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SECULAR FREE SOLUTION UP TO THIRD ORDER OF AN ELECTROMAGNETIC WAVE IN A COLD DISSIPATIVE MAGNETOACTIVE PLASMA MEDIUM.

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Abstract: The perturbation method of Lindstedt is applied to study the relativistic nonlinear effects of a circularly polarized transverse monochromatic wave in a cold dissipative plasma medium in the presence of static magnetic field which is along the direction of propagation of the wave. Amplitude dependent wave length and frequency shifts including relativistic corrections are derived.

1. Introduction

Nonlinear wave length and frequency shifts for various types of monochromatic waves travelling in a plasma medium have recently been obtained by several authors[1-5]. Boyd[4] obtained amplitude dependent frequency shift for standing extraordinary waves with propagation vector perpendicular to the direction of the magnetic field, whereas Das[5] obtained wave length and frequency shifts for propagating wave of extraordinary mode when the static magnetic field is perpendicular to the direction of propagation. All of them have taken non-dissipative plasma medium, moreover they have used the method of Bogoljubov, Krylov and Mitropolsky. Here we obtained the nonlinear wavelength and frequency shifts of a circularly polarized transverse monochromatic wave in a cold dissipative plasma including relativistic corrections in the presence of static magnetic field which is along the direction of propagation. Though the earlier authors have invariably used the Bogoljubov, Krylov and Mitropolsky method, we have adopted, for the first time perhaps in the plasma theories, the method of Lindstedt[6-8] which seems to be more convenient than others particularly for lengthy calculations such as those of these papers. Here we obtained two different frequency shifts or two wave number shifts which agree with our previous result* in the absence of static magnetic field. Again it is found that the frequency shifts further towards the red and the wave length contracts more due to the presence of small external magnetic field.

2. *Fundamental equations*

The set of cold plasma equations including relativistic effects and dissipation and Maxwell's equations are

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\vec{v}) = 0, \tag{2}$$

$$\left\{\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right\} \frac{\vec{v}}{(1 - v^2/c^2)^{1/2}} = -\frac{e}{m}\vec{E} - \frac{e}{mc}[\vec{v}\times\vec{H}] - \nu\vec{v}, \tag{2}$$

$$\operatorname{Curl}\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t}, \tag{3}$$

$$\operatorname{Curl}\vec{H} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} - \frac{4\pi e n v}{c}, \tag{4}$$

$$\operatorname{dir}\vec{E} = -4\pi e(n - n_0), \tag{5}$$

$$\operatorname{dir}\vec{H} = 0. \tag{6}$$

Here n is the electron number density, $\vec{v}$ is the electron velocity, $\vec{E}$ and $\vec{H}$ are the electric and magnetic field intensities, c is the velocity of light, n_0 is the uniform background ion density and ν is the collisional frequency, which is constant and greater than zero.

Since $\frac{v^2}{c^2} << 1$, the Equ. (2), correct up to third order reduces to

$$\left\{\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right\}\left(\vec{v} + \frac{\vec{v}\,v^2}{2c^2}\right) = -\frac{e\vec{E}}{m} - \frac{e}{mc}[\vec{v}\times\vec{H}] - \nu\vec{v}. \tag{7}$$

* Secular free solution upto third order of relatistic cold dissipative plasma equations for electromagnetic field (Unpublished).

The Maxwell's equations reduce to

$$\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \nabla^2 \vec{E} = 4\pi e \frac{\partial}{\partial t}(n \vec{v}) - c^2 \text{ grad div } \vec{E}, \tag{8}$$

$$\frac{\partial^2 \vec{H}}{\partial t^2} - c^2 \nabla^2 \vec{H} = -4\pi e c \text{ Curl } (n \vec{v}). \tag{9}$$

Series solution of these Equs. is sought in the form

$$\begin{aligned} \vec{v} &= \varepsilon \vec{v}_1 + \varepsilon^2 \vec{v}_2 + \varepsilon^3 \vec{v}_3 + \ldots, \\ \vec{E} &= \varepsilon \vec{E}_1 + \varepsilon^2 \vec{E}_2 + \varepsilon^3 \vec{E}_3 + \ldots, \\ \vec{H} &= \vec{H}_0 + \varepsilon \vec{H}_1 + \varepsilon^2 \vec{H}_2 + \varepsilon^3 \vec{H}_3 + \ldots, \\ n &= n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \varepsilon^3 n_3 + \ldots. \end{aligned} \tag{10}$$

The variables having ε as coefficients are in general functions of space and time In initial value problem the space variable is not expanded, the time variable t is expressed in power series as

$$t + \vec{a} = s(1 + \varepsilon a_1 + \varepsilon^2 a_2 + \varepsilon^3 a_3 + \ldots), \tag{11}$$

where $\vec{a}$ is included to adjust the phase and $a_1, a_2, a_3, \ldots$ are constants. These constants should be chosen appropriately so that terms fielding secular behaviour after integration are removed.

We then have

$$\frac{\partial^r}{\partial t^r} = \frac{1}{(1 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots)^2} \frac{\partial^r}{\partial^r s}, \qquad r = 1, 2. \tag{12}$$

3. *Approximation up to third order*

For transverse monochromatic wave, using the Equs. (7) to (12) we obtained the first order approximations as

$$\frac{\partial \vec{v}_1}{\partial s} + \frac{e \vec{E}_1}{m} + \frac{e}{m c} [\vec{v}_1 \times \vec{H}_0] + \nu \vec{v}_1 = 0,$$

$$\frac{\partial^2 \vec{E}_1}{\partial s^2} - c^2 \nabla^2 \vec{E}_1 = \frac{m\,\omega_0^2}{e} \frac{\partial \vec{v}_1}{\partial s}, \tag{13}$$

and

$$\frac{\partial^2 H_1}{\partial s^2} - c^2 \nabla^2 \vec{H}_1 = - \frac{m\omega_0^2 c}{e} [\nabla \times \vec{v}_1],$$

where ω_0 is the plasma frequency.

For secular free behaviour in the case of second order approximation as with the earlier authors we find that $a_1 = 0$ and so the second order approximation reduces to

$$\frac{\partial^2 \vec{v}_2}{\partial s^2} + \omega_0^2 \vec{v}_2 + \nu \frac{\partial \vec{v}_2}{\partial s} + \frac{e}{mc}\left[\frac{\partial \vec{v}_2}{\partial s} \times \vec{H}_0\right] = - \frac{e}{mc}\frac{\partial}{\partial s}[\vec{v}_1 \times \vec{H}_1],$$

$$\frac{\partial^2 \vec{E}_2}{\partial s^2} - c^2 \nabla^2 E_2 = \frac{m\,\omega_0^2}{e}\frac{\partial \vec{v}_2}{\partial s} - c^2 \text{ grad div } \vec{E}_2, \tag{14}$$

and

$$\frac{\partial^2 \vec{H}_2}{\partial s^2} - c^2 \nabla^2 \vec{H}_2 + \frac{m\,c\,\omega_0^2}{e}[\vec{\nabla} \times \vec{v}_2] = 0.$$

This leads to the third order approximation for $\vec{E}_3$ as

$$\begin{aligned}
&\frac{\partial^2 \vec{E}_3}{\partial s^2} - c^2 \nabla^2 \vec{E}_3 + \omega_0^2 \vec{E}_3 + \frac{\omega_0^2}{c}[\vec{v}_3 \times \vec{H}_0] + \frac{m\,\omega_0^2}{e}\nu \vec{v}_3 + \\
&+ c^2 \text{ grad div} \vec{E}_3 = \frac{m\,\omega_0^2}{e}\left\{- \frac{\partial}{\partial s}\left(\frac{\vec{v}_1 v_1^2}{2c^2}\right) - (\vec{v}_2 \cdot \vec{\nabla})\vec{v}_1 - \right. \\
&- \frac{e}{mc}[\vec{v}_2 \times \vec{H}_1] + \left.\frac{1}{n_0}\frac{\partial}{\partial s}(n_2 \vec{v}_1)\right\} + 2a_2 \{c^2 \nabla^2 \vec{E}_1 - \omega_0^2 \vec{E}_1 - \\
&- \frac{\omega_0^2}{c}[\vec{v}_1 \times \vec{H}_0] - \frac{m\,\omega_0^2}{e}\nu \vec{v}_1\}.
\end{aligned} \tag{15}$$

4. *Result up to third order and conclusion*

When the static magnetic field lies along the direction of propagation of the wave, let the solution of the first order approximation for $\vec{v}_1$ in a dissipative plasma medium, along the positive direction of z axis be

$$\vec{v}_1 = v_0\, e^{r's} (\hat{e}_x \cos \psi + \hat{e}_y \sin \psi),$$

where

$$\psi = \vec{k} \cdot z - \omega, s, \qquad v_0 \text{ is real constant and}$$

$$\vec{H}_0 = \hat{e}_z\, H_0, \qquad \vec{k} = \hat{e}_z\, k. \tag{16}$$

$\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$ are unit vectors along the three perpendicular axes.

The dispersion relations are found to be

$$c^2 \varkappa^2 = \omega_1^2 - \frac{\omega_0^2\, \omega_1}{\omega_1 \pm \Omega},$$

and (17)

$$r_1 = - \frac{\nu\, \omega_0}{2\,(\omega_1 \pm \Omega)^2},$$

where $\pm$ denotes left hand and right hand rotations and Ω is the cyclotron frequency.

Then

$$\vec{E}_1 = - \frac{m\, v_0\, e^{r's}}{e} \{(r_1 + \nu)(\hat{e}_x \cos \psi + \hat{e}_y \sin \psi) + (\omega_1 + \Omega)(\hat{e}_x \sin \psi - \hat{e}_y \cos \psi)\}, \tag{18}$$

$$\vec{H}_1 = - \frac{m\, c\, v_0\, \vec{k}\, e^{r's}}{e(r_1^2 + \omega_1^2)} \{(r_1^2 + r\,\nu + \omega_1^2 + \omega_1\, \Omega) \cdot (\hat{e}_y \operatorname{Sin} \psi + \hat{e}_x \cos \psi) +$$

$$+ (\omega_1\, \nu - r_1\, \Omega)(\hat{e}_y \cos \psi - \hat{e}_x \sin \psi)\},$$

$n_1 = 0,$

and

$$\vec{v}_2 = \frac{2r_1\, v_0^2\, \vec{k}\, (\omega_1 \nu - r_1\, \Omega)\, e^{2r_1 s}}{(\omega_1^2 + r_1^2)\,(4r_1^2 + 2r_1 \nu + \omega_0^2)},$$

$$\vec{E}_2 = \frac{v_0^2\, \omega_0^2\, m\, \vec{k}\, (\omega_1\, \nu - r_1\, \Omega)\, e^{2r_1 s}}{e\,(\omega_1^2 + r_1^2)\,(4r_1^2 + 2r_1\, \nu + \omega_0^2)}, \tag{19}$$

$$n_2 = 0.$$

We find that the right hand side of the Equ. (15) is linear combination of terms proportional to $\hat{e}_x \cos\psi$, $\hat{e}_y \sin\psi$, $\hat{e}_x \cos\psi$ and $\hat{e}_y \sin\psi$. In order to avoid secular behaviour in the third order solution, the co-efficient of $\hat{e}_x \cos\psi$, $\hat{e}_y \sin\psi$, $\hat{e}_x \cos\psi$ and $\hat{e}_y \sin\psi$ must vanish. We get two different expressions for a_2. Consequently two different frequency shifts result

$$\delta\omega_2' = -\frac{\omega_0^2 v_0^2 \omega_1 r_1 e^{2r_1 s}}{4c^2\{(c^2k^2+\omega_0^2)(r_1+\nu)-\omega_0^2\nu\}}\Bigg\{3 + \frac{4c^2k^2\omega_1^2\nu^2}{(r_1^2+\omega_1^2)^2(4r_1^2+2r_1\nu+\omega_0^2)} + \frac{4c^2k^2 r_1\Omega(r_1\Omega-2\omega_1\nu)}{(r_1^2+\omega_1^2)^2(4r_1^2+2r_1\nu+\omega_0^2)}\Bigg\}$$

$$\delta\omega_2'' = -\frac{\omega_0^2 v_0^2 \omega_1^2 e^{2r_1 s}}{4c^2\{(c^2k^2+\omega_0^2)(\omega_1+\Omega)-\omega_0^2\Omega\}}\Bigg\{1 + \frac{4c^2k^2\nu^2 r_1^2}{(r_1^2+\omega_1^2)^2(4r_1^2+2r_1\nu+\omega_0^2)} + \frac{4k^2c^2 r_1\Omega(\omega_1^2\nu-r_1^2\nu-r_1\omega_1\Omega)}{\omega_1(r_1^2+\omega_1^2)^2(4r_1^2+2r_1\nu+\omega_0^2)}\Bigg\}. \tag{20}$$

The first part in the brackets is due to the relativistic effects, the third part to the external magnetic field. In the absence of the external magnetic field, the result obtained, totally agrees with our previous result. Moreover we find that the circularly polarized wave splits up into two polarized waves propagating with different phase and group velocities, coming both from relativistic and non-relativistic effect. Again, in the case of small external magnetic field, the last two terms in the bracket in both the equations are positive. Hence the frequency shifts move towards the red due to the presence of small external magnetic field.

In the boundary value problem, the time variable is not expanded, the space variable $\vec{r}$ is expressed in power series as

$$\vec{r} + \vec{a} = \vec{R}(1 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots),$$

and

$$\vec{\nabla} = \frac{\partial}{\partial \vec{r}} = \frac{1}{(1 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots)}\vec{\nabla}_0,$$

where

$$\vec{\nabla}_0 = \frac{\partial}{\partial \vec{R}}. \tag{21}$$

The same procedure gives the following expression for the wave number shifts

$$\delta k_2' = \frac{\vec{k}\, v_0^2 r_1 \omega_0^2 e^{2r_1 t}}{4c^4 k^2 (r_1 + \nu)} \cdot \left\{ 3 + \frac{4c^2 k^2 \omega_1^2 \nu^2}{(\omega_1^2 + r_1^2)^2 (4r_1^2 + 2r_1 \nu + \omega_0^2)} + \right.$$

$$\left. + \frac{4c^2 k^2 r_1 \Omega (r_1 \Omega - 2\omega_1 \nu)}{(\omega_1^2 + r_1^2)^2 (4r_1^2 + 2r_1 \nu + \omega_0^2)} \right\},$$

and (22)

$$\delta k_2'' = \frac{\vec{k}\, v_0^2 \omega_1 \omega_0^2 e^{2r_1 t}}{4c^4 k^2 (\omega_1 + \Omega)} \left\{ 1 + \frac{4c^2 k^2 r_1^2 \nu^2}{(\omega_1^2 + r_1^2)^2 (4r_1^2 + 2r_1 \nu + \omega_0^2)} + \right.$$

$$\left. + \frac{4c^2 k^2 r_1 \Omega (\omega_1^2 \nu - r_1^2 \nu - r_1 \omega_1 \Omega)}{\omega_1 (r_1^2 + \omega_1^2)^2 (4r_2^2 + 2r_1 \nu + \omega_0^2)} \right\}.$$

The first part in the brackets is due to the relativistic effects the third part to the external magnetic field. In the absence of the external magnetic field, we obtained the same result as previously. Again in the case of small external magnetic field, the last two terms in the bracket taken together in both the equations are positive. Hence we conclude that the wave length contracts more due to the presence of a small external magnetic field.

Since $\omega_1 \gg \omega_0$, r_1 and ν are small, the Equs. (20) and (22) reduce to

$$\delta k_2' \approx -\frac{\vec{k}}{\omega_1} (\delta \omega_2') \text{ and } \delta k_2'' \approx -\frac{\vec{k}}{\omega_1} (\delta \omega_2''),$$

where Equ. (11) gives the relation between t and s. Under this restriction the relation $\delta(\vec{k}\omega) = 0$ can be profitably used to obtain the increment in $\vec{k}$ if the increment in ω is known and vice-versa.

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