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## LETTERS TO THE EDITOR

# THE ANGULAR MOMENTUM LAW IN SPECIAL RELATIVITY

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In relativistic mechanics difficulties are met in formulating the angular momentum law for extended bodies, e. g. for a stressed lever[1,2,3]. It is sometimes overlooked that in special relativity even for a point particle the angular momentum law cannot be formulated in the familiar form for a moving pivot. To realize this we first define angular momentum and torque for a particle as generally as possible and then try to get »the angular momentum law for a moving pivot«. The conclusion is that this law is not very useful since one had to give up the covariant definition of quantities and, furthermore, it has not the simple familiar form.

Let a particle with rest mass $m$ be moving in a inertial reference frame $S$. (We put $c = 1$ and $t = x_0$, $x = x_1$, $y = x_2$, $z = x_3$; greek indices run over 0, 1, 2, 3 and latin ones over 1, 2, 3). A point $P$ should be given so that its motion in $S$ is known. The angular momentum of the particle at the event $x_\mu$ on the world line of the particle in respect to an event $X_\mu$ on the world line of point $P$ is defined as

$$j_{\mu\nu} = m\,(x_\mu - X_\mu)\,\vartheta_\nu - m\,(x_\nu - X_\nu)\,\vartheta_\mu\,. \tag{1}$$

Here $\vartheta_\mu$ is the relative four-velocity, i .e. the four-velocity of the particle measured in a reference frame in which the point $P$ is at rest. The torque acting on the particle at event $x_\mu$ in respect to event $X_\mu$ is defined as

$$h_{\mu\nu} = (x_\mu - X_\mu)\,f_\nu - (x_\nu - X_\nu)\,f_\mu\,. \tag{2}$$

Here $f_\mu = m\mathrm{d}v_\mu/\mathrm{d}\tau$ is the four-force, $v_\mu = \mathrm{d}x_\mu/\mathrm{d}\tau$ being the four-velocity and $\tau$ the proper time of the particle.

(1) and (2) can be viewed upon as generalizations of the corresponding Newtonian equations

$$^{N}j_{kn} = m(x_k - X_k)(\dot{x}_n - \dot{X}_n) - m(x_n - X_n)(\dot{x}_k - \dot{X}_k),$$

$$^{N}h_{kn} = (x_k - X_k)(m\ddot{x}_n) - (x_n - X_n)(m\ddot{x}_k).$$

While in Newtonian mechanics the angular momentum law $\mathrm{d}(^{N}j_{nk})/\mathrm{d}t = {}^{N}h_{kn}$ is valid, provided $\ddot{X}_k = 0$, this is not true in relativistic mechanics. The proper time rate of change of angular momentum (1), $\mathrm{d}j_{\mu\nu}/\mathrm{d}\tau$, is not equal to the torque (2), $h_{\mu\nu}$, even if $\dot{X}_k \neq 0$, $\ddot{X}_k = 0$.

Let us introduce another inertial reference frame S′. The axes $y'$ and $z'$ of frame are mutually parallel to the $y$ and $z$ axes, the $x'$ and $x$ axes coincide and the origin O′ is moving with the constant velocity $w$ along the $x$-axis in frame S. The origin O of frame S and the origin O′ coincide at the instant $t' = t = 0$. In this case the familiar Lorentz transformations apply. According to the relativity principle the observer in frame S′ defines angular momentum and torque in the same way as the observer in frame S. Hereby he uses a point Q′ moving in S′ independently of point P. He defines the angular momentum of the particle at event $x'_\mu$ (which in this frame corresponds to the event $x_\mu$ in frame S) in respect to an event $Y'_\mu$ on the world line of point Q′. In his definition the relative four-velocity $\tilde{\vartheta}'_\mu$ enters, measured in an inertial frame in which the point Q′ is at rest.

A suitable choice is the event $Y'_\mu$ in frame S′ which corresponds to the event $X_\mu$ in frame S (thus $Y'_\mu$ being identical with $X'_\mu$). This means that the event lying at the intersection of the world lines of point P and Q′ is referred to in both frames. It is natural that the origins O and O′ are chosen as points P and Q′. The world lines of the origins intersect at the event $X_\mu = X'_\mu = (0, 0, 0, 0)$. The relative four-velocities $\vartheta_\mu$ and $\tilde{\vartheta}'_\mu$ in this case become the four-velocities $v_\mu$ and $v'_\mu$ of the particle in frames S and S′, respectively. The definitions of angular momentum and torque in both frames simplify to

$$j_{\mu\nu} = mx_{\mu}v_{\nu} - mx_{\nu}v_{\mu}, \tag{3}$$

$$j'_{\mu\nu} = mx'_{\mu}v'_{\nu} - mx'_{\nu}v'_{\mu}, \tag{3'}$$

$$h_{\mu\nu} = x_{\mu}f_{\nu} - x_{\nu}f_{\mu}, \tag{4}$$

$$h'_{\mu\nu} = x'_{\mu}f'_{\nu} - x'_{\nu}f'_{\mu}. \tag{4'}$$

$j_{\mu\nu}$ and $h_{\mu\nu}$ are transformed to $j'_{\mu\nu}$ and $h'_{\mu\nu}$ as skew-symmetric tensors. The angular momentum law is valid in both frames generally

$$\mathrm{d}j_{\mu\nu}/\mathrm{d}\tau = h_{\mu\nu}, \tag{5}$$

$$\mathrm{d}j'_{\mu\nu}/\mathrm{d}\tau = h'_{\mu\nu}. \tag{5'}$$

The definition (1) is not common in the literature. Often, however, the angular momentum in respect to a point P at rest in frame S is defined[4].

In the definitions (3, 3′) and (4, 4′) each of the two observers uses a point at rest in his reference frame. This does not appear convenient if a well defined pivot exists, e. g. a center to which the particle is bound. In this case both observers try to place points P and Q′ at the pivot. As the world lines of points P and Q′ merge into a single world line of the pivot there is no fixed event to refer to. So the observer in S can use for $X_{\mu}$ the event on the world line of the pivot happening for him simultaneously with the event $x_{\mu}$ while the observer in S′ can use for $Y'_{\mu}$ the event on the world line of the pivot happening for him simultaneously with the event $x'_{\mu}$. (The events $X_{\mu}$ and $Y_{\mu}$ are not simultaneous for either of the two observers).

Let the pivot move in frame S from the origin O at $t = 0$ with the constant velocity $W$ ($W < w$) along the $x$-axis.*. Then in the frame S′ the pivot is moving with the velocity $W' = (W - w)/(1 - wW)$ ($W' < 0$) along the $x'$-axis. The $z$-components of angular momentum and torques are

$$j_{12}(t) = m(x_1 - Wt)\,v_2 - m\gamma_W x_2 (v_1 - W\gamma_u), \tag{6}$$

$$h_{12}(t) = (x_1 - Wt)\,f_2 - x_2 f_1. \tag{7}$$

$u_k = \mathrm{d}x_k/\mathrm{d}t$ is the three-velocity, $\gamma_W = (1 - W^2)^{-1/2}$ and $\gamma_u = v_0 = [1 - (u_1^2 + u_2^2 + u_3^2)]^{-1/2}$. The proper time rate of change of angular momentum (6) is equal to

* The generalization to the motion of the pivot in an arbitrary direction is not trivial and incorporates the Thomas precession[5].

$$\mathrm{d}j_{12}(t)/\mathrm{d}\tau = h_{12} + \gamma_W \gamma_u^4 W p x_2 - (\gamma_W - 1)\,[x_2 f_1 + m v_2 (v_1 - W \gamma_u)]. \qquad (8)$$

with $p = m\,(u_1 \mathrm{d}u_1/\mathrm{d}t + u_2 \mathrm{d}u_2/\mathrm{d}t + u_3 \mathrm{d}u_3/\mathrm{d}t)$. (The $y$-components are obtained replacing index 2 with index 3, the $x$-components are the same as in (3), (4), (5).) Equation (8) indicates that the angular momentum law is not valid in the familiar form. The additional terms in (8), though, contain a velocity at least quadratically and tend to zero in the Newtonian limit.

The quantities $j'_{12}(t')$, $h'_{12}(t)$ and $\mathrm{d}j'_{12}(t)/\mathrm{d}\tau$ introduced by the observer in frame S′ are symmetrical: they are obtained replacing all quantities in $j_{12}(t)$ (6), $h_{12}(t)$ (7) and $\mathrm{d}j_{12}(t)/\mathrm{d}\tau$ (8) with the corresponding dashed ones. Certainly, this does not mean that $j_{12}(t)$, $h_{12}(t)$ and $\mathrm{d}j_{12}(t)/\mathrm{d}\tau$ are transformed into $j'_{12}(t')$, $h'_{12}(t')$ and $\mathrm{d}j'_{12}(t')/\mathrm{d}\tau$ in going over from frame S to frame S′. So these quantities do not transform covariantly.

If the pivot is at rest in frame S′ ($W' = 0$, $W = w$) the equations (3′) (4′) and (5′) follow immediately. If, furthermore, the particle is instantaneously at rest in frame S′ ($v_1 = \gamma_W$, $v_2 = v_3 = 0$)

$$\mathrm{d}j_{12}(t)/\mathrm{d}\tau = h_{12} + (1 - \gamma_W^{-1})\, x_2 f_1 \,. \qquad (9)$$

There are two ways to study in the moving frame S an extended body, e. g. a stressed lever, at rest in frame S′. One either defines all quantities in respect to the origin O and uses equations (3), (4) and (5) or defines all quantities in respect to the pivot at O′ and uses equations (7), (8) and (9). In both cases the hidden angular momentum[6], which is different in the two cases, has to be taken into account. The consequent treatment, especially for the second way, is rather lengthy and seems somewhat artificial**.

## References

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** In this connection one appreciates the extreme viewpoint that thermodynamic systems are to be considered in the rest frame only[7].

Tisak: »Riječka tiskara« — Rijeka