k$	$i - i_k$	$i - i_k$	$i - i_k$
0	1.571	0.250	—0.013	0.130	—0.023	0.124
1	1.278	0.166	0.000	—0.058	—0.005	—0.039
2	0.901	0.082	—0.002	—0.035	0.001	—0.019
3	0.664	0.045	—0.002	—0.021	—0.000	—0.015
4	0.519	0.027	—0.002	—0.013	—0.001	—0.008
5	0.424	0.018	—0.002	—0.010	—0.001	—0.004
6	0.357	0.013	—0.002	—0.007	—0.001	—0.006
7	0.308	0.010	—0.001	—0.006	—0.001	—0.002
8	0.271	0.007	—0.002	—0.006	—0.002	—0.002
9	0.242	0.006	—0.002	—0.005	—0.002	—0.002
10	0.218	0.005	—0.002	—0.005	—0.002	—0.002
11	0.198	0.004	—0.002	—0.005	—0.004	—0.002
12	0.182	0.003	—0.004	—0.006	—0.002	—0.003
13	0.168	0.003	—0.001	—0.007	—0.014	—0.004
14	0.156	0.002	—0.014	—0.020	—0.012	—0.004

Table 3

Off-axis peak type of distribution given by Equ. (13).

Test curve			Frie[4]	Edels et al.[7]	Berge et al.[5]	Barr[9]
k	$I(x)$	$i(r)$	$i - i_k$	$i - i_k$	$i - i_k$	$i - i_k$
0	1.188	0.750	—0.055	—0.424	0.008	—0.078
1	1.192	0.798	—0.052	—0.420	0.012	—0.053
2	1.193	0.900	—0.066	—0.374	0.001	0.020
3	1.165	0.985	—0.056	—0.403	0.014	0.046
4	1.098	0.993	—0.073	—0.442	—0.013	0.033
5	0.987	0.949	—0.062	—0.398	0.004	0.022
6	0.852	0.862	—0.057	—0.366	0.003	0.023
7	0.700	0.747	—0.039	—0.305	0.012	0.007
8	0.545	0.616	—0.044	—0.269	—0.001	0.002
9	0.388	0.475	—0.026	—0.193	0.016	—0.004
10	0.250	0.337	—0.009	—0.129	0.013	—0.010
11	0.140	0.210	—0.012	—0.085	0.002	—0.015
12	0.057	0.105	0.001	—0.036	0.007	—0.021
13	0.013	0.033	0.005	—0.012	0.007	—0.024
14	0.000	0.000	0.000	0.000	0.000	—0.018

using the five methods and then at each point the difference between the calculated and actual value is computed.

Table 4

Off-axes peak type distribution given by Equ. (14).

Test curve			Frie[4]	Edels et al.[7]	Berge et al.[5]	Barr[9]
k	$I(x)$	$i(r)$	$i - i_k$	$i - i_k$	$i - i_k$	$i - i_k$
0	0.173	0.0	0.002	—0.020	0.005	—0.008
1	0.177	0.007	—0.005	—0.025	—0.003	—0.006
2	0.186	0.027	—0.001	—0.008	0.001	0.000
3	0.198	0.054	—0.004	—0.026	—0.002	0.003
4	0.207	0.083	—0.006	—0.033	0.000	0.006
5	0.210	0.106	—0.010	—0.048	—0.003	0.006
6	0.203	0.120	—0.012	—0.056	—0.004	—0.006
7	0.186	0.123	—0.010	—0.054	—0.002	0.005
8	0.162	0.116	—0.009	—0.051	—0.002	0.003
9	0.134	0.101	—0.008	—0.045	—0.002	0.000
10	0.105	0.083	—0.008	—0.039	—0.003	—0.001
11	0.077	0.063	—0.005	—0.029	—0.005	—0.004
12	0.036	0.046	—0.005	—0.022	0.003	—0.004
13	0.054	0.031	0.000	—0.018	—0.030	—0.006
14	0.023	0.020	—0.020	—0.034	—0.018	—0.004

Table 5

Standard deviations obtained by using three place values of $I(x)$ as input data for different methods and various distribution shapes.

Test curve Equs.	N	Frie[4]	Edels et al.[7]	Berge-Richter[5]	Barr[9]	Bockasten[8]
(11)	11	0.058	0.566	0.006	0.022	0.063
	15	0.036	0.219	0.009	0.016	
	21	0.017	0.010	0.009	0.006	0.031
(12)	11	0.005	0.079	0.009	0.019	0.009
	15	0.005	0.036	0.008	0.034	
	21	0.005	0.020	0.006	0.018	0.007
(13)	11	0.072	0.728	0.008	0.069	0.073
	15	0.045	0.303	0.009	0.032	
	21	0.031	0.028	0.017	0.013	0.036
(14)	11	0.009	0.080	0.009	0.009	0.005
	15	0.008	0.036	0.009	0.006	
	21	0.007	0.006	0.010	0.003	0.005

Table 6

Standard deviations obtained by using one place values of $I(x)$ as input data for different methods and various distribution shapes.

Test curve Equs.	N	Frie[4]	Edels et al.[7]	Berge-Richter[5]	Barr[9]	Bockasten[8]
(11)	11	0.138	0.413	0.138	0.073	0.114
	15	0.108	0.337	0.126	0.020	
	21	0.242	0.528	0.136	0.025	0.100
(12)	11	0.008	0.087	0.015	0.061	0.013
	15	0.008	0.037	0.005	0.032	
	21	0.017	0.026	0.011	0.019	0.011
(13)	11	0.111	0.717	0.319	0.058	0.319
	15	0.115	0.234	0.090	0.021	
	21	0.137	0.094	0.090	0.021	0.137
(14)	11	0.046	0.109	0.045	0.017	0.040
	15	0.039	0.064	0.039	0.013	
	21	0.054	0.071	0.054	0.016	0.041

The standard deviation of each set of results is calculated from

$$\sigma = \left\{ \frac{\sum_{n=1}^{N} [\Delta i(r_n)]^2}{N} \right\}^{1/2} . \tag{15}$$

The results of comparison of four methods using as input, fifteen data points good to three decimal places are given in Tables 1, 2, 3 and 4. These tables are a good illustration of how well various methods agree.

In Tables 5 and 6 standard deviations are given for various numbers of intervals and for data points good to three (Table 5) and respectively one (Table 6) decimal place.

6. Conclusions

Several conclusions may be drawn from these tables. First if input data is expected to be accurate and to conform to a bell shaped analytical functions, the method of Berge-Richter[5] gives the best accuracy for Equ. (11) type, whereas Frie' s method[4] for Equ. (12) type. For off-axis peak type Equ. (13) the method of Berge-Richter[5] is the best one while for Equ. (14) type Barr's method[9] is most accurate. Although the method by Berge-Richter[5] is in average the best one for all types, especially when working with the smaller number of the input data.

However if there is an appreciable scatter (input data good to one decimal place — Table 6) Barr's method[9] gives in average best final results.

The accuracy of the methods for each particular distribution type and for different number of input data is well illustrated by the standard deviation values given in Table 5 and 6.

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PRIMENA ABELOVE INTEGRALNE JEDNAČINE NA OPTIČKI TANKE IZVORE PLAZME

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Sadržaj

U radu je dato kritičko poređenje pet najčešće korišćenih metoda za izračunavanje radijalne raspodele emitera ili indeksa prelamanja u cilindričnom, optički tankom izvoru plazme.

Testiranje metoda izvršeno je za četiri analitički rešive raspodele, poređenjem rezultata numeričkog i analitičkog proračuna.

Pokazano je, da se za svaki oblik funkcije raspodele može izabrati najpogodniji metod u zavisnosti od tačnosti ulaznih podataka.