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NONISOMORPHISM OF TWO QUANTUM MECHANICAL OPERATORS

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Abstract: It is proved that coordinates and momenta of particles with and without spin are defined by nonisomorphic operators and this fact is used to distinguish two possible interpretations of Hamiltonians of linear harmonic oscilator with double spectrum.

1. *Introduction*

It is well known that Quantum mechanics of particles (one- dimensional motion) without spin defines coordinate of the particle as a multiplication operator X in the space $L_2(R)$, where R is the set of all real numbers, and the momentum as a differential operator P in the same space. For particles with spin coordinate X and momentum $\mathscr{P}$ are defined as operator matrices

$$X = \begin{Bmatrix} X & 0 \\ 0 & X \end{Bmatrix} \quad \mathscr{P} = \begin{Bmatrix} P & 0 \\ 0 & P \end{Bmatrix} \tag{1}$$

in the space $L_2(R) \oplus L_2(R)$. Since both spaces, $L_2(R)$ and $L_2(R) \oplus L_2(R)$, are Hilbert spaces they are isomorphic. Are the operators X and P isomorphic to the operators X and $\mathscr{P}$?

2. *Symmetries in Hilbert space*

A symmetry S is a unitary and self-adjoint operator defined on Hilbert space H. If S is a symmetry, then it can be easily verified that the operators

$$P_M = \frac{1}{2}(E + S) \qquad P_{\overline{M}} = \frac{1}{2}(E - S), \tag{2}$$

where E is the identity operator, are two projections on, let us say, the subspaces M and its orthogonal complement $\overline{M}$ and

$$S = P_M - P_{\overline{M}}. \tag{3}$$

Conversely, if M is a subspace of Hilbert space H, then the difference $P_M - P_{\overline{M}}$ is a symmetry.

A symmetry S has two proper values, $+1$ and -1, and the corresponding proper subspaces are M for the first and $\overline{M}$ for the second proper value. The elements that belong to subspaces M and $\overline{M}$ we shall call symmetric and antisymmetric elements, respectively. We shall call rank of symmetry S the value $r = \min\{\dim M, \dim \overline{M}\}$. If $r > +\infty$, symmetry S is called π-symmetry. In the case $r = +\infty$, symmetry S is called ϱ-symmetry.

We shall now prove the following theorem.

If S_1 and S_2 are two arbitrary ϱ-symmetries in Hilbert spaces H_1 and H_2 respectively, then they are isomorphic.

Let M_1 and M_2 be two subspaces and $S_1 = P_{M1} - P_{\overline{M}1}$ and $S_2 = P_{M2} - P_{\overline{M}2}$ two ϱ-symmetries on the spaces H_1 and H_2. Let $\{x_0, x_2, \ldots\}$, $\{x_1, x_3, \ldots.\}$, $\{y_0, y_2, \ldots.\}$, $\{y_1, y_3, \ldots..\}$ be bases of subspaces M_1, M_1, M_2, M_2 respectively. Let I be a mapping of the space H_1 on the space H_2 that assigns to arbitrary element $x = \sum_{n=0}^{\infty} f_n x_n$ of the space H_1 an element $y = \sum_{n=0}^{\infty} f_n y_n$ of the space H_2. I is obviously an isomorphism. Since for any element $y = \sum_{n=0}^{\infty} f_n y_n$ of the space H_2, we have

$$I S_1 I^{-1} = I S_1 \sum_{n=0}^{\infty} f_n x_n = I \sum_{n=0}^{\infty} (-1)^n f_n x_n = \sum_{n=0}^{\infty} (-1)^n f_n y_n = S_2 y. \tag{4}$$

Therefore,

$$S_2 = I S_1 I^{-1}, \tag{5}$$

and the theorem is proved.

The following theorem is now obvious.

If the operator S_2 is isomorphic to a ϱ-symmetry S_1, then S_2 is a symmetry too.

Let us try now to solve our problem. If M and $\overline{M}$ are subspaces of the space $L_2(R) \oplus L_2(R)$ consisting of all elements of the form $< f(x), 0 >$ and $< 0, f(x) >$, $f(x) \in L_2(R)$, respectively, then the operator $\sigma = \varrho_M - \varrho_{\overline{M}}$ is Pauli operator

$$\sigma = \begin{Bmatrix} E & 0 \\ 0 & -E \end{Bmatrix}. \tag{6}$$

Obviously, σ is a ϱ-symmetry on the space $L_2(R) \oplus L_2(R)$.

Suppose that I is an isomorphism between spaces $L_2(R)$ and $L_2(R) \oplus L_2(R)$ such that $I^{-1} X I = X$ and $I^{-1} \mathscr{P} I = P$. Since

$$X \sigma = \sigma X, \qquad \mathscr{P} \sigma = \sigma \mathscr{P} \tag{7}$$

t follow s

$$X S = S X, \qquad P S = S P, \tag{8}$$

where $S = I^{-1} \sigma I$ is a ϱ-symmetry on the space $L_2(R)$.

If $\psi_n(a)$, $n = 0, 1, 2, \ldots$, are Hermite functions, then the equations (8) imply

$$X S \psi_n(x) = \sqrt{\frac{n}{2x}}\, S \psi_{n-1}(x) + \sqrt{\frac{n+1}{2x}}\, S \psi_{n+1}(x)$$

$$P S \psi_n(x) = -i\hbar \sqrt{\frac{nx}{2}}\, S \psi_{n-1}(x) + i\hbar \sqrt{\frac{(n+1)x}{2}}\, S \psi_{n+1}(x). \tag{9}$$

The equations (9) show that the functions $S\psi_n(x)$ satisfy the same differential equations as the Hermite functions $\psi_n(x)$. Since S is a unitary and self-adjoint operator, it follows

$$S \psi_n(x) = \varepsilon_n \psi_n(x), \qquad \varepsilon_n = \pm 1. \tag{10}$$

If we take $\varepsilon_0 = +1$, we see from the first of the equations (9) that $\varepsilon_n = +1$ for all n. If we take $\varepsilon_0 = -1$, then $\varepsilon_n = -1$ for all n. So we have finally

$$S = \pm E. \tag{11}$$

Since the identity operator is not ϱ-symmetry, we can conclude that there is no isomorphism I such that the equations $X = I^{-1} x I$ and $P = I^{-1} \mathscr{P} I$ are valid.

We have proved that $\mathscr{P}$ and P (x and X) are substantialy different operators and we are now ready for the next problem closely related to our present conclusion

3. *Linear harmonic oscillator with double spectrum*

Let us investigate the operator matrix

$$\mathscr{H} = \begin{Bmatrix} H & 0 \\ 0 & H \end{Bmatrix}, \tag{12}$$

where $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} k x^2$ is the energy operator of linear harmonic oscillator. Operator $\mathscr{H}$ has, obviously, the same proper values as the operator H, while the corresponding proper subspaces of H and $\mathscr{H}$ are one- and two-dimensional, respectively. Since the multiplicity of the spectrum is an isomorphic invariant[1)] we conclude that H and $\mathscr{H}$ are not isomorphic operators. This means that H and $\mathscr{H}$ are the energy operators of two different mechanical systems. H is energy operator of linear harmonic oscillator without spin. And $\mathscr{H}$?

If we have an operator A in H that corresponds to a definite physical quantity of particle without spin, then is generally accepted that the corresponding operator $\mathscr{A}$ of particle with spin is defined by operator matrix in $H \oplus H$ with A on the main diagonal.[2)] In this way we defined coordinate and momentum operators (1) of particle with spin. Therefore, we can consider operator (12) as energy operator of linear harmonic oscillator with spin. But, the difference between operators H and $\mathscr{H}$ cannot be interpreted as a consequence of spin only. It is possible that the multiplicity of the spectrum of $\mathscr{H}$ is caused by some other reason. If so, how to discriminate these possibilities?

If Π is operator of parity in the space $L_2(R)$ defined by the equation $\Pi f(x) = f(-x)$, then we can easily see that Π is a ϱ-symmetry and, therefore, isomorphic to the operator σ. We can, as well, easily calculate that

$$\sigma = I \Pi I^{-1}, \tag{13}$$

where $I\psi_{2n}(x) = \,>\psi_n(x), 0>$, $I\psi_{2n+1}(x) = \,>0, \psi_n(x)>$, and $\psi_n(x)$ are Hermite functions.

The isomorphism I and a simple calculation give

$$H_{is} \equiv I^{-1}\mathscr{H} I = \frac{1}{2}\left(H + \frac{1}{2}\hbar\,\omega\,\Pi\right). \tag{14}$$

Since the operator H_{is} is isomorphic to the operator $\mathscr{H}$, the proper values of H_{is} are $E_n = (n + 1/2)\,\hbar\,\omega, n = 0, 1 \ldots$, and to each proper value E_n corresponds a two-dimensional subspace spanned by the functions $\psi_{2n}(x)$ and $\psi_{2n+1}(x)$. If we, in first approximation, take into account the relativistic effects, then we have to add to non-relativistic Hamitonian a term of the form[3)]

$$H_{cor} = -\frac{p^4}{8\,m^3\,c^2}. \tag{15}$$

So, we finally get

$$H_{rel} = \frac{1}{2}\left(H + \frac{1}{2}\hbar\,\omega\,\Pi\right) - \frac{\varrho^4}{16 m^3 c^2}. \tag{16}$$

If we return to the space $L_2(R) \oplus L_2(R)$, then we get

$$\mathscr{H}_{rel} = \mathscr{H} - \frac{\mathscr{P}_{is}^4}{16\, m^3\, c^2}, \tag{17}$$

where $\mathscr{P}_{is} \equiv I\,P\,I^{-1} \neq \mathscr{P}$.

If the operator $\mathscr{H}$ is energy operator of linear harmonic oscillator with spin, then we obtain by standard procedure

$$\mathscr{H}^*_{rel} = \mathscr{H} - \frac{\mathscr{P}^4}{8\, m^3\, c^2}. \tag{18}$$

Let us take the last terms on the right sides of the equation (16) and (18) as perturbation. In the first case, as we can easily calculate, the energy correction is

$$\triangle\, E_n = (\psi_n(x), -\frac{P^4}{16 m^3 c^2}\psi_n(x)) = -\frac{3}{64}\frac{\hbar^2\,\omega^2}{m c^2}(2n^2 + 2n + 1). \tag{19}$$

This result shows that the multiplicity of energy levels of the unpertubed system is removed by relativistic effects. The energy of two states, $\psi_{2n}(x)$ and $\psi_{2n+1}(x)$, are displaced in the same direction and the energy level of antisymmetric state $\psi_{2n+1}(x)$ becomes lower than the energy level of the symmetric state $\psi_{2n}(x)$.

In the socond case, on the contrary, the perturbation term of the equation (18), as the consequence of the equation

$$\left(\begin{Bmatrix}\psi_n\\ 0\end{Bmatrix}, \begin{Bmatrix}P^4 & 0\\ 0 & P^4\end{Bmatrix}\begin{Bmatrix}\psi_n\\ 0\end{Bmatrix}\right) = \left(\begin{Bmatrix}0\\ \psi_n\end{Bmatrix}, \begin{Bmatrix}P^4 & 0\\ 0 & P^4\end{Bmatrix}\begin{Bmatrix}0\\ \psi_n\end{Bmatrix}\right) \tag{20}$$

does not remove the multiplicity of energy level of unperturbed system.

The sistuation is now this. We cannot decide whether the operator $\mathscr{H}$ is energy operator of a particle without or with spin if we do not include in the problem an operator of a physical quantity that is differently defined in both cases under consideration. We also see, that the multiplicity of the spectrum of $\mathscr{H}$ is caused by a special parity dependence of $\mathscr{H}$ in the first case and by spin in the second case.

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IMPULS I KOORDINATA ČESTICE SA SPINOM

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U radu je pokazano da multiplikativni i izvodni operator iz prostora $L_2(R)$ nisu izomorfni sa operatorima (1) iz prostora $L_2(R) \oplus L_2(R)$. Dokaz je izveden na ovaj način.

U radu je najpre pokazano da su sve simetrije beskonačnog ranga u Hilbertovom prostoru izomorfne i da je svaki operator izomorfan sa takvom simetrijom i sam simetrija beskonačnog ranga, ili ϱ-simetrija. Jedna od ϱ-simetrija u prostoru $L_2(R) \oplus L_2(R)$ jeste operator (6) koji komutira sa operatorima (1). Ako pretpostavimo egzistenciju izomorfizma, onda mora postojati ϱ-simetrija u prostoru $L_2(R)$ takva da su zadovoljene jedn. (8). Ako levu i desnu stranu jedn. (8) primenimo na Ermitove funkcije dobićemo jedn. (9) iz kojih sledi da funkcije $S\psi_n(x)$ zadovoljavaju istu diferencijalnu jednačinu kao i Ermitove funkcije $\psi_n(x)$, odakle sledi jedn. (11). Pošto jedinični operator nije ϱ-simetrija sledi kontradikcija čime je dokaz završen.

Činjenica da se impuls i koordinata čestica sa spinom definišu kao dijagonalne operatorske matrice ne opravdava zaključak da se svi operatori takvog oblika odnose isključivo na čestice sa spinom. Tako se operator (12) može smatrati kao operator energije kako čestice sa spinom tako i čestice bez spina. Tek uzimanjem relativističke korekcije u račun ove dve interpretacije postaju različite i to zbog toga što se u oba slučaja različito definiše operator impulsa.