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LETTERS TO THE EDITOR

A NOTE ON THE EQUILIBRIUM OF EXTENDED BODIES IN SPECIAL RELATIVITY

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It was pointed out in a recent letter, that the angular momentum law for a point particle can be formulated in the special theory of relativity in two ways[1]. In the familiar covariant formulation the law reads as

$$\mathrm{d}\, j_{\mu\nu}/\mathrm{d}\tau = h_{\mu\nu} \tag{1}$$

where the angular momentum and the torque of the particle at event x_μ (μ, $\nu = 0, 1, 2, 3$) are defined in respect to the origin 0 of an inertial reference frame S:

$$j_{\mu\nu} = m x_\mu v_\nu - m x_\nu v_\mu\,, \qquad h_{\mu\nu} = x_\mu f_\nu - x_\nu f_\mu\,. \tag{2}$$

m is the rest mass, $v_\mu = \mathrm{d}\,x_\mu/\mathrm{d}\tau$ is the four-velocity, $f_\mu = m\,\mathrm{d}\,v_\mu/\mathrm{d}\tau$ the four-force and τ the proper time of the particle. In the second not covariant formulation, based on classical ideas, the angular momentum and the torque are defined in respect to a pivot moving uniformly in frame S. The purpose of the present note is to treat according to both formulations the equilibrium of an extended body and to compare the corresponding results. So the second formulation is needed for a special case only, the particle is instantaneously at rest in respect to the pivot which is moving with velocity w along the x — axis in frame S, coinciding with origin 0 at time $t = 0$. In this case the four-velocity of the particle in frame S is $v_\mu = (\gamma, \gamma w, 0, 0)$ with $\gamma = (1 - w^2)^{-1/2}$, $c = 1$, the angular momentum is zero and the torque is

$$h_{1k}(t) = (x_1 - w\,t)\, f_k - x_k f_1 = x_1 (t = 0)\, f_k - x_k f_1, \qquad k = 2{,}3$$
$$h_{23}(t) = 0\,, \tag{1'}$$

whereas the law takes on the unfamiliar form

$$\frac{\mathrm{d}\, j_{1k}(t)}{\mathrm{d}\tau} = h_{1k}(t) + (1 - 1/\gamma)\, x_k f_1, \qquad \frac{\mathrm{d}\, j_{23}(t)}{\mathrm{d}\tau} = h_{23}(t). \tag{2'}$$

In obtaining Equ. (1′) the equation $x_1 = x_1(t = 0) + w\,t$ for the motion of the pivot was used.

Consider now an extended body, e. g. a lever, at rest in another inertial reference frame $S'^{2)}$. The axes of this »rest« frame are parallel to the corresponding axes of frame S and the origin $0'$ of frame S' is the pivot. In frame S' the body is in equilibrium: stationary external four-forces acting on the body at discrete points $x'_{r\alpha}$ ($\alpha = 1, 2, \ldots N$) give zero resultant force and torque

$$F'_r = \sum_\alpha f'_{r\alpha} = 0, \quad H'_{rs} = \sum_\alpha (x'_{r\alpha} f'_{s\alpha} - x'_{s\alpha} f'_{r\alpha}) = 0,$$
$$r, s = 1, 2, 3\,.$$

In the rest frame the angular momentum of the body J'_{rs} is zero and the angular momentum law $\mathrm{d}\,J'_{rs}/\mathrm{d}\tau = H'_{rs}$ applies.

I) Let the observer in the »moving« frame S use the first formulation of the angular momentum law in respect to the origin 0. The angular momentum of the body J_{rs} is constant and its derivative $\mathrm{d}\,J_{rs}/\mathrm{d}\tau$ is zero. The resultant torque in respect to the origin 0, however, is not equal to zero:

$$H_{1k} = \sum_\alpha [x_{1\alpha}(t = 0) + w\,t]\, f_{k\alpha} - \sum_\alpha x_{k\alpha} f_{1\alpha} = \sum x'_{1\alpha} f'_{k\alpha}/\gamma +$$
$$+ w\,t \sum_\alpha f'_{k\alpha} - \sum_\alpha x'_{k\alpha} f'_{1\alpha} = - w^2 \sum x_{k\alpha} f_{1\alpha}, \quad H_{23} = 0. \tag{3}$$

The Lorentz transformation was used to get $\gamma\, x_{1\alpha}(t = 0) = x'_{1\alpha}$ and it was supposed that four-forces acting on extended bodies transform as four-forces acting on point particles, i.e. $f_{1\alpha} = \gamma f'_{1\alpha}$. To upheld in equilibrium the angular momentum law in respect to the origin 0 in the moving frame S the well-known additional term has to be introduced

$$\frac{d J_{1k}}{d\tau} = H_{1k} + w^{-2} \sum_{\alpha} x_{k\alpha} f_{1\alpha}, \quad \frac{d J_{23}}{d\tau} = H_{23} \tag{4}$$

II) Let the observer in the moving frame S use the formulation of the angular momentum law in respect to the moving pivot 0′. Now the angular momentum of the body is zero, whereas the torque is

$$H_{1k}(t) = \sum_{\alpha} x_{1\alpha}(t=0) f_{k\alpha} - \sum_{\alpha} x_{k\alpha} f_{1\alpha} = H_{1k}, \quad H_{23}(t) = 0. \tag{3'}$$

To upheld in equilibrium the angular momentum law in respect to the moving pivot in the frame S another additional term has to be introduced

$$\frac{d J_{1k}(t)}{d\tau} = H_{1k}(t) + \gamma^{-2}(\gamma - 1) \sum_{\alpha} x_{k\alpha} f_{1\alpha}, \quad \frac{d J_{23}(t)}{d\tau} = H_{23}.\ (t). \tag{4'}$$

For $w \ll 1$ the additional term in (4′) is equal to one half of the additional term in (4).

Equations (4) and (4′) with the additional terms give a rather unphysical impression.* Usually the additional term in (4) according to M. von Laue is interpreted as the hidden torque due to an elastic energy current. The appearance of another additional term in (4′) casts some doubt on the appropriateness of this interpretation. Indeed, Aranoff has uttered arguments against

*Formally, the additional terms are brought about as follows:
Equs. (2) and (2′) are written down for all point particles the extended body can be thought composed of and summed up to give Equs. (4) and (4′). In Equs. (3) and (3′) the summation goes over N discrete points, however, at which the external forces are acting. Thus, the transmission of the action of external forces is assumed with infinite velocity, an assumption completelly unjustified in special relativity[4].
V. N. Streltsov has tried to redefine the relativistic length in order to get rid of the additional term in (4) (P2-5555, Joint Institute for Nuclear Research, Dubna, 1971).

the use of the angular momentum law in the moving frame and the elastic energy current interpretation of additional terms[3]. In Aranoff's opinion the equilibrium condition in the moving frame should be formulated on the basis of nonsimultaneity of acting forces as observed in the moving frame. Finally, it should be emphasized that both formulations of the angular momentum law refer to equilibrium only. So far the dynamical problem has not been tackled generally. Only the solution for the rotating ring has been hinted at[4]. These difficulties reflect the unsatisfactory state of present attempts to formulate the relativistic dynamics (and statics) of moving exended bodies[5].

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