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A CONSTRUCTION OF FREE MAXWELL'S FIELD FROM AN EIGHT COMPONENT DIRAC'S FIELD

J. BRANA and K. LJOLJE

University of Sarajevo, Sarajevo

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Abstract: The free electromagnetic field is considered in a spinor notation and a construction of the free electromagnetic field from a zero mass Dirac's field is given.

1. Introduction

A spinor description of the electromagnetic field has been investigated by several authors (Laporte and Uhlenbeck[1]), Oppenheimer[2]), Archibald[3]), Ohmura[4]), Good[5], Moses[6]), Perkins[7]), Pestov[8])). There were two main subjects related to this problem: spinor notation of the electromagnetic field[2,5,6]) and structure of electron[4]). Here we want to show how a free electromagnetic field can be constructed from a Dirac's field.

In the first part we derive a spinor notation of the free electromagnetic field in an eight component form and give its extension to a Dirac's field. The second part contains three unitary transformations of which two give this field in the form of the previous works[4,6]). Considerations of Perkins[7]) and Pestov[8]) can be similarly connected to this work. In the third part we give a construction of a free electromagnetic field from a zero mass Dirac's field. The procedure which we apply is oriented to the final conclusion.

2. *A Dirac's field as an extension of Maxwell's field*

The Maxwell's equation of free field

$$\begin{aligned}
&\operatorname{rot} \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \\
&\operatorname{div} \vec{H} = 0, \\
&\operatorname{rot} \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}, \\
&\operatorname{div} \vec{E} = 0,
\end{aligned} \tag{1}$$

can be written in the form

$$\frac{\partial}{\partial x_a} \bar{\eta}_a \psi = 0. \qquad (a = 0, 1, 2, 3), \tag{2}$$

where

$$x_0 = ct,\ x_1 = x,\ x_2 = y,\ x_3 = z,$$

$$\bar{\eta}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \bar{\eta}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\eta}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \bar{\eta}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

and

$$\psi = \begin{bmatrix} E_x \\ E_y \\ E_z \\ F \\ H_x \\ H_y \\ H_z \\ G \end{bmatrix} \tag{4}$$

The F and G are arbitrary functions.

Now we want that the Equ. (2) becomes a Dirac's equation. In order to achieve it we add »missing« 1 to the matrices $\bar{\eta}_a$ and obtain new matrices

$$\bar{\eta}_0 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = 1$$

$$\bar{\eta}_1 \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \eta_1$$

(5)

$$\bar{\eta}_2 \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \eta_2$$

$$\bar{\eta}_3 \to \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \eta_3$$

It is easy to prove that matrices η_i satisfy commutations rules

$$\eta_i \eta_k + \eta_k \eta_i = 2\delta_{ik} \, 1 . \tag{6}$$

The Equ. (2) then becomes

$$\left(\frac{\partial}{\partial x_0} + \vec{\eta} \, \nabla \right) \bar{\psi} = 0, \tag{7}$$

where $\vec{\eta} = (\eta_1, \eta_2, \eta_3)$ and

$$\tilde{\psi} = \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \\ \tilde{F} \\ \tilde{H}_x \\ \tilde{H}_y \\ \tilde{H}_z \\ G \end{bmatrix} \tag{8}$$

The mark on components of E and H means that there are new functions not equivalent to those of electromagnetic field. The functions F and G are now not arbitrary. They are connected to others and play a definite role in the new field. This is a consequence of the new η_i matrices.

Introducing matrices

$$\gamma_i = \eta_0 \eta_i \; ; \; \gamma_0 = \eta_0 \qquad \text{where} \qquad \eta_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{8 \times 8} \tag{9}$$

the equation (7) becomes

$$\partial_\mu \gamma_\mu \tilde{\psi} = 0. \tag{10}$$

In the usual vector notation the Equ. (7) has the form

$$\begin{aligned}
\operatorname{rot}\vec{H} &= \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \nabla G,\\
\operatorname{div}\vec{H} &= -\frac{1}{c}\frac{\partial F}{\partial t},\\
\operatorname{rot}\vec{E} &= -\frac{1}{c}\frac{\partial H}{\partial t} - \nabla F,\\
\operatorname{div}\vec{E} &= -\frac{1}{c}\frac{\partial G}{\partial t}.
\end{aligned} \tag{11}$$

The components of this field satisfy the equations

$$\Box\, \psi_i = 0. \qquad (i = 1, 2, \ldots, 8) \tag{12}$$

If we put $G = F = 0$ in Equs. (11) they go over to the Equs. (1). We may say the Dirac's equations (10) plus the conditions $G = 0$, $F = 0$ are the Maxwell's equations (1).

However, if we leave $G \neq 0$, $F \neq 0$, the Equ. (10) or the Equs. (11) gives a Dirac's field different from the Maxwell's field. The new field is very »close« to the free Maxwell's field. This fact can be used for various physical reasonings. In this paper our goal is to make use of this equation in order to show how Maxwell's field can be constructed from a Dirac's field.

The Equ. (10) contains real as well as complex solutions.

3. *Some other representations of the Dirac's field*

In order to compare the Equ. (7) with the work of Ohmura[4)] and Moses[6)] and to see some properties of this field we give two other representations of the Equ. (7).

First let's perform a transformation by the unitary operator

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix}
i & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & i & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & i & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & -i & 0 & 0 & 0 & 1\\
-i & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & -i & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & -i & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & i & 0 & 0 & 0 & 1
\end{bmatrix}. \quad U^{-1} = U^{+} \tag{13}$$

The new matrices $\triangle_i = U\eta_i U^{-1}$ are

$$\triangle_i = -\begin{pmatrix}\delta_i & 0\\ 0 & \delta_i^*\end{pmatrix} \quad (i = 1, 2, 3), \tag{14}$$

where δ_i matrices from Ohmura's work[4)]

$$\delta_1 = \begin{bmatrix} 0 & 0 & 0 & -i\\ 0 & 0 & -i & 0\\ 0 & i & 0 & 0\\ i & 0 & 0 & 0\end{bmatrix}, \quad \delta_2 = \begin{bmatrix} 0 & 0 & i & 0\\ 0 & 0 & 0 & -i\\ -i & 0 & 0 & 0\\ 0 & i & 0 & 0\end{bmatrix}, \quad \delta_3 = \begin{bmatrix} 0 & -i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{bmatrix}. \tag{15}$$

The new wave function is

$$\psi_{\text{new}} = U\psi = \begin{bmatrix} \tilde{\vec{H}} + i\tilde{\vec{E}}\\ -iF + G\\ \tilde{\vec{H}} - i\tilde{\vec{E}}\\ iF + G\end{bmatrix}, \tag{16}$$

and the new equation

$$\frac{1}{c}\frac{\partial\,\psi_{\text{new}}}{\partial\,t} + \sum_i^3 \triangle_i \frac{\partial}{\partial x_i}\psi_{\text{new}} = 0. \tag{17}$$

The Equ. (17) is separated in two parts

$$\frac{1}{c}\frac{\partial}{\partial\,t}\psi_0 - \sum_i^3 \delta_i \frac{\partial}{\partial x_i}\psi_0 = 0, \tag{18}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\psi_0^* - \sum_{i=1}^3 \delta_i^* \frac{\partial}{\partial x_i}\psi_0^* = 0, \tag{19}$$

where is

$$\psi_0 = \begin{bmatrix} \tilde{\vec{H}} - i\tilde{\vec{E}}\\ ih - e\end{bmatrix} \tag{20}$$

and $e = -G,\ h = -F.$

The Equ. (18) is that which Ohmura[4)] used but for the free field ($\vec{j} = 0, \varrho = 0$).

Similarly, the corresponding equation in the work by Moses[6] one can get with transformation

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & i \\ i & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad S^{-1} = S^{+}. \tag{21}$$

The next representation wich we give here separates the Equ. (7) in two parts of usual Dirac's form. The unitary operator for this purpose is

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & -1 \\ 0 & 0 & -1 & -i & 0 & 0 & 0 & 0 \\ -1 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & -1 \\ 0 & 0 & 0 & 0 & -i & 1 & 0 & 0 \end{bmatrix} \quad S^{-1} = S^{+}. \tag{22}$$

The new matrices $a_i = S\eta_i S^{-1}$ are

$$a_i = 1 \otimes \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = 1 \otimes \sigma_i \otimes \sigma_i, \tag{23}$$

where σ_i are Pauli's matrices. The new wave function ψ' is

$$\psi' = S\psi = \begin{bmatrix} -\tilde{E}_x + i\tilde{E}_y \\ \tilde{E}_z - iF \\ -i\tilde{H}_x - \tilde{H}_y \\ i\tilde{H}_z - G \\ -\tilde{E}_z - iF \\ -\tilde{E}_x - i\tilde{E}_y \\ -i\tilde{H}_z - G \\ -i\tilde{H}_x + \tilde{H}_y \end{bmatrix}, \tag{24}$$

and the new equation

$$\frac{1}{c}\frac{\partial}{\partial t}\psi' + \vec{a}\cdot\nabla\psi' = 0. \tag{25}$$

Due to quasidiagonal form of a_i matrices this equation is separated in two equations

$$\begin{aligned}&\frac{1}{c}\frac{\partial\psi'_I}{\partial t} + \vec{a}\cdot\nabla\psi'_I = 0,\\&\frac{1}{c}\frac{\partial\psi'_{II}}{\partial t} + \vec{a}\cdot\nabla\psi'_{II} = 0.\end{aligned} \tag{26}$$

4. A connection of the free Maxwell field and an eight component Dirac field

A comparison of the Equs. (11) and the Equs. (1) shows a simple connection between the fields $(\vec{\tilde{E}}, F, \vec{\tilde{H}}, G)$ and $(\vec{E}, \vec{H})$

$$\begin{aligned}\vec{E} &= \vec{\tilde{E}} + c\,\nabla\int G(\vec{r}, t)\,\mathrm{d}t,\\\vec{H} &= \vec{\tilde{H}} + c\,\nabla\int F(\vec{r}, t)\,\mathrm{d}t.\end{aligned} \tag{27}$$

Let us emphasize that $(\vec{\tilde{E}}, F, \vec{\tilde{H}}, G)$ is a Dirac field and $(\vec{E}, \vec{H})$ a free Maxwellian field. Therefore, the Equs. (27) give a construction of a free Maxwellian field from a massless Dirac field. This result can be recognised also in opposite way.

Let us start from the Maxwell Equs. (1) and let us decompose $\vec{E}$ and $\vec{H}$ in the following way

$$\begin{aligned}\vec{E} &= \vec{\tilde{E}} + c\,\nabla\,\tilde{G},\\\vec{H} &= \vec{\tilde{H}} + c\,\nabla\,\tilde{F}.\end{aligned} \tag{28}$$

Inserting (28) into (1) we get

$$\begin{aligned}
\operatorname{rot} \vec{\tilde{H}} &= \frac{1}{c}\frac{\partial \vec{\tilde{E}}}{\partial t} + \nabla \frac{\partial}{\partial t}\tilde{G},\\
\operatorname{div} \vec{\tilde{H}} &= -\frac{1}{c}\frac{\partial^2}{\partial t^2}\tilde{F},\\
\operatorname{rot} \vec{\tilde{E}} &= -\frac{1}{c}\frac{\partial \vec{\tilde{H}}}{\partial t} - \nabla \frac{\partial}{\partial t}\tilde{F},\\
\operatorname{div} \vec{\tilde{E}} &= -\frac{1}{c}\frac{\partial^2}{\partial t^2}\tilde{G}.
\end{aligned} \tag{29}$$

Denoting $\frac{\partial \vec{\tilde{G}}}{\partial t} = G$ and $\frac{\partial \vec{\tilde{F}}}{\partial t} = F$ the system (29) goes over to the system (11). Thus the relations (28) and (29) give a decomposition of a free Maxwellian field.

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KONSTRUKCIJA SLOBODNOG MAXWELLOVOG POLJA IZ OSMO-KOMPONENTNOG DIRACOVOG POLJA

J. BRANA i K. LJOLJE

Univerzitet u Sarajevu, Sarajevo

Sadržaj

Spinorsko opisivanje elektromagnetskog polja razmatrano je od više autora (Laporte i Uhlenbeck[1]), Oppenheimer[2]), Archibald[3]), Ohmura[4]), Good[5]), Moses[6]), Perkins[7]), Pestov[8])). U ovim razmatranjima bila su dva glavna problema:

formalna spinorska notacija elektromagnetskog polja i struktura elektrona. U radu se pokazuje, kako se slobodno elektromagnetsko polje može konstruirati iz jednog Diracovog polja.

U prvom dijelu je dan spinorski prikaz elektromagnetskog polja u osmokomponent nom obliku i prijelaz na Diracovo polje. Drugi dio sadrži transformacije dobivenih jednadžbi na oblike od Ohmure[4)] i Mosesa[6)] i separaciju na četverokomponentna Diracova polja. U trećem dijelu je data izgradnja slobodnog elektromagnetskog polja iz Diracovog polja bez mase.