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LORENTZ-INVARIANT EXPANSION OF THE SCATTERING AMPLITUDE FOR PARTICLES OF ANY SPIN*

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Abstract: The Lorentz-invariant expansion of the scattering amplitude for particles of arbitrary spin is given explicitly. The conditions for P, T and CPT invariance are also given. πN and NN scatterings are considered as an illustration of the method.

1. Introduction

The problem of expansion of the scattering amplitude for particles of arbitrary spin in covariant structures arose at the very beginning of the development of quantum field theory. In the usual approach[1,2,3], such covariant expansions are derived by a procedure specially devised for any combination of spins encountered in practical calculations.

On the basis of the full Lorentz invariance (including invariance under C, P and T) Hepp[4] proposed a general method for constructing covariant expansions. Hepp's solutions, however, was not quite explicit, and the problem still attracts attention (see, for instance, Refs.[5,6]).

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In the present paper, following the methods given in an unpublished paper of Oksak and Todorov, we approach the general problem of covariant expansions by exploiting the manifestly covariant technique given in Refs.[7,8]. Roughly speaking, this technique consists in substituting spin tensors with two-component indices, used in Ref.[4], by homogeneous polynomials of pairs of complex variables associated with each particle with spin.

The Lorentz-invariant expansion is written as a sum of

$$(2s_1 + 1)(2s_2 + 1)(2s_3 + 1)(2s_4 + 1)$$

terms (for binar processes). The complete Lorentz invariance leads to further reduction of the number of independent invariant amplitudes.

In Sec. 2 the problem of the Lorentz-covariant expansion of the scattering amplitude is reduced to the problem of a Lorentz-invariant expansion of a homogeneous function of two-component complex spinors. The degree of homogeneity is related to the spins of the incoming and outgoing particles. Finite-dimensional representations of the SL (2, C) group in the space of two-component complex spinors were used. From the point of view of representation theory, the general formula for the Lorentz-invariant expansion of the scattering amplitude is a decomposition of the direct product of SL (2, C) finite-dimensional representations $[2\, s_a, 0]$ into irreducible representations (this fact was noticed by Hepp[4]).

Further we find identities for invariant structures, with the help of which the general expansion can be written as a function of structures associated with a single channel (s, t or u).

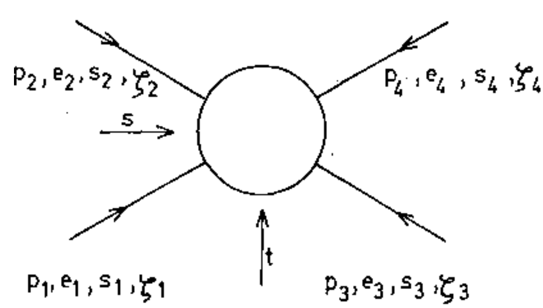

Fig. 1

In Sec. 3 the C, P and CPT invariant conditions for the amplitude are derived.

In Sec. 4 an illustration of the developed method is given for the cases of pion-nucleon and nucleon-nucleon scattering. As expected, our results agree with the wellknown formulae for these processes.

In Appendix I. some useful differentiation formulae and identities are given.

In Appendix II. we give an explicit realization of the Dirac and Bargmann--Wigner formalism used in the main body of the paper.

Appendix III. contains a table of transformation laws for the Lorentz-invariant structures under P operation.

2. *Lorentz-invariant expansion of the scattering amplitude*

Proof of the main formula. Let us consider a process shown in Fig. 1. Here P, e, s, ζ denote the four-momentum, charge, spin and spin projection of the a^{th} particle (in our case $a = 1, 2, 3, 4$). The conservation of the momentum gives

$$P_1 + P_2 + P_3 + P_4 = 0. \tag{1}$$

All particles are on the mass shell

$$P_a^2 \equiv P_{0a}^2 - \vec{P}_a^2 = M\hat{a}^2. \tag{2}$$

We introduce the Mandelstam variables

$$\begin{aligned} S_1 &\equiv S = P_1 + P_2, & S^2 &= s \equiv s_1, \\ S_2 &\equiv T = P_1 + P_3, & T^2 &= t \equiv s_2, \\ S_3 &\equiv U = P_1 + P_4, & U^2 &= u \equiv s_3, \end{aligned} \tag{3}$$

with the well-known property

$$\sum_{i=1}^{3} s_i^2 = s + t + u = \sum_a M_a^2.$$

We denote the physical scattering amplitude of the process by

$$T_{s_1 s_2 s_3 s_3}(P_1, e_1, e_1 \zeta_1; \quad P_2, e_2, e_2 \zeta_2; \quad P_3, e_3, e_3 \zeta_3; \quad P_4, e_4, e_4 \zeta_4).$$

Let $U_{s,\zeta}^{(e)\,a_1 \,..\, a_{2s}}(P)$ be a Bargmann-Wigner spinor for a particle of four-momentum P, charge e, spin s and spin projection $e\zeta$ ($a_1, \ldots, a_{2s} = 1, 2, 3, 4$ are Dirac indices), which satisfies the equation*⁾

$$(\overset{(2s)}{\hat{\mathrm{P}}} - \mathrm{eM})\, \mathrm{U}_{s,\zeta}^{(e)}(P) = 0,$$

* We omit the spinor indices in this and similar formulae.

where

$$\overset{(2s)}{\hat{\mathrm{P}}} \equiv \hat{\mathrm{P}} \times \overset{(2s)}{\ldots} \times \hat{\mathrm{P}} \tag{4}$$

and

$$\underline{S}^2\, U^{(e)}_{s,\zeta}(P) = s(s+1)\, U^{(e)}_{s,\zeta}(P),$$

$$S_3\, U^{(e)}_{s,\zeta}(P) = e_\zeta\, U^{(e)}_{s,\zeta}(P).$$

(We have used a somewhat unusual notation for the spin projection, which allows us to write down all four solutions of the free Dirac equation in one formula.)

In analogy with the case of particles with spin $\frac{1}{2}$ we introduce a Dirac conjugated spinor:

$$\widetilde{U^{(e)}_{s,\zeta}}(P) = \overline{U^{(e)}_{s,\zeta}(P)}\, \overset{(2s)}{\gamma^0} \tag{5}$$

(the bar stands for complex conjugation).

On the basis of the spinor space used in Appendix 2, in which γ_5 is diagonal, we also introduce the four-component complex spinors

$$\xi = \binom{z}{0}, \qquad \tilde{\xi} = \xi\, C,$$

where $z = \binom{z_1}{z_2}$ and z_1, z_2 are complex numbers.

We associate a Bargmann-Wigner spin-tensor with the physical amplitude in a given channel $(12 \to 34)$

$$T_{s_1 s_2 s_3 s_4}(P_1, e_1;\ P_2, e_2;\ P_3, e_3;\ P_4, e_4) =$$

$$= U^{(e)}_{s_1\zeta_1}(P_1)\, U^{(e)}_{s_2\zeta_2}(P_2)\, T_{s_1 s_2 s_3 s_4}(P_1, e_1, e_1\zeta_1;\quad P_2, e_2, e_2\zeta_2;\quad P_3, e_3, e_3\zeta_3;$$

$$P_4, e_4, e_4\zeta_4)\ \widetilde{U}^{(e_3)}_{s_3\zeta_3}(-P_3)\, \widetilde{U}^{(e_4)}_{s_4\zeta_4}(-P_4),$$

where a summation is to be performed over each repeated index ζ_a from $-s_a$ to s_a. The minus signs of the arguments of the Bargmann-Wigner spinors are determined by the choice of the channel.

Further, after multiplying from the left and right by the necessary number of $\tilde{\xi}$ and ξ, respectively, we obtain the scalar amplitude

$$T_{s_1 s_2 s_3 s_4}(P_1, e_1, z_1;\ P_2, e_2, z_2;\ P_3, e_3, z_3;\ P_4, e_4, z_4) =$$

$$= \overset{(2s_1)}{\tilde{\xi}_1}\ U^{(e_1)}_{s_1 \zeta_1}(P_1)\ \overset{(2s_2)}{\tilde{\xi}_2}\ U^{(e_2)}_{s_2 \zeta_2}(P_2),$$

$$T_{s_1 s_2 s_3 s_4}(P_1, e_1, e_{11};\ P_2, e_2, e_{22};\ P_3, e_3, e_{33}\ ;\ P_4, e_4, e_{44})$$

$$\tilde{U}^{(e_3)}_{s_3 \zeta_3}(-P_3)\ \overset{(2s_3)}{\xi_3}\ \tilde{U}^{(e_4)}_{s_4 \zeta_4}(-P_4)\ \overset{(2s_4)}{\xi_4}. \tag{6}$$

The invariance of the amplitude (6) under Lorentz transformations means

$$T_{s_1 s_2 s_3 s_4}(P_1, e_1, z_1; \ldots) = T_{s_1 s_2 s_3 s_4}(\Lambda(A)\, P_1, e_1, A\, z_1; \ldots),$$

where $A \,\varepsilon\, \mathrm{SL}(2, C)$

and

$$A\, P\, A^{\star} = \Lambda(A)\, P.$$

The amplitude (6) is a homogeneous polynomial of degree $2s_a$ with respect to z_a.

From the theory of finite-dimensional representations of the Lorentz group in the space of two-component complex spinors it follows that the amplitude (6) is a polynomial of invariants of the type

$$u_0(z_a, z_b) = z_a\, \varepsilon\, z_b,$$

$$u_j(z_a, z_b) = \frac{1}{2}\, \varepsilon_{jkl}\, z_a\, \varepsilon\, (\underset{\sim}{S_k}\, \tilde{S}_l - S_l\, \tilde{S}_k)\, z,$$

$$(\gamma = 1, 2, 3).$$

So we write

$$T_{s_1 s_2 s_3 s_4}(P_1, e_1, z_1;\ P_2, e_2, z_2;\ P_3, e_3, z_3;\ P_4, e_4, z_4) =$$

$$= \sum_{J_1 = |s_1 - s_3|}^{s_1 + s_3}\ \sum_{J_2 = |s_2 - s_4|}^{s_2 + s_4}\ \sum_{j = |j_1 - j_2|}^{J_1 + J_2} P^{J_1}_{s_1 s_3}(z_1, z_3;\ \partial_u) \cdot$$

$$\cdot\, P^{J_2}_{s_2 s_4}(z_2, z_4;\ \partial_v)\, P^{J}_{J_1 J_2}(u, v;\ \partial_z) \cdot F^{(e_1, e_2, e_3, e_4)}_{J_1 J_2 J}(z;\ S, T, U). \tag{7}$$

The monomials P and the functions F satisfy the homogeneity conditions

$$P^{J}_{J_1 J_2}(\lambda_1 u,\ \lambda_2 v;\ \lambda_3 \partial_z) = \lambda_1^{2J_1}\, \lambda_2^{2J_2}\, \lambda_3^{2J}\, P^{J}(u, v;\ \partial_z),$$

$$F^{(e_1, e_2, e_3, e_4)}_{J_1 J_2 J}(\lambda_z;\ S, T, U) = \lambda^{2J}\, F^{(e_1, e_2, e_3, e_4)}_{J_1 J_2 J}(z;\ S, T, U).$$

The monomials P have the form[7]

$$P_j^{j_1 j_2}(u, v;\ \partial_z) = N_{j_1 j_2}^j (u\ \varepsilon\ v)^{j_1+j_2-j} (u\ \partial_z)^{j_1-2+j} (v\ \partial_z)^{j_2-j_1+j}.$$

The normalization constant $N_{j_1 j_2}^j$ can be determined from the following requirement: If $F_j(z)$ and $F_{j_1 j_1}(u, v)$ are homogeneous polynomials of z and u, v of degree $2j$ and $2j_1$, $2j_2$, respectively, and

$$F_{j_1 j_2}(u, v) = \sum_{j=j_1-j_2}^{j_1+j_2} P_{j_1 j_2}^j (u, v;\ \partial_z)\, F_j(z), \tag{8a}$$

then $F_j(z)$ can be expressed conversely in terms of $F_{j_1 j_2}(u, v)$ by the formula

$$F_j(z) = P_{j_1 j_2}^j (\partial_u, \partial_v;\ z)\, F_{j_1 j_2}(u, v). \tag{8b}$$

So with the help of formulae (I, 1) we obtain

$$N_{j_1 j_2}^j = \left(\frac{(2j+1)}{(j_1+j_2+j+1)!\,(j_1+j_2-j)!\,(j_1-j_2+j)!\,(j_2-j_1+j)!}\right)^{\frac{1}{2}}.$$

Further, the u and v differentiation in Equ. (7) gives after simple and lengthy calculations

$$T_{s_1 s_2 s_3 s_4}(P_1, e_1, z_1;\ P_2, e_2, z_2;\ P_3, e_3, z_3);\ P_4, e_4, z_4) =$$
$$= \sum_{j_1, j_2, j} P_{s_1 s_2 s_3 s_4}^{j_1 j_2 j}(z_1, z_1; z_3, z_4;\ \partial_z) \cdot F_{j_1 j_2 j}^{(e_1, e_2, e_3, e_4)}(z;\ S, T, U), \tag{9}$$

where

$$P_{s_1 s_2 s_3 s_4}^{j_1 j_2 j}(z_1, z_2, z_3, z_4;\ \partial_z) = N_{s_1 s_2 s_3 s_4}^{j_1 j_2 j}(z_1\,\varepsilon\, z_3)^{s_1+s_3-j_1}(z_2\ \varepsilon\ z_4)^{s_2+s_4-j_2}\ \cdot$$
$$\cdot \sum_{\substack{\alpha+\beta+\gamma+\delta=\\ =s_1+s_3-j}} \frac{(z_1\ \varepsilon\ z_2)^\alpha (z_1\ \varepsilon\ z_4)^\beta (z_3\ \varepsilon\ z_2)^\gamma (z_3\ \varepsilon\ z_4)^\delta}{\alpha!\ \varrho!\ \gamma!\ \delta!}\ \cdot$$
$$\cdot \frac{(z_1\ \partial_z)^{s_1-s_3+j_1-\alpha-\beta}(z_2\ \partial_z)^{s_2-s_4+j_2-\alpha-\gamma}}{(s_1-s_3+j_1-\alpha-\beta)!\ (s_2-s_4+j_2-\alpha-\gamma)!}\ \cdot$$
$$\cdot \frac{(z_3\ \partial_z)^{s_3-s_1+j_1-\gamma-\delta}(z_4\ \partial_z)^{s_4-s_2-j_2-\delta-\beta}}{(s_3-s_1+j_1-\gamma-\delta)!\ (s_4-s_2-j_2-\delta-\beta)!}, \tag{10}$$

$$N^{j_1 j_2 j}_{s_1 s_2 s_3 s_4} = \left(\frac{(2j_1+1)(s_1-s_3+j_1)!\,(s_3-s_1+j_2)!\,(2j_2+1)(s_2-s_4+j_2)!}{(s_1+s_3+j_1+1)!\,(s_1+s_3-j_2)!\,(s_2+s_4+j_2+1)!} \cdot \right.$$

$$\left. \frac{(s_4-s_2+j_2)!\,(2j+1)(j_1+j_2-j)!\,(j_1-j_2+j)!\,(j_2-j_1+j)!}{(s_2+s_4-j_2)!\,(j_1+j_2+j+1)!} \right)^{\frac{1}{2}}.$$

The function $F^{(e_1, e_2, e_3, e_4)}_{j_1 j_2 j}(z;\ S, T, U)$ is determined from the homogeneity condition and the possibilities to form Lorentz invariants from z and S, T, U

$$F^{(e_1, e_2, e_3, e_4)}_{j_1 j_2 j}(z;\ S, T, U) = \sum_{k=0}^{j} f^{(e_1, e_2, e_3, e_4)}_{j_1 j_2 jk}(s, t, u)\ u_1^k(z)\, u_2^{j-k}(z) +$$

$$+ u_3(z) \sum_{k=0}^{j-1} g^{(e_1, e_2, e_3, e_4)}_{j_1 j_2 jk}(s, t, u)\ u_1^k(z)\, u_2^{j-1-k}(z), \tag{11}$$

where $u_j(z) = \frac{1}{2}\varepsilon_{jk}\, z\varepsilon\, (\underset{\sim}{S_k}\widetilde{S}_l - \underset{\sim}{S_l}\widetilde{S}_k)\, z$. Higher powers of $u_3(z)$ do not appear in the right-hand side of Equ. (11), since they can be expressed in terms of powers of $u_1(z)$ and $u_2(z)$ from the identity

$$[S_1 u_1(z) + S_2 u_2(z) + S_3 u_3(z)]^2 = 0. \tag{12}$$

The z-differentiation in Equ. (9) can be performed with the help of formulae (I, 2).

Identities for invariant structures. The way we perform the transition from the spinor amplitude to the scalar one shows that the amplitude should be a function of structures associated with a given channel (s, t or u channel, respectively). The remaining structures can be eliminated with the help of the identities*

$$u_0(13)\,u_0(24) - u_0(12)\,u_0(34) - u_0(14)\,u_0(23) = 0,$$

$$u_0(13)\,u_j(24) - u_0(12)\,u_j(34) - u_j(14)\,u_0(23) = 0,$$

$$\varepsilon_{kjl}\, S_k^2\, u_0(13)\,u_k(24) - u_j(12)\,u_l(34) + u_j(23)\,u_l(14) = 0, \tag{13a}$$

$$S_l^2 S_k^2\, u_0(13)\,u_0(24) - u_j(12)\,u_j(34) + u_j(14)\,u_j(23) = 0,$$

$$(k, j, l = 1, 2, 3),$$

* We use the abbreviations $u_\mu(z_a, z_b) \equiv u_\mu(a, b)$.

which follow from the modified identity (I, 3)

$$(z_1\, \varepsilon \sigma_{\mu_1}\, \varrho^{\mu_3}\, z_3)(z_2\, \varepsilon \sigma_{\mu_2}\, \sigma^{\mu_4}\, z_4) \cdot (z_1\, \varepsilon \sigma_{\mu_1}\, \sigma^{\mu_2}\, z_2)(z_3\, \varepsilon \sigma_{\mu_3}\, \sigma^{\mu_4}\, z_4) \cdots$$
$$(z_1 \varepsilon \sigma_{\mu_1}\, \sigma^{\mu_4}\, z_4)(z_2\, \varepsilon \sigma_{\mu_2} \sigma^{\mu_3}\, z_3) = 0,$$
$$(\mu_1, \mu_2, \mu_3, \mu_4 = 0, 1, 2, 3).$$

Another necessary identity is found when the operator

$$(z_1\, \partial_z)(z_2\, \partial_z)(z_3\, \partial_z)(z_4\, \partial_z)$$

acts on Equ. (12)

$$\sum_{j=1}^{3} S_j^2 \,|\, u_j(13)\, u_j(24) + u_j(14)\, u_j(23) + u_j(12)\, u_j(34) \,| +$$
$$+ \sum_{j,k,l=1}^{3} (\varepsilon_{jkl})^2 (S_k, S_l) \,|\, u_k(13)\, u_l(34) + \tag{13b}$$
$$+ u_k(13)\, u_l(24) + u_k(14)\, u_l(23) \,| = 0.$$

Obviously, taking different differential operators and higher powers of Equ. (12), one can obtain identities relating more general structures.

3. *Discrete operations*

To obtain a CPT-invariant expansion, let us analyze the right-hand side of Equ. (6). Here we have two types of scalar products one in the spin space coming from the summation over ζ and the other in the Dirac space. So the product

$$\overset{(2s)}{\xi}\, U^{(e)}_{s,\zeta}(P)\, (\widetilde{U^{(e)}_{s,\zeta}}(P) \overset{(2s)}{\xi})$$

is a scalar in the Dirac space and a spin vector in the spin space.

So, the transformation properties of each single z under space and time reflection will be determined from the requirement

$$\widetilde{I \overset{(2s)}{\xi}}\, U^{(e)}_{s,\zeta}(P) = \overset{(2s)}{\widetilde{\xi}}\, I\, U^{(e)}_{s,\zeta}(P),$$

or

$$\widetilde{I \overset{(2s)}{\xi}}\, U^{(e)}_{s,\zeta}(P) = \overline{I\, U^{(e)}_{s,\zeta}}(P) \overset{(2s)}{\xi},$$

when the operator I involves time reversal.

Space reflection. The physical amplitude transforms under space reflection as

$$I_s\, T_{s_1 s_2 s_3 s_4}(P_1, e_1, e_1 \zeta_1;\; P_2, e_2, e_2 \zeta_2;\; P_3, e_3, e_3 \zeta_3;\; P_4, e_4, e_4 \zeta_4) =$$

$$= \eta_s (T(I_s P_1, e_1, e_1 \zeta_1;\; I_s P_2, e_2, e_2 \zeta_2;\; I_s P_3, e_3, e_3 \zeta_3;\; I_s P_4, e_4, e_4 \zeta_4),$$

where η_s is the phase factor of space reflection (which is a product of four such factors, one for each particle) and

$$I_s P = (P_0, -\vec{P}).$$

Using the space-invariant condition

$$\overset{(2s)}{\widetilde{I_s \xi}}\; U^{(e)}_{s,\zeta}(P) = \overset{(2s)}{\widetilde{\xi}}\, I_s\, U^{(e)}_{s,\zeta}(P),$$

with the help of the identity for the Bargmann-Wigner spinors (II, 1), we derive

$$I_s Z_a = \frac{\widetilde{P}_a}{M_a} Z_a. \tag{14}$$

The amplitude will be invariant under space reflection if and only if

$$T_{s_1 s_2 s_3 s_4}(P_1, e_1, z_1;\; P_2, e_2, z_2;\; P_3, e_3, z_3;\; P_4, e_4, z_4) =$$

$$= (-1)^{2s_3+2s_4}\, \eta_s\, T_{s_1 s_2 s_3 s_4}(I_s P_1, e_1, I_s z_1;\; I_s P_2, e_2, I_2 z_2;$$

$$I_s P_3, e_3, I_s z_3;\; I_s P_4, e_4, I_s z_4), \tag{15}$$

where the factor $(-1)^{2s_3+2s_4}$ results from the minus signs of the arguments in Equ. (6).

Time reversal. The physical amplitude transforms under time-reversal as

$$I_t\, T_{s_1 s_2 s_3 s_4}(P_1, e_1, e_1 \zeta_1;\; P_2, e_2, e_2 \zeta_2;\; P_3, e_3, e_3 \zeta_3;\; P_4, e_4, e_4 \zeta_4) =$$

$$= \eta_t \prod_{a=1}^{4} (-1)^{s_a - \zeta_a}\, T_{s_3 s_4 s_1 s_2}\, (I_s P_3, e_3, -e_3 \zeta_3;\; I_s P_4, e_4, -e_4 \zeta_4;$$

$$I_s P_1, e_1, -e_1 \zeta_1;\; I_s P_2, e_2, -e_2 \zeta_2),$$

where η_t is the phase factor of time reversal. Using the T-invariance condition

$$\widetilde{I_t \zeta}\; U^{(e)}_{s,\zeta}(P) = \overline{I_t\, U^{(e)}_{s,\zeta}(P)}\, \overset{(2s)}{\zeta},$$

with the help of the identity for the Bargmann-Wigner spinor (II, 2), we derive the time-invariant condition for the amplitude

$$T_{s_1 s_2 s_3 s_4}(P_1, e_1, z_1;\ P_2, e_2, z_2;\ P_3, e_3, z_3;\ P_4, e_4, z_4) =$$
$$= (-1)^{2s_3+2s_4}\ \eta_t\ T_{s_3 s_4 s_1 s_2}(I_s P_3, e_3, I_s z_3;\ I_s P_4, e_4, I_s z_4; \tag{16}$$
$$;I_s P_1, e_1, I_s z_1;\ I_s P_2, e_2, I_s z_2).$$

CPT transformation. Under CPT transformation the amplitude transforms as

$$I_{CPT}\ T_{s_1 s_2 s_3 s_4}(P_1, e_1, e_1 \zeta_1;\ P_2, e_2, e_2 \zeta_2;\ P_3, e_3,\ e_3 \zeta_3;\ P_4, e_4, e_4 \zeta_4) =$$
$$\eta_{CPT} \prod_{a=1}^{4} (-1)^{s_a - \zeta_a}\ T_{s_3 s_4 s_1 s_2}(P_3 - e_3, e_3 \zeta_3;\ P_4, -e_4, e_4 \zeta_3;$$
$$;P_1, -e_1, e_1 \zeta_1;\ P_2, -e_2, e_2 \zeta_2).$$

Using the identity (II, 3) in the same way as in the preceding subsections, we find that the amplitude will be invariant under CPT transformation if and only if

$$T_{s_1 s_2 s_3 s_4}(P_1, e_1, z_1;\ P_2, e_2, z_2;\ P_3, e_3, z_3;\ P_4, e_4, z_4) =$$
$$= \eta_{CPT}\ T_{s_3 s_4 s_1 s_2}(P_3, -e_3, z_3;\ P_4, -e_4, z_4;\ P_1, -e_1, z_1;\ P_2, -e_2, z_2). \tag{17}$$

4. *Application to the case of πN and NN scattering*

Pion-nucleon scattering. Using formulae (9), (10) and (11) for $s_1 = s_3 = \frac{1}{2}, s_2 = = s_4 = 0$, we find

$$T_{\pi N}(P_1, z_1;\ P_2;\ P_3, z_3;\ P_4) =$$
$$= f_0(s, t, u)\, u_0(13) + f_1(s, t, u)\, u_1(13) + f_2(s, t, u)\, u_2(13) +$$
$$+ f_3(s, t, u)\, u_3(13).$$

If we apply the space-invariant condition (15) we shall see that only two of the invariant scalar functions $f(s, t, u)$ are linearly independent

$$T_{\pi N}(P_1, z_1;\ P_2;\ P_3, z_3;\ P_4) =$$
$$= \frac{A}{4}[2(s + u)\, u_0(13) + u_1(13) - u_3(13)] + \tag{18}$$
$$+ \frac{MB}{2}[2(s - u)\, u_0(13) - u_1(13) - u_3(13)],$$

where

$$A = 2(f_1 - f_3),\ MB = -(f_1 + f_3),$$

and M is the nucleon mass.

Equ. (18) agrees exactly with the familiar result (see Refs.[2,3])

$$T_{\pi N}(P_1, P_2, P_3, P_4) = A + B\hat{Q},$$

where

$$\hat{Q} = \frac{\hat{P}_2 + \hat{P}_4}{2},$$

as it is easy to verify with the help of the formula

$$T_{\pi N}(P_1, z_1;\ P_2;\ P_3, z_3;\ P_4) = \tilde{\xi}_1 (\hat{P}_1 + M)(A + B\,\hat{Q})(-\hat{P}_3 + M)\,\xi_3.$$

Of course, the amplitude (17) is also T-invariant.

Nucleon-nucleon scattering. In the case of $s_1 = s_2 = s_3 = s_4 = \frac{1}{2}$ we find from Equ. (5)

$$T_{NN}(P_1, z_1;\ P_2, z_2;\ P_3, z_3;\ P_4, z_4) =$$

$$= \frac{1}{2} f_1\, u_0(13)\, u_0(24) + \frac{1}{2\sqrt{3}} f_2\, [u_0(12)\, u_0(34) - u_0(14)\, u_0(23)] +$$

$$+ \frac{1}{2} f_3\, u_0(13)\, u_1(24) + \frac{1}{2} f_4\, u_0(13)\, u_2(24) + \frac{1}{2} f_5\, u_0(13)\, u_3(24) +$$

$$+ \frac{1}{2} f_6\, u_1(13)\, u_0(24) + \frac{1}{2} f_7\, u_2(13)\, u_0(24) + \frac{1}{2} f_8\, u_3(13)\, u_0(24) +$$

$$+ \frac{1}{4\sqrt{2}} f_9\, [u_0(12)\, u_1(34) + u_1(12)\, u_0(34) - u_0(23)\, u_1(14) + u_1(23)\, u_0(14)] +$$

$$+ \frac{1}{4\sqrt{2}} f_{10}\, [u_0(12)\, u_2(34) + u_2(12)\, u_0(34) - u_0(23)\, u_2(14) + u_2(23)\, u_0(14)] +$$

$$+ \frac{1}{4\sqrt{2}} f_{11}\, [u_0(12)\, u_3(34) + u_3(12)\, u_0(34) - u_0(23)\, u_3(14) + u_3(23)\, u_0(14)] +$$

$$+ \frac{1}{2\sqrt{6}} f_{12}\, [u_1(12)\, u_1(34) + u_1(13)\, u_1(24) + u_1(23)\, u_1(14)] +$$

$$+ \frac{1}{2\sqrt{6}} f_{13}\, [u_2(12)\, u_2(34) + u_2(13)\, u_2(24) + u_2(23)\, u_2(14)] +$$

$$+ \frac{1}{4\sqrt{6}} f_{14}\, [u_1(12)\, u_2(34) + u_1(13)\, u_2(24) + u_1(14)\, u_2(23) +$$

$$+ u_2(12)u_1(34) + u_2(13)u_1(24) + u_2(14)u_1(23)] +$$

$$+ \frac{1}{4\sqrt{6}} f_{15} [u_2(12)u_3(34) + u_2(13)u_3(24) + u_2(14)u_3(23) +$$

$$+ u_3(12)u_2(34) + u_3(13)u_2(24) + u_3(14)u_2(23)] +$$

$$+ \frac{1}{4\sqrt{6}} f_{16} [u_3(12)u_1(34) + u_3(13)u_1(24) + u_3(14)u_1(23) +$$

$$+ u_1(12)u_3(34) + u_1(13)u_3(24) + u_1(14)u_3(23)].$$

As can be seen, all three kinds of structures (13) (24), (12) (34) and (14) (23) appear. However, since we are dealing with the t channel, our amplitude should only be a function of structures (13) (24). With the help of identities (13) all structures in (33) can be expressed in terms only of sixteen linearly independent structures $u_\mu(13)u^\nu(24)$

$$T_{NN} = \frac{1}{2} f_1 u_0(13)u_0(24) +$$

$$+ \frac{1}{2\sqrt{3}} \frac{f_2}{s_1 s_2 s_3} [s_1 u_1(13)u_1(24) + s_2 u_2(13)u_2(24) + s_3 u_3(13)u_3(24)] +$$

$$+ \frac{1}{2} f_3 u_0(13)u_1(24) + \frac{1}{2} f_4 u_0(13)u_2(24) + \frac{1}{2} f_5 u_0(13)u_3(24) +$$

$$+ \frac{1}{2} f_6 u_1(13)u_0(24) + \frac{1}{2} f_7 u_2(13)u_0(24) + \frac{1}{2} f_8 u_3(13)u_0(24) +$$

$$+ \frac{1}{2\sqrt{2}} \frac{f_9}{s_1} [u_2(13)u_3(24) - u_3(13)u_2(24)] +$$

$$+ \frac{1}{2\sqrt{2}} \frac{f_{10}}{s_2} [u_3(13)u_1(24) - u_1(13)u_3(24)] +$$

$$+ \frac{1}{2\sqrt{2}} \frac{f_{11}}{s_3} [u_1(13)u_2(24) - u_2(13)u_1(24)] +$$

$$+ \frac{1}{2\sqrt{6}} \frac{f_{12}}{s_1^2} [2s_1^2 u_1(13)u_1(24) + s_2^2 u_2(13)u_2(24) + s_3^2 u_3(13)u_3(24)] +$$

$$+ \frac{1}{2\sqrt{6}} \frac{f_{13}}{s_2^2} [s_1^2 u_1(13)u_1(24) + 2s_2^2 u_2(13)u_2(24) + s_3^2 u_3(13)u_3(24)] +$$

$$+ \frac{1}{4\sqrt{6}} f_{14} [u_1(13)u_2(24) + u_2(13)u_1(24)] +$$

$$+ \frac{1}{4\sqrt{6}} f_{15} [u_2(13)u_3(24) + u_3(13)u_2(24)] +$$

$$+ \frac{1}{4\sqrt{6}} f_{16} [u_3(13)u_1(24) + u_1(13)u_3(24)].$$

The requirement for P invariance (15) leads to such relations between scalar functions $f_k(s, t, u)$ $(k = 1, 2, \ldots, 16)$ that only eight of them $(G_m(s, t, u), m = 1, 2, \ldots 8)$ are linarly independent

$$
\begin{aligned}
T_{NN} = {} & G_1 \{ (s + u)^2 u_0(13) u_0(24) + \\
& + \frac{1}{2}(s + u) [u_0(13) u_1(24) + u_1(13) u_0(24)] + \\
& + \frac{1}{2}(s + u) [u_0(13) u_3(24) - u_3(13) (u_0 24)] + \\
& + \frac{1}{4} [u_1(13) u_3(24) - u_3(13) u_1(24) + u_1(13) u_1(24) - u_3(13) u_3(24)]\} + \\
& + MG_2 \{ (s^2 - u^2) u_0(13) u_0(24) + \\
& + \frac{1}{2} u [u_0(13) u_1(24) + u_1(13) u_0(24)] + \\
& + \frac{1}{2} s [u_0(13) u_3(24) - u_3(13) u_0(24)] + \\
& - \frac{1}{4} [u_1(13) u_1(24) + u_3(13) u_3(24)] \} + \\
& + M^2 G_3 u_2(13) u_2(24) + M^2 G_4 \{(s - u)^2 u_0(13) u_0(24) + \qquad (19) \\
& - \frac{1}{2}(s - u) [u_0(13) u_1(24) + u_1(13) u_0(24)] + \\
& + \frac{1}{2}(s - u) [u_0(13) u_3(24) - u_3(13) u_0(24)] + \\
& + \frac{1}{4} [- u_1(13) u_3(24) + u_3(13) u_1(24) + u_1(13) u_1(24) - u_3(13) u_3(24)] \} + \\
& + G_5 \left\{ \frac{1}{4} t^2 u_0(13) u_0(24) + \frac{1}{2} t [- u_0(13) u_1(14) - u_1(13) u_0(24) - \right. \\
& - u_0(13) u_3(24) + u_3(13) u_0(24)] + \\
& + \frac{1}{4} [u_1(13) u_3(24) - u_3(13) u_1(24) + u_1(24) + u_1(13) u_1(24) - u_3(13) u_3(24)]\} + \\
& + MG_6 \left\{ \frac{1}{2} s [u_0(13) u_1(24) - u_1(13) u_0(24)] - \frac{1}{2} u \,|\, u_0(13) u_3(24) + \right. \\
& + u_3(13) u_0(24)] - \frac{1}{4} [u_1(13) u_3(24) + u_3(13) u_1(24)]\} + \\
& + G_7 \{[- 2t u_0(13) + u_1(13) - u_3(13)] u_2(24)\} + \\
& + G_8 \{u_2(13) [- 2t u_0(24) + u_1(24) + u_3(24)]\}.
\end{aligned}
$$

The T-invariance condition (16) implies the vanishing of the functions G_7 and G_8. Further, the structure with the coefficient G_8 does not satisfy the symmetry property for identical particles, i. e., the coefficient changes its sign under the substitutions $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. So, it follows that $G_6 = 0$.

Thus, the full Lorentz reflection-invariant amplitude for NN scattering has the form (19) with

$$G_6 = G_7 = G_8 = 0.$$

This expression agrees with the well-known result for NN scattering[1])

$$T_{NN}(P_1, P_2, P_3, P_4) =$$

$$= G\, 1 \times 1 + G_2 \,|\, \hat{K} \times 1 + 1 \times \hat{P} \,| +$$

$$+ G_3 \hat{K} \times \hat{P} + G_4 + \gamma_5 \hat{K} \times \mathrm{i}\, \gamma \hat{P} + G_5 \,\mathrm{i}\, \gamma_5 \times \mathrm{i}\, \gamma_5,$$

where $K = P_2 + P_4$, $P = P_1 + P_3$, as can be seen with the help of the formula

$$T_{NN}(P_1, z_1;\ P_2, z_2;\ P_3, z_3;\ P_4, z_4) =$$

$$= \tilde{\xi}_1 (\hat{P}_1 + M) \tilde{\xi}_2 (\hat{P}_2 + M)\, T_{NN}(P_1, P_2, P_3, P_4)(-\hat{P}_3 + M)\, \xi_3 (-\hat{P}_4 + M)\, \xi_4.$$

5. *Conclusions*

Summarizing, we may draw the conclusion that a prescription for the invariant expansion of the scattering amplitude for particles of arbitrary spin has been found.

Furthermore, we are able to construct the modulus of the amplitude which is related to the cross-section of a scattering process

$$| T_{s_1 s_2 s_3 s_4}(P_1, e_1;\ P_2, e_2\ P_3, e_3) |^2 = \frac{1}{(2s_1)!\,(2s_2)!\,(2s_3)!\,(2s_4)!} \cdot$$

$$\cdot\ \overline{T_{s_1 s_2 s_3 s_4}}(P_a, e_a, \partial_{z_a})\, T_{s_1 s_2 s_3 s_4}(P_a, e_a, z_a),$$

where the complex conjugation concerns only the invariant scalar amplitudes.

In order to find an explicit covariant expansion of the amplitude and hence the cross-section for particles of any spin, it might be useful to apply the so-called terminal systems of computer technique (see Ref.[8]).

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Appendix I.

Some useful differentiation formulae and identities. Let z, a, b, c, . . . be two-component spinors, A, B, . . . are $2 \cdot 2$ matrices and $\partial_z = \begin{pmatrix} \partial_{z_1} \\ \partial_{z_2} \end{pmatrix}$.

We have

$$(a\,\partial_z)(bz) = (ab),$$

$$\frac{(a\,\partial_z)^n (bz)^m}{n!\,m!} = \frac{(ab)^n (bz)^{m-n}}{n!\,(m-n)!},$$

$$\frac{(a\,\partial_z)^\alpha (bz)^\beta (cz)^\gamma}{\alpha!\,\varrho!\,\gamma!\,\gamma!} = \sum_{k=0}^{\alpha} \frac{(ba)^k (ca)^{\alpha-k} (bz)^{\beta-k} (cz)^{\gamma-\alpha-k}}{k!\,(\alpha-k)!\,(\varrho-k)!\,(\gamma-\alpha-k)!},$$

$$\frac{(a\,\partial_z)^\alpha (bz)^\beta (cz)^\gamma (dz)^\delta}{\alpha!\,\varrho!\,\gamma!\,\delta!} \tag{I.1}$$

$$= \sum_{s+t+u=\alpha} \frac{(ba)^s (ca)^t (da)^u (bz)^{\varrho-s} (cz)^{\gamma-t} (dz)^{\delta-u}}{s!\,t!\,u!\,(\varrho-s)!\,(\gamma-t)!\,(\delta-u)!},$$

and

$$\frac{(a\,\partial_z)^n (zAz)^\alpha}{n!\,\alpha!} \quad \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(zAz)^{\alpha-n+k} (aAz+zAa)^{n-2k} (aAa)^k}{(\alpha-n+k)!\,(n-2k)!\,k!}$$

$$\frac{(a\,\partial_z)^n (zAz)^\alpha (zBz)^\beta}{n!\,\alpha!\,\varrho!} \quad \sum_{k=0}^{n} \sum_{l=0}^{\left[\frac{n-k}{2}\right]} \sum_{m=0}^{\left[\frac{k}{2}\right]} \frac{(zAz)^{\alpha-n+k+l}}{(\alpha-n+k+l)!} \times \tag{I.2}$$

$$\times \frac{(aAz+zAa)^{n-k+2l} (aAa)^l (aBz+zBa)^{k+2m} (aBa)^m}{(n-k+2l)!\;\; l!\,(k+2m)!\;\; m!}$$

It is also useful to note the binomial equality and the important identity

$$(u\,\varepsilon\,v)\,(u\,\varepsilon\,b) = (ua)\,(vb) - (ub)\,(va). \tag{I.3}$$

Appendix II.

The Dirac formalism and identities for Bargmann-Wigner spinors. Let

$$\{\gamma_\mu\,\gamma_\nu\} = 2g_{\mu\nu},$$

where

$$(g_{\mu\nu}) = \begin{pmatrix} I & 0 \\ 0 & -\underline{I} \end{pmatrix}.$$

For the Dirac matrices we choose the representation

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}, \; \gamma^5 = \gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = -i\begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}.$$

We also use the matrices

$$C = -C^{-1} = -{}^tC = -\overset{*}{C} = \mathrm{i}\,\gamma^0\,\gamma^2 = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$$

$$V_T = -V_T^{-1} = -{}^tV_T = -\overset{*}{V}_T = \mathrm{i}\,\gamma^0\,\gamma^5\,C = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix},$$

with the properties

$$\gamma^0\,\gamma^\mu\,\gamma^0 = g^{\mu\mu}\,\gamma^\mu = \overset{*}{\gamma}{}^\mu,$$

$$C\,\gamma^\mu\,C^{-1} = -{}^t\gamma^\mu,$$

$$V_T\,\gamma^\mu\,V_T^{-1} = \overline{\gamma}{}^\mu.$$

As usual, we denote

$$\hat{P} \equiv P_\mu\,\gamma^\mu = \begin{pmatrix} O & P \\ \widetilde{P} & \widetilde{O} \end{pmatrix},$$

where

$$\underset{\sim}{P} \equiv P^\mu\,\sigma_\mu = P_0 - \underline{\vec{P}}\cdot\underline{\vec{\sigma}},\ \widetilde{P} \equiv g_{\mu\mu}\,P^\mu\,\sigma_\mu = P_0 + \underline{P}\cdot\underline{\sigma}.$$

The two-dimensional representation of the Lorentz boost is

$$B(\underset{\sim}{P}) = \frac{M + \underset{\sim}{P}}{\sqrt{2M(\omega + M)}},$$

where

$$\omega = +(M^2 + P^2)^{\frac{1}{2}} = |P_0|.$$

It is seen that the boost matrix is positive definite. One readily verifies that

$$|B(\underset{\sim}{P})|^2 = \frac{P}{M},$$

or equivalently

$$\underset{\sim}{P} B(\tilde{P}) = MB(\underset{\sim}{P}).$$

The useful relation

$$\varepsilon \underset{\sim}{P} \varepsilon^{-1} = {}^t\underset{\sim}{P} = \overset{\approx}{P}$$

follows from

$$t_{\sigma_\mu} = \varepsilon \sigma^\mu \varepsilon^{-1}.$$

The solution of the Dirac equation

$$(\hat{P} - eM) U^e_{1/2,\zeta}(P) = 0,$$

where the charge $e = \pm 1$ and the spin projection $\zeta = \pm \frac{1}{2}$, have the form

$$U^e_{1/2,\zeta}(P) = \begin{pmatrix} B(\underset{\sim}{P}) l_{e\zeta} \\ eB(\tilde{P}) l_{e\zeta} \end{pmatrix}.$$

The two-component spinors e_ζ are determined from the equations

$$\sigma_3 l_\zeta = \zeta l_\zeta,$$

$$\varepsilon l_\zeta = -(-1)^{\frac{1}{2}-\zeta} l_{-\zeta},$$

with the normalization

$$\overset{*}{l}_\zeta l_{\zeta'} = M \delta_{\zeta\zeta'}.$$

It follows that the spinors $U^{(e)}_{1/2,\zeta}(P)$ satisfy the conditions

$$\overset{*}{U}{}^{(e)}_{1/2,\zeta}(P)\, U^{(e)}_{1/2,\zeta'}(P) = 2\, P_0\, \delta_{\zeta\zeta'},$$

$$\widetilde{U}{}^{(e)}_{1/2,\zeta}(P)\, U^{(e)}_{1/2,\zeta'}(P) = 2Me\, \delta_{\zeta\zeta'}$$

and the completeness relation

$$\sum_{\zeta} U^{(e)}_{1/2,\zeta}(P) \times \widetilde{U}{}^{(e)}_{1/2,\zeta}(P) = \hat{P} + eM.$$

It is not difficult to verify the following identities:

$$U^{(e)}_{1/2,\zeta}(P) = \gamma^0 \frac{\overset{*}{\hat{P}}}{M} U^{(e)}_{1/2,\zeta}(I_s\, P), \tag{II.1}$$

$$\widetilde{U}{}^{(e)}_{1/2,\zeta}(P) = \widetilde{U}{}^{(e)}_{1/2,\zeta}(P) \frac{\overset{*}{\hat{P}}}{M} \gamma^0;$$

$$U^{(e)}_{1/2,\zeta}(P) = (-1)^{1/2-\zeta}\, \widetilde{U}{}^{(e)}_{1/2,-\zeta}(I_s\, P) \frac{\overset{*}{\hat{P}}}{M} {}^tV_T, \tag{II.2}$$

$$\widetilde{U}{}^{(e)}_{1/2,\zeta}(P) = (-1^{1/2-\zeta}\, V_T \frac{\overset{*}{\hat{P}}}{M} U^{(e)}_{1/2,-\zeta}(I_s\, P);$$

$$U^{(e)}_{1/2,\zeta}(P) = (-1)^{1/2-\zeta}\, C\, \widetilde{U}{}^{(-e)}_{1/2,\zeta}(P) \tag{II.3}$$

$$\widetilde{U}{}^{(e)}_{1/2,\zeta}(P) = (-1)^{1/2-\zeta}\, {}^tC\, U^{(-e)}_{1/2,\zeta}(P).$$

One can easily check that the above identities are also satisfied for the Bargmann--Wigner spinors if the Dirac algebra is constructed as in Equ. (4).

Appendix III

Table of P-transformation laws for invariant structures. If

$$M_1 = M_3 = M,\ M_2 = M_4 = M,$$

we have the following transformation properties for space reflection:

$$I_s u_0 (13) = \frac{1}{4M^2} [(-2 + t - u) u_0 (13) - u_1 (13) + u_3 (13)],$$

$$I_s u_1 (13) = \frac{1}{4M^2} [-4t (u + SU) u_0 (13) - (s + t - u) u_1 (13) - 2(u + SU) u_3 (13)]$$

$$I_s u_2 (13) = u_2 (13),$$

$$I_s u_3 (13) = \frac{1}{4M^2} [4t (s + SU) u_0 (13) - 2 (s + SU) u_1 (13) + (s - t - u) u_3 (13)];$$

$$I_s u_0 (24) = \frac{1}{4m^2} [(s - t - u) u_0 (24) - u (24) - u_3 (24)],$$

$$I_s u_1 (24) = \frac{1}{4m^2} [-4t (u + SU) u_0 (24) - (s + t - u) u_1 (24) + 2 (u + SU) u_3 (24)],$$

$$I_s u_2 (24) = u_2 (24),$$

$$I_s u_3 (24) = \frac{1}{4m^2} [-4t (s + SU) u_0 (24) + 2 (s + SU) u_1 (24) + (s - t - u) u_3 (24)].$$

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RAZVOJ AMPLITUDE RASPRŠENJA ZA ČESTICE PROIZVOLJNOG SPINA UZ OČUVANJE LORENTZ INVARIJANTNOSTI

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Sadržaj

Razvoj amplitude raspršenja za čestice proizvoljnog spina uz očuvanje Lorentz invarijantnosti dan je eksplicitno. Uvjeti za P, T i CPT invarijantnost su pokazani. Kao primjeri primjene metode, razmotreni su slučajevi πN i NN raspršenja.