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RELATIVISTIC MOTION OF AN ELECTRON IN UNIFORM EXTERNAL ELECTRIC AND MAGNETIC FIELDS

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Abstract: In this paper, we find the eigenfunctions and eigenvalues of the Hamiltanian for relativistic electron moving in a uniform external electric and magnetic fiield.

Starting from the solution of Dirac equation in a frame S' in which the electron moves in a uniform magnetic field, we perform a Lorentz transformation on this solution to obtain a solution of Dirac equation in a new frame S in which an electric field also appears.

To solve the problem we employ the Jannussis Schrauben-functions.

1. Introduction

We consider an electron moving relativisticaly in a uniform external magnetic field $\vec{H}'$ in the reference system S'.

The Dirac equation is

$$\{\vec{\gamma}'(\vec{P}' - \frac{e}{c}\vec{A}'(\vec{r}')) + \gamma_4' P_4' - i m_0 c\} \psi'(\vec{r}', t') = 0, \tag{1}$$

where

$$\vec{P}' = -i\hbar\vec{\Delta}', \; P_4' = -i\hbar\frac{\partial}{\partial x_4'}, \; \vec{H}' = \text{rot}\,\vec{A}'(\vec{r}'), \; \vec{A}'(\vec{r}')$$

is the vector potential, and the γ'_μ -matrices obey the rule

$$\gamma'_\mu \gamma'_\nu + \gamma'_\nu \gamma'_\mu = 2 I \delta_{\mu\nu}, \quad (\mu, \nu = 1, 2, 3, 4) \tag{2}$$

with $\gamma'_i = - i \beta' \alpha'_i \ (i = 1, 2, 3)$, $\gamma'_4 = \beta'$,

$$\beta' = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \text{ and } \vec{\alpha}' = \begin{pmatrix} 0 & \vec{\sigma}' \\ \vec{\sigma}' & 0 \end{pmatrix}.$$

If the vector potential has the symmetric form

$$\vec{A}'(\vec{r}') \equiv \left(-\frac{H'}{2} y', \ \frac{H'}{2} x', \ 0\right), \tag{3}$$

(where the magnetic field is parallel to the z-axis) the eigenfunctions $\psi'(\vec{r}', \vec{t}')$ take the following form[1]

$$\psi'(\vec{r}', \vec{r}') = \begin{bmatrix} \dfrac{-\hbar (k'_z A_3 + i\sqrt{2 B' n}\, A_4)}{\dfrac{E'}{c} + m_0 c} \psi'_{k',n}(\vec{r}', \vec{r}'_m) \\ \dfrac{\hbar (k'_z A_4 + i\sqrt{2 B' n}\, A_3)}{\dfrac{E'}{c} + m_0 c} \psi'_{k',n-1}(\vec{r}', \vec{r}'_m) \\ A_3 \quad \psi'_{k',n}(\vec{r}', \vec{r}'_m) \\ A_4 \quad \psi'_{k',n-1}(\vec{r}', \vec{r}'_m) \end{bmatrix} e^{-\frac{i}{\hbar} E' t'}, \tag{4}$$

where $E' = \pm c \sqrt{m_0^2 c^2 + \hbar^2 (k'^2_z + 2 B' n)}$ is the energy of the electron, $B' = \dfrac{e H}{h c}$ and $\psi'_{k',n}(\vec{r}', \vec{r}'_m)$ is the Schrauben Function[2,3]. This function in polar coordinates is

$$\psi'_{k',n}(\vec{r}', \vec{r}'_m) = \sqrt{\frac{B'}{2}\left(\frac{B'}{2}\right)^n \frac{1}{n!\, L_z}}\; e^{ik'_z z' - \frac{B'}{4}(\vec{r}' - \vec{r}'_m)^2 + \frac{ie}{\hbar c}(A'(\vec{r}')\vec{r}')} \,|r' - r'_m|^n \, e^{-in\Phi'}, \tag{6}$$

where $r'_m \equiv (x'_m, y'_m, 0)$ with $x'_m = \dfrac{2}{B'} k'_y$, $y'_m = -\dfrac{2}{B'} k'_x$ is the centre of the spiral orbit of an electron at the $x' y'$ projection plane.

The coefficients A_3, A_4 take arbitrary values and the quantity B' has dimensions of inverse surface.

Other papers which are referred in the study of an electron in a uniform external magnetic field using the Dirac equation are[4-6].

2. *Eigenfunctions and eigenvalues of the Dirac equation with electric and magnetic field*

We consider that the S' system is moving with respect to the system S relativisticaly, with velocity $\vec{V}$ paralel to the xy — plane of the system S. The velocities $\vec{v}$ and $\vec{v}'$ of the electron with respect to S and S' are related by the equation

$$\vec{v} = \frac{\vec{v}' + \gamma \vec{V} + \frac{\gamma - 1}{V^2} (\vec{u}' \vec{V}) \vec{V}}{\gamma \left(1 + \frac{\vec{v}' \vec{V}}{c^2}\right)}, \tag{7}$$

whereas the Lorentz transformation connecting the two systems is

$$\vec{r} = \vec{r}' + \frac{(\gamma - 1)}{V^2} \vec{V} (\vec{V} \vec{r}') - i \gamma \frac{\vec{V} x_4'}{c} \tag{8}$$

$$x_4 = \gamma \left(x_4' + i \frac{\vec{V} \vec{r}'}{c}\right), \tag{9}$$

with $\gamma = \left(1 - \frac{V^2}{c^2}\right)^{-1/2}$ and $x_4 = i c t$.

The vector potential $\vec{A}(\vec{r})$ and the operators $\vec{\gamma}, \vec{P}$ are transformed according to (8). The quantities γ_4 and $P_4 = -i h \frac{\partial}{\partial x_4}$ are transformed according to (9). Inserting the above transformations of these quantities into (1) using the relationship

$$A_4 = \gamma \left(A_4' + i \frac{\vec{V} \vec{A}' (\vec{r}')}{c}\right), \tag{10}$$

with $A_4' = i \Phi' = 0$ and setting $\psi' (\vec{r}', \vec{t}') = \hat{\psi} (\vec{r}, \vec{t})$ we finally have

$$\left\{\vec{\gamma} \left(\vec{P} - \frac{e}{c} A (\vec{r})\right) + \gamma_4 \left(P_4 - \frac{e}{c} A_4\right) - i m_0 c\right\} \hat{\psi} (\vec{r}, \vec{t}) = 0. \tag{11}$$

This equation is identical to the Dirac equation satisfied by the spinor $\psi(\vec{r}, \vec{t})$ in the system S hence $\hat{\psi}(\vec{r}, t) \equiv \psi(\vec{r}, t)$.

We thus see that starting from the spinor $\psi'(\vec{r}', t')$ describing the electron in the system S' and transforming $\vec{r}'$ and t' according to (8) and (9), we find the corresponding spinor $\psi(\vec{r}, t)$ in the system S.

The antisymmetric electromagnetic field tensor F_{ps} is given in the S system by the relationship

$$F_{ps} = \sum_{i=1}^{4} \sum_{i=1}^{4} F_{ik} \frac{\partial x^{i'}}{\partial x^p} \frac{\partial x^{k'}}{\partial x^s}, \quad (p, s = 1, 2, 3, 4) \tag{12}$$

where only the component $F'_{12} = H'$ is non-zero.

From (12), with the aid of the inverse transformations of (8) and (9) we obtain

$$\vec{H} = \gamma \vec{H}' \tag{13}$$

and

$$\vec{\mathscr{E}} = \frac{\gamma}{c} | \vec{H}' x \vec{V} | + \frac{\gamma(\gamma - 1)}{c V^2} H' V_x V_y (-V_x \vec{c} + V_y \vec{j}) =$$

$$= \frac{\gamma}{c} | \vec{H}' x \vec{V} | + \frac{\gamma(\gamma - 1)}{c V^2} H' V_x V_y (2 V_y \vec{j} - \vec{V})$$

or

$$\vec{\mathscr{E}} = \frac{1}{c} | \vec{H} x \vec{V} | + (\gamma - 1) \frac{H}{2c} V \sin \vartheta \vec{j} - \vec{V}) \sin 2 \vartheta, \tag{14}$$

where ϑ is the angle between the velocity $\vec{V}$ and the x-axis in the S coordinate system.

For $V << C$ equations (13) and (14) tend to the corresponding non-relativistic equations.

The term $\vec{A}'(\vec{r}'_m) \vec{r}'$ of the relationship (6) is transformed as

$$\vec{A}'(\vec{r}'_m) \vec{r}' = \vec{A}(\vec{r}_m) \vec{r} + \left| \frac{H}{2} x \vec{V} \right| \vec{r}_m t. \tag{15}$$

Inserting (8), (9), (14) and (15) into (4) we take for the eigenfunctions of the equation (11)

$$\psi(\vec{r}, t) = \begin{bmatrix} \dfrac{-\hbar\,(k_z A_3 + i\sqrt{2nB\gamma^{-1}}\, A_4}{\dfrac{E'}{c} + m_0 c}\psi_{k,n}(\vec{r}, \vec{r}_m) \\ \dfrac{\hbar\,(k_z A_4 + i\sqrt{2nB\gamma^{-1}}\, A_3}{\dfrac{E'}{c} + m_0 c}\psi_{k,n-1}(\vec{r}, \vec{r}_m) \\ A_3 \qquad \psi_{k,n}(\vec{r}, \vec{r}_m) \\ A_4 \qquad \psi_{k,n-1}(\vec{r}, \vec{r}_m) \end{bmatrix} e^{\frac{i}{\hbar}\gamma\frac{E'}{c^2}(\vec{r}\,\vec{V}) - \frac{i}{\hbar}\gamma E' t}, \quad (16)$$

where

$$E' = \pm c\sqrt{m_0^2 c^2 + \hbar^2 (k_z^2 + 2nB\gamma^{-1})}. \quad (17)$$

The function $\psi_{k,n}(\vec{r}, \vec{r}_m)$ has the form

$$\psi_{k,n}(r, r_m) = \sqrt{\frac{B}{2\pi}\left(\frac{B}{2}\right)^n \frac{\gamma^{-(n+1)}}{n!\,L_z}}\left(1 + \frac{1}{2}(\gamma^{-1} - 1)(1 + e^{2i(\varphi-\theta)})\right)^n e^{ik_z z}$$

$$- e^{\frac{B\gamma^{-1}}{4c^2}((\vec{r} - \vec{r}_m)\vec{V})^2 - \frac{\gamma^{-1}B}{4}\{\vec{r} - \vec{r}_m\}^2 + \frac{ie}{\hbar c}(\vec{A}(\vec{r}_m)\vec{r})} \cdot \quad (18)$$

$$\cdot\, e^{\frac{ie}{c\hbar}\left\{\frac{c}{2}\vec{\mathscr{E}}\,\vec{r}_m - \frac{(\gamma-1)}{4}H(2V\sin\theta\,\vec{j} - \vec{V})\,\vec{r}_m\sin 2\theta\right\}t}\,|r - r_m|^n e^{-in\varphi},$$

with $\vec{\mathscr{E}} \equiv (\mathscr{E}_x\, \mathscr{E}_y, 0)$, because the terms $\{\vec{r}' - \vec{r}'_m\}^2$, $|\vec{r}' - \vec{r}'_m|^n e^{in\varphi'}$ are given by the relatioships

$$\{\vec{r}' - \vec{r}'_m\}^2 = \{\vec{r}\ \vec{r}_m\}^2 - \frac{1}{c^2}\left((\vec{r} - \vec{r}_m)\vec{V}\right)^2 \quad (19)$$

and

$$|\vec{r}' - \vec{r}'_m|^n e^{-in\varphi'} = |r - r_m|^n e^{-in\varphi}\left(1 + \frac{1}{2}(\gamma^{-1} - 1)(1 + e^{2i(\varphi-\theta)})\right)^n. \quad (20)$$

The difference $\{\vec{r} - \vec{r}_m\}$ is time-independent.

The term $e^{\frac{i}{\hbar}\gamma\frac{E'}{c}(\vec{r}\ \vec{V})}$ includes the drift-velocity.

The eigenvalues of the energy in the system S are given by the equation

$$E = \pm c\gamma\sqrt{m_0^2 c^2 + \hbar^2 (k_z^2 + 2nB\gamma^{-1})} -$$
$$- \frac{e}{2}\vec{\mathscr{E}}\vec{r}_m + (\gamma - 1)\frac{eH}{4c}(2V\sin\vartheta\vec{j} - \vec{V})\vec{r}_m \sin 2\vartheta. \tag{21}$$

The energy depends not only on the magnitude of $\vec{V}$ but also on the angle ϑ.

3. *Probability density and current*

For the probability density we get

$$\varrho = \psi(\vec{r}, t)\psi^*(\vec{r}, t) \tag{22}$$

or

$$\varrho = \left\{A_3 A_4^* + \frac{\hbar^2}{\left(\frac{E'}{c} + m_0 c\right)^2}(k_z^2 A_3 A_3^* + 2nB\gamma^{-1} A_4 A_4^*) + \right.$$
$$\left. + i k_z \sqrt{2nB\gamma^{-1}}\,(A_3^* A_4 - A_4^* A_3)\right\}\psi_{k,n}(\vec{r}, \vec{r}_m)\psi_{k,n}(\vec{r}\,\vec{r}_m)\psi_{k,n}(\vec{r}, \vec{r}_m) +$$
$$+ \left\{A_4 A_4^* + \frac{\hbar^2}{\left(\frac{E'}{c} + m_0 c\right)^2}(k_z^2 A_4 A_4^* + 2nB\gamma^{-1} A_3 A_3^*) - \right.$$
$$\left. - i k_z \sqrt{2nB\gamma^{-1}}\,(A_3^* A_4 - A_4^* A_3)\right\}\cdot\psi_{k,n-1}(\vec{r}, \vec{r}_m)\psi_{k,n-1}^*(\vec{r}, \vec{r}_m). \tag{23}$$

The above equation for small magnetic field and large n can be written

$$\varrho = (A_3 A_3^* + A_4 A_4^*)\frac{2E'}{E' + m_0 c^2}\psi_{k,n}(\vec{r}, \vec{r}_m)\psi_k^*(\vec{r}, \vec{r}_m). \tag{24}$$

From the normalization relationship

$$\sum_{i=1}^{4}\int \psi_i \psi_i^*\, dr = 1 \tag{25}$$

we obtain that

$$-A_3 A_3^* + A_4 A_4^* = \frac{\gamma^{-1}(E' + m_0 c^2)}{2E'}. \tag{26}$$

Inserting (26) into (24) we have

$$\varrho = \frac{\gamma^{-2} B}{2\pi n'! L_z} \left(\frac{\gamma^{-1} B}{2} R^2 \left(1 - \frac{V^2}{c^2} \cos^2 (\varphi - \vartheta)\right)\right)^n e^{-\frac{\gamma^{-1} B}{2} R^2 \left(1 - \frac{V^2}{c^2} \cos^2 (\varphi - \vartheta)\right)}, \tag{27}$$

where $R = \vec{r} - \vec{r}_m$. The above equation represents the Poisson distribution with the parameter

$$\xi = \frac{\gamma^{-1} B}{2} R^2 \left[1 - \frac{V^2}{c^2} \cos^2 (\varphi - \vartheta)\right]. \tag{28}$$

Points for which the density becomes a maximum are such that

$$\xi = n \tag{29}$$

or

$$R^2 = R'^2 \left(1 - \frac{V^2}{c^2} \cos^2 (\varphi - \vartheta)\right)^{-1} = R'^2 q, \tag{30}$$

with $R'^2 = \frac{2n}{B'}$ and $\left(1 - \frac{V^2}{c^2} \cos^2 (\varphi - \vartheta)\right)^{-1} = q.$

For velocities $V << c$ the above equation is in agreement with the classical results.

The transformation of R'^2 from the S' system to the S system is according to (8)

$$R^2 = R'^2 + \frac{\gamma^2}{c^2} (\vec{R}' \vec{V})^2, \tag{31}$$

where $\vec{R}' \vec{V} = R' V \cos (\varphi' - \vartheta)$. Then we have

$$\text{for} \quad \varphi' = 2k\pi + \vartheta \pm \frac{\pi}{2} \quad \text{that} \quad R^2 = R'^2 \quad \text{and}$$

$$\text{for} \quad \varphi' = 2k\pi + \vartheta \qquad \text{that} \quad R^2 = \gamma^2 R'^2. \tag{32}$$

The above equations show that the projection of the electron orbit on the $x'y'$-plane of the S' system is an ellipse with minor axis $b = \gamma^{-1} R$ parallel to the velocity $\vec{V}$ and major axis $a = R$ perpendicular to it.

From (27) we conclude that for $V << c$ the probability density tends to the nonrelativistic limit. For $V \to C$, the density ϱ tends to zero.

The expression of the probability density in K-space gives some interesting results

if we set $K_x = \frac{B}{2} R_y$, $K_y = -\frac{B}{2} R_x$ and $K_x^2 + K_y^2 = \left(\frac{K_0}{2}\right)^2$, (33)

then equation (27) becomes

$$\varrho = \frac{\gamma^{-2} B}{2\pi n ! L_z} \left(\frac{\gamma^{-1} B}{2} \left(\frac{K_0}{B}\right)^2 \left(1 - \frac{V^2}{c^2} \cos^2 (\varphi - \right.\right.$$

$$\left.\left. - \vartheta\right)\right)^n e^{-\frac{\gamma^{-1} B}{2} \left[\left(\frac{K_0}{2}\left(1 - \frac{V^2}{c^2} \cos^2 (\varphi - \vartheta)\right)\right]\right.}. \tag{34}$$

The quantity (34) becomes maximum whenever

$$K_0^2 = B^2 R^2 = 2nB\gamma q. \tag{35}$$

The above relationship for the classical case where $\gamma \cong q \cong 1$, expresses the connection between the surface of a unit cell in Harpes K-space and B.

The relationship (35) may be written as follows

$$\hbar^2 K_0^2 = 4 n \gamma m_0 \mu H q, \tag{36}$$

where $\mu = \frac{\hbar^2 B}{2 m_0 H}$ is the Bohr magneton.

The probability current density is calculated by the form

$$\vec{j} = \psi^\star (\vec{r}, t) \, c \, \vec{a} \, \psi (r, t). \tag{37}$$

4. *Energy and angle between $\vec{\mathscr{E}}$ and $\vec{V}$*

Equation (21) gives the energy of the electron in the system S. If the direction of $\vec{r}_m$ coincides with the direction of $\vec{V}$ Equ. (21) by (14) reduces to

$$E = \pm c \gamma \sqrt{m_0^2 c^2 + \hbar^2 (k_z^2 + 2 n B \gamma^{-1})}. \tag{38}$$

Substituting (36) in (38) and for small kinetic energy and $V < C$ we have

$$E = \pm m_0 c^2 \left(1 - \frac{V^2}{c^2}\right)^{-1/2} \left(1 + \frac{\hbar^2 k_z + 4 n \mu H \gamma^{-1} m_0}{m_0^2 c^2}\right)^{1/2} =$$

$$= m_0 c^2 + \frac{\hbar^2 k_z^2}{2 m_0} + 2 n \mu H + \frac{1}{2} m_0 V^2, \qquad (39)$$

if we chose the positive square root.

The above equation connects the energy of the electron to the magnetic energy $a \mu H$, where is equal to the distance between two successive Landau energy levels[3].

The equation (21) for $V < < C$ gives

$$E = \pm c \sqrt{m_0^2 c^2 + \hbar^2 (k_z^2 + 2 n B)} - \frac{e}{2} \mathscr{E} r_m + \frac{1}{2} m V^2, \qquad (40)$$

with $m = \frac{E'}{c^2}$. Now if we consider that the electron has a small kinetic energy $T <$ $< m_0 c^2$ in the S' system we have $m \cong m_0$ and we obtain

$$E' = \pm c \sqrt{m_0^2 c^2 + \hbar^2 (k_z^2 + 2 n B)} = \pm m_0 c^2 \left(1 + \frac{\hbar^2 (k_z^2 + 2 Bn)}{m_0{}^2 c^2}\right)^{1/2} =$$

$$= m_0 c^2 + n \hbar \omega_c + \frac{\hbar^2 k_z^2}{2m}, \qquad (41)$$

where we chose the positive square root and $\omega_c = \frac{eH}{m_0 c}$ is the Larmor frequency.

Substituting the equation (41) to (40) we take[7]

$$E = n \hbar \omega_c + \frac{P^2{}_z}{2m} - \frac{e}{\hbar} \mathscr{E} r_m + \frac{1}{2} m V^2. \qquad (42)$$

The angle between $\vec{\mathscr{E}}$ and $\vec{V}$ is given by

$$\cos \lambda = \frac{\vec{\mathscr{E}} \vec{V}}{|\mathscr{E}| |V|}, \qquad (43)$$

which by the aid of equation (14) can be written:

$$\cos \lambda = \frac{(\gamma - 1) \sin 2 \vartheta \cos (\pi + 2 \vartheta)}{((\gamma^2 + 2\gamma - 3) \sin^2 2 \vartheta + 4)^{1/2}}. \qquad (44)$$

From (44) for $V << C$ we get $|\lambda| = \frac{\pi}{2}$ which coincides to the classical result. For $V \cong C$ dividing the equation (44) by $(\gamma - 1)$, we have

$$\cos \lambda = \cos (\pi + 2\vartheta)$$

or

$$|\lambda| = \pi + 2\vartheta.$$

In the particular case $\vartheta = 0$ (i. e when S' moves along the axis Ox) it follows that $|\lambda| = \pi$, that is the electric field becomes then antiparallel to $\vec{V}$.

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