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SOME ADVANTAGES OF SPINOR NOTATION OF ELECTROMAGNETIC FIELD

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Abstract: The spinor notation of the electromagnetic field[1] offers a simple way of introduction of potentials in the theory. Separation in field components, and charge conservation, is also simple.

1. Introduction

In the paper[1] it was shown that the Maxwell's equations for free electromagnetic field can be written in a spinor notation in the following way:

$$\left(\frac{1}{c}\,\partial_t + \vec{\nabla}\,\vec{\eta}\right)\Psi = 0, \tag{1.1}$$

$$F = G = 0; \tag{1.2}$$

where η_i $(i = x, y, z)$ are the Dirac's matrices and

$$\Psi = \begin{bmatrix} E_x \\ E_y \\ E_z \\ F \\ H_x \\ H_y \\ H_z \\ G \end{bmatrix} \tag{1.3}$$

Equation (1.1) is the Dirac's equation

$$\left(\frac{1}{c}\,\partial_t + \vec{\nabla}\,\vec{\eta} + i\varkappa\beta\right)\Psi = 0 \qquad (1.4)$$

for massless particles ($\varkappa = 0$).

The spinor notation of electromagnetic field has been investigated in several other papers[2-6]. Here, we rely on the paper[1].

Let's write equation (1.1) in explicit form also

$$\begin{bmatrix} \frac{1}{c}\partial_t & 0 & 0 & 0 & 0 & \partial_z & -\partial_y & \partial_x \\ 0 & \frac{1}{c}\partial_t & 0 & 0 & -\partial_z & 0 & \partial_x & \partial_y \\ 0 & 0 & \frac{1}{c}\partial_t & 0 & \partial_y & -\partial_x & 0 & \partial_z \\ 0 & 0 & 0 & \frac{1}{c}\partial_t & \partial_x & \partial_y & \partial_z & 0 \\ 0 & -\partial_z & \partial_y & \partial_x & \frac{1}{c}\partial_t & 0 & 0 & 0 \\ \partial_z & 0 & -\partial_x & \partial_y & 0 & \frac{1}{c}\partial_t & 0 & 0 \\ -\partial_y & \partial_x & 0 & \partial_z & 0 & 0 & \frac{1}{c}\partial_t & 0 \\ \partial_x & \partial_y & \partial_z & 0 & 0 & 0 & 0 & \frac{1}{c}\partial_t \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \\ F \\ H_x \\ H_y \\ H_z \\ G \end{bmatrix} = 0, \qquad (1.5)$$

or

$$\begin{aligned} \operatorname{rot}\vec{H} &= \frac{1}{c}\frac{\partial E}{\partial t} + \nabla G, \\ \operatorname{div}\vec{H} &= -\frac{1}{c}\frac{\partial F}{\partial t} \\ \operatorname{rot}\vec{E} &= -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} - \nabla F, \\ \operatorname{div}\vec{E} &= -\frac{1}{c}\frac{\partial G}{\partial t}. \end{aligned} \qquad (1.6)$$

Taking $F = G = 0$ we see that equations (1.6) go over to the Maxwell's equations for free electromagnetic field. The notation $\vec{E}, \vec{H}$ and similarly further is convenient but these quantities are not generally vectors.

Equations (1.6) are not separated in unknown functions (E_i, H_i). In order to solve these equations one introduces potentials in such a way that equations for them are separated in new unknown functions. This procedure turns out to be very simple, almost immediately evident, in the spinor notation of electromagnetic field, and we want to present it here. At the same time it opens a different view to the Dirac's field what we will discuss at another place.

The operator

$$\frac{1}{c}\partial_t - \vec{\nabla}\vec{\eta} \tag{1.7}$$

in the product with the operator $\left(\frac{1}{c}\partial_t + \vec{\nabla}\vec{\eta}\right)$ gives the D'Alembertian operator $\square$ or the operator of the Klein-Gordon equation for massless particles,

$$\left(\frac{1}{c}\partial_t - \vec{\nabla}\vec{\eta}\right)\left(\frac{1}{c}\partial_t + \vec{\nabla}\vec{\eta}\right) = \left(\frac{1}{c}\partial_t + \vec{\nabla}\vec{\eta}\right)\left(\frac{1}{c}\partial_t - \vec{\nabla}\vec{\eta}\right) = \square. \tag{1.8}$$

Knowing it we may define a new quantity Φ by which the spinor Ψ is determined

$$\Psi = \left(\frac{1}{c}\partial_t - \vec{\nabla}\vec{\eta}\right)\Phi, \tag{1.9}$$

and, due to (1.1) and (1.8), satisfies the equation

$$\square\Phi = 0. \tag{1.10}$$

Let's mention that Φ given by (1.9) is arbitrary to a transformation

$$\Phi' = \Phi + \chi, \tag{1.11}$$

where χ satisfies the equation

$$\left(\frac{1}{c}\partial_t - \vec{\nabla}\vec{\eta}\right)\chi = 0. \tag{1.12}$$

One can easily see that (1.11) is the gauge transformation. Because of diagonal form of the operator $\square$, Eqs. (1.10) are separated in Φ_a.

In Section 2, we apply this procedure to the free electromagnetic field and in Section 3 to the electromagnetic field with sources. Section 4 contains separation in field components and charge conservation law.

2. Potentials of free electromagnetic field

According to (1.9) we have

$$
\begin{bmatrix} E_x \\ E_y \\ E_z \\ F \\ H_x \\ H_y \\ H_z \\ G \end{bmatrix}
=
\begin{bmatrix}
\frac{1}{c}\partial_t & 0 & 0 & 0 & 0 & -\partial_x & \partial_y & -\partial_x \\
0 & \frac{1}{c}\partial_t & 0 & 0 & \partial_z & 0 & -\partial_x & -\partial_y \\
0 & 0 & \frac{1}{c}\partial_t & 0 & -\partial_y & \partial_x & 0 & \partial_z \\
0 & 0 & 0 & \frac{1}{c}\partial_t & -\partial_x & -\partial_y & -\partial_z & 0 \\
0 & \partial_z & -\partial_y & -\partial_x & \frac{1}{c}\partial_t & 0 & 0 & 0 \\
-\partial_z & 0 & \partial_x & -\partial_y & 0 & \frac{1}{c}\partial_t & 0 & 0 \\
\partial_y & -\partial_x & 0 & -\partial_z & 0 & 0 & \frac{1}{c}\partial_t & 0 \\
-\partial_x & -\partial_y & -\partial_z & 0 & 0 & 0 & 0 & \frac{1}{c}\partial_t
\end{bmatrix}
\begin{bmatrix} -A_x^{(1)} \\ -A_y^{(1)} \\ -A_z^{(1)} \\ \varphi^{(2)} \\ -A_x^{(2)} \\ -A_y^{(2)} \\ -A_z^{(2)} \\ \varphi^{(1)} \end{bmatrix}
\tag{2.1}
$$

where we have taken the components of Φ with plus or minus sign for convenience of further calculation. The notation is chosen in accordance with final results and usual notation.

Equations (2.1) can be written also in the form

$$
\begin{bmatrix} \vec{E} \\ F \\ \vec{H} \\ G \end{bmatrix}
= \left(\frac{1}{c}\partial_t - \vec{\nabla}\vec{\eta}\right)
\begin{bmatrix} -\vec{A}^{(1)} \\ 0 \\ \vec{0} \\ \varphi^{(1)} \end{bmatrix}
+ \left(\frac{1}{c}\partial_t - \vec{\nabla}\vec{\eta}\right)
\begin{bmatrix} \vec{0} \\ \varphi^{(2)} \\ -\vec{A}^{(2)} \\ 0 \end{bmatrix}
\tag{2.2}
$$

Evaluating the right side we find

$$
\vec{E} = -\frac{1}{c}\partial_t \vec{A}^{(1)} - \nabla\varphi^{(1)} - \operatorname{rot}\vec{A}^{(2)},
$$

$$
F = \frac{1}{c}\partial_t \varphi^{(2)} + \operatorname{div}\vec{A}^{(2)},
$$

$$\vec{H} = \operatorname{rot} \vec{A}^{(1)} - \frac{1}{c} \partial_t \vec{A}^{(2)} - \nabla \varphi^{(2)}, \tag{2.3}$$

$$G = \frac{1}{c} \partial_t \varphi^{(1)} + \operatorname{div} \vec{A}^{(1)}.$$

Usual forms of the electric and magnetic vectors are

$$\vec{E} = -\frac{1}{c} \partial_t \vec{A} - \nabla \varphi,$$

$$\vec{H} = \operatorname{rot} \vec{A}. \tag{2.4}$$

Therefore, if we write

$$\vec{E}^{(i)} = -\frac{1}{c} \partial_t \vec{A}^{(i)} - \nabla \varphi^{(i)}.$$

$$\vec{H}^{(i)} = \operatorname{rot} \vec{A}^{(i)}, \qquad i = 1,\ 2, \tag{2.5}$$

equations (2.3) become

$$\vec{E} = \vec{E}^{(1)} - \vec{H}^{(2)},$$

$$F = \frac{1}{c} \partial_t \varphi^{(2)} + \operatorname{div} \vec{A}^{(2)}, \tag{2.6}$$

$$\vec{H} = \vec{H}^{(1)} + \vec{E}^{(2)},$$

$$G = \frac{1}{c} \partial_t \varphi^{(1)} + \operatorname{div} \vec{A}^{(1)}.$$

The components of Φ ($\vec{A}^{(i)}$, $\varphi^{(i)}$, $i = 1,2$) satisfy, according to (1.10), the equations

$$\Box A_l^{(i)} = 0, \qquad l = x, y, z$$

$$\Box \varphi^{(i)} = 0, \qquad i = 1, 2. \tag{2.7}$$

The field (2, 6) is still Dirac's field. Now, let's take $F = G = 0$, in order to get Maxwell's field

$$\vec{E} = \vec{E}^{(1)} - \vec{H}^{(2)},$$

$$\vec{H} = \vec{H}^{(1)} + \vec{E}^{(2)},$$

$$\frac{1}{c}\,\partial_t\,\varphi^{(1)} + \operatorname{div}\vec{A}^{(1)} = 0, \tag{2.8}$$

$$\frac{1}{c}\,\partial_t\,\varphi^{(2)} + \operatorname{div}\vec{A}^{(2)} = 0,$$

$$\Box\,A_l^{(i)} = 0, \qquad l = x, y, z, \tag{2.9}$$

$$\Box\,\varphi^{(i)} = 0.$$

The Maxwell's equations of free field are invariant to transformations

$$\vec{E} \to -\vec{H},$$

$$\vec{H} \to \vec{E}. \tag{2.10}$$

Consequently, if $(\vec{E}^{(1)}, \vec{H}^{(1)})$, $(\vec{E}^{(2)}, \vec{H}^{(2)})$ are solutions of the Maxwell's equations, the linear combination

$$(\vec{E}^{(1)} - \vec{H}^{(2)},\ \vec{H}^{(1)} + \vec{E}^{(2)})$$

is also solution. Thus, we have in (2.8) two solutions of the Maxwell's equation. The first solution is determined by the potentials $(\vec{A}^{(1)}, \varphi^{(1)})$ and the second by the potentials $(\vec{A}^{(2)}, \varphi^{(2)})$. Let us emphasize that $F = G = 0$ give for the potentials the Lorentz's conditions (first two equations of (2.9)).

Due to equivalence of the solutions $(\vec{E}^{(1)}, \vec{H}^{(1)})$, $(\vec{E}^{(2)}, \vec{H}^{(2)})$ we may drop one. We chose $\vec{E}^{(2)} = \vec{H}^{(2)} = 0$, or $\vec{A}^{(2)} = 0$, $\varphi^{(2)} = 0$, and have $(\vec{A}^{(1)} \to \vec{A},\ \varphi^{(1)} \to \varphi)$ standard relations

$$\vec{E} = -\frac{1}{c}\,\partial_t\,\vec{A} - \nabla\varphi,$$

$$\vec{H} = \operatorname{rot}\vec{A},$$

$$\Box\,A_l = 0, \qquad l = x, y, z, \tag{2.11}$$

$$\Box\,\varphi = 0,$$

$$\frac{1}{c}\,\partial_t\,\varphi + \operatorname{div}\vec{A} = 0.$$

or

$$\begin{bmatrix} \vec{E} \\ 0 \\ \vec{H} \\ 0 \end{bmatrix} = \left(\frac{1}{c}\,\partial_t - \vec{\nabla}\,\vec{\eta}\right) \begin{bmatrix} -\vec{A} \\ 0 \\ \vec{0} \\ \varphi \end{bmatrix}, \quad \Box \begin{bmatrix} -\vec{A} \\ 0 \\ \vec{0} \\ \varphi \end{bmatrix} = 0. \tag{2.12}$$

3. Potentials of electromagnetic field with sources

Considerations from the previous Section can be easily extended to the general case of an electromagnetic field with sources. It is necessary only to add the source term on the right side of equation (1.1). The source term which gives directly the Maxwell's equation ($F = G = 0$) is

$$4\Pi \begin{bmatrix} -\frac{1}{c}\,j_x \\ -\frac{1}{c}\,j_y \\ -\frac{1}{c}\,j_z \\ 0 \\ 0 \\ 0 \\ 0 \\ \varrho \end{bmatrix} \tag{3.1}$$

The equation (1.1) then reads

$$\left(\frac{1}{c}\,\partial_t + \vec{\nabla}\,\vec{\eta}\right) \begin{bmatrix} E_x \\ E_y \\ E_z \\ F \\ H_x \\ H_y \\ H_z \\ G \end{bmatrix} = 4\Pi \begin{bmatrix} -\frac{1}{c}\,j_x \\ -\frac{1}{c}\,j_y \\ -\frac{1}{c}\,j_z \\ 0 \\ 0 \\ 0 \\ 0 \\ \varrho \end{bmatrix} \tag{3.2}$$

The potentials for this case are again determined by (1.9) but now, due to (3.2), instead of (1.10) satisfy the equation

$$\Box \Phi = 4\Pi \begin{bmatrix} \frac{1}{c} \dot{j}_x \\ \frac{1}{c} \dot{j}_y \\ \frac{1}{c} \dot{j}_z \\ 0 \\ 0 \\ 0 \\ 0 \\ -\varrho \end{bmatrix} \tag{3.3}$$

General solution of this equation is given by the general solution of the homogeneous equation and a particular solution. The solution of the homogeneous equation is considered in the previous Section. A particular solution is

$$\Phi = \begin{bmatrix} -A_x \\ -A_y \\ -A_z \\ 0 \\ 0 \\ 0 \\ 0 \\ \varphi \end{bmatrix}, \tag{3.4}$$

where (A_l, φ) are particular solutions of the equations

$$\Box A_l = -\frac{4\Pi}{c} \dot{j}_l, \qquad l = x, y, z,$$

$$\Box \varphi = -4\Pi\varrho. \tag{3.5}$$

$F = G = 0$ give again the Lorentz's condition

$$\frac{1}{c}\,\partial_t\,\varphi + \operatorname{div}\vec{A} = 0.$$

We may consider the source term in general form

$$4\Pi \begin{bmatrix} -\frac{1}{c} j_x^{(1)} \\ -\frac{1}{c} j_y^{(1)} \\ -\frac{1}{c} j_z^{(1)} \\ \varrho^{(2)} \\ -\frac{1}{c} j_x^{(2)} \\ -\frac{1}{c} j_y^{(2)} \\ -\frac{1}{c} j_z^{(2)} \\ \varrho^{(1)} \end{bmatrix} \tag{3.6}$$

The particular solution of the equation

$$\Box\,\Phi = 4\Pi \begin{bmatrix} \frac{1}{c} j_x^{(1)} \\ \frac{1}{c} j_y^{(1)} \\ \frac{1}{c} j_z^{(1)} \\ -\varrho^{(2)} \\ \frac{1}{c} j_x^{(2)} \\ \frac{1}{c} j_y^{(2)} \\ \frac{1}{c} j_z^{(2)} \\ -\varrho^{(1)} \end{bmatrix} \tag{3.7}$$

is then

$$\Phi = \begin{bmatrix} -A_x^{(1)} \\ -A_y^{(1)} \\ -A_z^{(1)} \\ \varphi^{(2)} \\ -A_x^{(2)} \\ -A_y^{(2)} \\ -A_z^{(2)} \\ \varphi^{(1)} \end{bmatrix} \equiv \begin{bmatrix} -\vec{A}^{(1)} \\ \varphi^{(2)} \\ -\vec{A}^{(2)} \\ \varphi^{(1)} \end{bmatrix}, \tag{3.8}$$

where the componets of Φ are particular solutions of

$$\Box \vec{A}^{(i)} = -\frac{4\Pi}{c}\vec{j}^{(i)},$$

$$\Box \varphi^{(i)} = -4\Pi\varrho^{(i)}, \qquad i = 1, 2, \tag{3.9}$$

with two Lorentz's conditions (for $F = G = 0$)

$$\frac{1}{c}\partial_t \varphi^{(i)} + \operatorname{div}\vec{A}^{(i)} = 0 \qquad i = 1, 2. \tag{3.10}$$

Writing

$$\left(\frac{1}{c}\partial_t + \vec{\nabla}\vec{\eta}\right)\psi = 4\Pi \begin{bmatrix} -\frac{1}{c}\vec{j}^{(1)} \\ 0 \\ \vec{0} \\ \varrho^{(1)} \end{bmatrix} + 4\Pi \begin{bmatrix} \vec{0} \\ \varrho^{(2)} \\ \frac{1}{c}\vec{j}^{(2)} \\ 0 \end{bmatrix} = I^{(1)} + I^{(2)} \tag{3.11}$$

the solution Ψ, due to linearity of this equation, can be written in the form

$$\Psi = \Psi_1 + \Psi_2,$$

where the particular solution Ψ_1 is associated with the current (source) $I^{(1)}$ and Ψ_2 with the current $I^{(2)}$.

The explicit equations for Ψ_1 and Ψ_2 read

$$\operatorname{rot} \vec{H}^{(1)} = \frac{1}{c} \frac{\partial \vec{E}^{(1)}}{\partial t} + \frac{4\Pi}{c} \vec{j}^{(1)},$$

$$\operatorname{div} \vec{H}^{(1)} = 0, \tag{3.12}$$

$$\operatorname{rot} \vec{E}^{(1)} = -\frac{1}{c} \frac{\partial \vec{H}^{(2)}}{\partial t},$$

$$\operatorname{div} \vec{E}^{(1)} = 4\Pi \varrho^{(1)},$$

$$\operatorname{rot} \vec{H}^{(2)} = \frac{1}{c} \frac{\partial \vec{E}^{(2)}}{\partial t},$$

$$\operatorname{div} \vec{H}^{(2)} = 4\Pi \varrho^{(2)},$$

$$\operatorname{rot} \vec{E}^{(2)} = -\frac{1}{c} \frac{\partial \vec{H}^{(2)}}{\partial t} - \frac{4\Pi}{c} \vec{j}^{(2)},$$

$$\operatorname{div} \vec{E}^{(2)} = 0. \tag{3.13}$$

Changing

$$\vec{E} \to -\vec{H},$$

$$\vec{H} \to \vec{E} \tag{3.14}$$

the system (3.13) goes over to (3.12). From here follows that Ψ_1, Ψ_2 are the Maxwell's fields with sources $I^{(1)}$, $I^{(2)}$ respectively. Consequently we can drop one source and use equation (3.2).

4. *Separation in field components and charge conservation law*

Applying $\left(\frac{1}{c}\partial_t - \vec{\nabla}\vec{\eta}\right)$ to Equ. (3.2) we get

$$-\Box \begin{bmatrix} \vec{E} \\ F \\ \vec{H} \\ G \end{bmatrix} = \frac{4\Pi}{c} \begin{bmatrix} -\frac{1}{c}\dot{\vec{j}} - c\nabla\varrho \\ 0 \\ \operatorname{rot}\vec{j} \\ \dot{\varrho} + \operatorname{div}\vec{j} \end{bmatrix}. \tag{4.1}$$

Transition to the electromagnetic field $F = G = 0$, gives

$$- \Box \begin{bmatrix} \vec{E} \\ O \\ \vec{H} \\ O \end{bmatrix} = \frac{4\pi}{c} \begin{bmatrix} -\frac{1}{c}\dot{\vec{j}} - c\nabla\varrho \\ 0 \\ \operatorname{rot}\vec{j} \\ \dot{\varrho} + \operatorname{div}\vec{j} \end{bmatrix} \tag{4.2}$$

This is a set of separated equations in field components. The last row is the charge conservations law

$$\frac{\partial}{\partial t}\varrho + \operatorname{div}\vec{j} = 0. \tag{4.3}$$

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NEKE PREDNOSTI SPINORSKOG PRIKAZA ELEKTROMAGNETSKOG POLJA

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Sadržaj

U radu[1] je izložen spinorski prikaz slobodnog elektromagnetskog polja. Ovakav način pisanja jednadžbi elektromagnetskog polja omogućava i jednostavno uvođenje potencijala. Kako se to čini pokazano je u ovom radu. Osnovne jednažbe su (1.9) i (1.10). Gradijentna invarijantnost je data jednadžbama (1.11) i (1.12).

U drugom dijelu rada postupak je primijenjen na slobodno elektromagnetsko polje a u trećem na polje sa izvorima.

Ista metoda omogućava i separaciju Maxwellovih jednadžbi po komponentama polja. Ova separacija je izložena u četvrtom dijelu. Zakon o sačuvanju naboja je jedna od separiranih jednadžbi (4.2).