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REMARKS ON THE LIOUVILLE EQUATION

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Abstract

In the present paper we deal with certain remarks on the Liouville equation and we define general solutions depending also on time. From these solutions result such ones that have physical meaning and close relation with the phase for any Hamiltonian function. Then with the help of the normalization conditions of a distribution in the phase space for quadratic forms of the Hamiltonian function we also find time fluctuation.

1. Introduction

In a previous paper of Jannussis - Papaloucas [1] the Poisson equation of motion

$$\frac{dF}{dt} = \{F, H\} \qquad (1.1)$$

was studied, and certain solutions of the following form

$$F(q,p) = F_0(H)e^{iU(q,p)} \qquad (1.2)$$

were found where the function $U(q,p) = \omega \int dq(\frac{\partial H}{\partial p})^{-1}$ is the phase and the integral extends over the surface of $H = \text{const}$.

According to the method applied in [1] for the Poisson equation we shall work in the present paper with the solution of the Liouville equation

$$\frac{\partial F}{\partial t} + \{F, H\} = 0 \qquad (1.3)$$

Jannussis and Brodimas [2] as well as Contopoulos [3] have already worked on the above equation asking to define integrals of motion for a special case in which the Hamiltonian function is a periodic function of time and the coefficients are polynomials in (q,p) of the same degree as the order of perturbation.

In the case where the Hamiltonian function does not depend on time we consider q,p,t as independent variables. Then the Liouville equation

(1.3) has a solution of the form

$$F(q,p,t) \sim e^{i\omega t} f(q,p) \tag{1.4}$$

and the new function, f(q,p) satisfies the eigenvalue equation

$$i\omega f(q,p) + \{f,H\} = 0 , \tag{1.5}$$

which is a linear differential equation of first order with partial derivatives, the solution of which will be of the form

$$f(q,p) = F_0(H) e^{-i\omega \int dq \left(\frac{\partial H}{\partial p}\right)^{-1}} = F_0(H) e^{i\omega \int dp \left(\frac{\partial H}{\partial q}\right)^{-1}} , \tag{1.6}$$

where H = const.

Therefore a solution of the Liouville equation depending on time will be of the form

$$F(q,p,t) = F_0(H) e^{i\omega\left[t - \int dq \left(\frac{\partial H}{\partial p}\right)^{-1}\right]} . \tag{1.7}$$

Now we can easily prove that for an arbitrary Hamiltonian function H(q,p) the solution of the Liouville equation (1.3) has the following form

$$F(q,p,t) = F_0(H) \cdot U\left(\omega t - \omega \int dq \left(\frac{\partial H}{\partial p}\right)^{-1}\right) , \tag{1.8}$$

where U is an arbitrary function.

Solutions of the form (1.7) have physical meaning because the expression $\omega \int dq \left(\frac{\partial H}{\partial p}\right)^{-1}$ represents the phase.

2. Phase properties

To make things simpler, we will restrict our problem to the simple classical description starting from the energy theorem

$$E = \frac{p^2}{2m} + V(q) \tag{2.1}$$

For the Hamiltonian (2.1), the phase φ is given by the following relation

$$\varphi = \omega m \int \frac{dq}{p} \tag{2.2}$$

which because of (2.1) takes the form

$$\varphi = \omega\sqrt{\frac{m}{2}} \int \frac{dq}{\sqrt{E - V(q)}} = \omega\sqrt{2m}\, \frac{d}{dE} \int \sqrt{E - V(q)}\, dq = \omega \frac{d}{dE} \int p dq \tag{2.3}$$

or

$$\varphi = \omega \frac{d\Phi}{dE}\,, \tag{2.4}$$

where

$$\Phi = \int p dq \tag{2.5}$$

is the phase integral which was of great importance for the old quantum-mechanics [4].

In the case we consider the phase is an integer multiple of 2π, that is

$$\varphi = 2\pi n\ , \qquad n = \text{integer} \tag{2.6}$$

then the eigenvalues of ω are given by the relation

$$\omega = \frac{2\pi n}{\sqrt{\frac{m}{2}}} \frac{1}{\int_{q_1}^{q_2} \frac{dq}{\sqrt{E - V(q)}}} \tag{2.7}$$

and the limits in the above integral are taken from the relation

$$E = V(q_1) = V(q_2)\ , \quad q_1 < q_2\ . \tag{2.8}$$

For the simple case of the harmonic oscillator $V(q) = \frac{m}{2}\omega_0^2 q^2$, the relation (2.6) lead to the following result

$$\omega = 2\pi \frac{n\omega_0}{\int_{q_1}^{q_2} \frac{d(\sqrt{\frac{m}{2E}}\,\omega_0 q)}{\sqrt{1-(\sqrt{\frac{m}{2E}}\,\omega_0 q)^2}}} \tag{2.9}$$

or finally

$$\omega = n\omega_0\ . \tag{2.10}$$

We come to the same conclusion, if we consider the eigenvalues Liouville equation (1.5) for the case of the harmonic oscillator, that is

$$i\omega f - m\omega_0^2\left(q\frac{\partial}{\partial p} - \frac{p}{m^2\omega_0^2}\frac{\partial}{\partial q}\right)f = 0 \tag{2.11}$$

which in polar coordinates

$$m\omega_0 q = P\cos\vartheta \quad , \quad p = P\sin\vartheta \tag{2.12}$$

takes the form

$$i\omega f - \omega_0\frac{\partial f}{\partial\vartheta} = 0 \tag{2.13}$$

and has the normalized solution

$$f = \frac{1}{\sqrt{2\pi}}e^{in\vartheta} \tag{2.14}$$

with the eigenvalues

$$\omega = n\omega_0 . \tag{2.15}$$

Generally, the condition of the eigenvalues does not hold, and the Liouville equation for the case of the harmonic oscillator has the following solution according to (1.8)

$$F(q,p,t) = F_0(H)\ U\ (\omega t + \frac{\omega}{\omega_0}\ \mathrm{arctg}\ \frac{p}{m\omega q}) . \tag{2.16}$$

For the special case $\omega = \omega_0$ and

$$U \sim e^{i\omega_0 t + i\ \mathrm{arctg}\ \frac{p}{m\omega_0 q}} \tag{2.17}$$

we come to known solution [5], which for the case of quantum mechanics leads to the phase operators [5-7].

If except the above found solutions of the form $F(q,p,t)$ we want to define such ones that will aldo satisfy the condition of normalization on the phase space and will depend on the time. Then we can assume certain time fluctuations for the case when the Hamiltonian function is a polynomial in second order as for as q and p are concerned. Such solutions depending on time in the phase space and being also normalizable, was applied recently in the "Rate Theory of Solids" [8] not only classicaly but also quantum mechanically.

Since the Liouville equation for the case of quadratic forms coincidens with the Wigner equation [9], it is expected that the results can be explained both in classical or quantum mechanical way.

3. Time fluctuation

In order to find time fluctuations we will examine both the case of the harmonic oscillator and the case of electrons in a uniform magnetic field.

For the harmonic oscillator the Liouville equation is of the form (2.11) and admits, according to the known facts [10], Boltzmann distribution as a stationary solution that is $e^{-\frac{H}{kT}}$, where k is Boltzmann constant and T is the absolute temperature.

If for the equation (2.11) we ask a solution of the form

$$F(p,q,t) = c\,e^{-\frac{H}{kT} + a(t)p + b(t)q}, \tag{3.1}$$

which will fulfill the normalization condition

$$\int F(p,q,t)\, dp dq = 1 \tag{3.2}$$

then by substituting (3.1) to (2.11) it follows that the functions a(t) and b(t) satisfy the Hamilton's equations of motion of the harmonic oscillator, that is

$$\frac{da}{at} = -\frac{1}{m} b, \quad \frac{db}{dt} = m\omega^2 a \tag{3.3}$$

with the known solutions

$$a(t) = a_0 \cos(\omega t + \varphi), \quad b(t) = a_0 \omega m \sin(\omega t + \varphi) \tag{3.4}$$

which yield to the following relations

$$a^2+\frac{b^2}{m^2\omega^2}=a_0^2\ ,\qquad \frac{b^2}{2m}+\frac{m}{2}\,\omega^2a^2=\frac{m}{2}\,\omega^2a_0^2\ . \tag{3.5}$$

In the same way from the normalization condition (3.2) the constant c takes the value

$$c=\frac{\omega}{2\pi kT}\,e^{-\frac{m}{2}kTa_0} \tag{3.6}$$

and the solution we seek (3.1) takes the form

$$F(p,q,t)=\frac{\omega}{2\pi kT}\,e^{-\frac{1}{kT}\left[\frac{1}{2m}(p-mkTa(t))^2+\frac{m}{2}\,\omega^2(q-\frac{kT}{m\omega^2}\,b(\dot{t}))^2\right]}, \tag{3.7}$$

where the quantities $kTa(t)$ and $\frac{kT}{m\omega^2}\,\dot{b}(t)$ correspond to the average value of the velocity and the mean path and their values are periodic functions of time and fluctuate between $|kTa_0|$ and $\left|\frac{kTa_0}{\omega}\right|$

In the same way we also study the case of electrons in an uniform magnetic field H parallel to the z axis and with a symmetric vector potential.

The Hamiltonian function in this case os given by the relation

$$H=\frac{\vec{P}^2}{2m}\ ,\qquad \vec{P}=(\vec{p}+\frac{e}{c}\,\vec{A}\ (\vec{q}))\ , \tag{3.8}$$

where $\vec{P}$ represents the generalized momentum.

The Liouville equation takes the form

$$\frac{\partial F}{\partial t}=\omega\left(P_1\,\frac{\partial F}{\partial P_2}-P_2\,\frac{\partial F}{\partial P_1}\right)-\frac{1}{m}\,P_3\,\frac{\partial F}{\partial q_3} \tag{3.9}$$

where $\omega=\frac{eH}{2mc}$ is the Larmor frequency.

Following the same way we used for the harmonic oscillator (3.1), we find that the normalized solution of (3.7) is the following

$$F(\vec{p},t)=\left(\frac{1}{2m\pi kT}\right)^{3/2}\exp\left\{-\frac{1}{2m\pi kT}\left[\left(P_1-mkTa_0\ \sin(\omega t+\varphi)\right)^2+\right.\right.$$
$$\left.\left.+\left(P_2-mkTa_0\ \cos(\omega t+\varphi)\right)^2+P_3^2\right]\right\}\ . \tag{3.10}$$

The above distribution is a generalized Maxwell distribution in the space of generalized momentum, the average velocity of which is a periodic function of time and has coordinates $(kTa_0\sin(\omega t+\varphi),\ kTa_0\cos(\omega t+\varphi),0)$ and changes from $-a_0kT$ to a_0kT.

The distributions we found in (3.7) and (3.10) are positive and hold for any time.

There is a lot of interest for the case of the harmonic oscillator disturbed by a function of the form $a(t)p+b(t)q$, that is the Hamiltonian function depends on time, and is of the form

$$H(p,q,t)=\frac{p^2}{2m}+\frac{m}{2}\omega^2q^2+a(t)p+b(t)q\,, \tag{3.11}$$

where the correspondent Liouville equation is

$$\frac{\partial F}{\partial t}=\left(m\omega^2q+b(t)\right)\frac{\partial F}{\partial p}-\left(\frac{p}{m}+a(t)\right)\frac{\partial F}{\partial q}\,. \tag{3.12}$$

The equation (3.12) has the following solution

$$F(p,q,t)=e^{-\frac{1}{kT}\left(\frac{p^2}{2m}+\frac{m}{2}\omega^2q^2\right)+A(t)p+B(t)q+c(t)} \tag{3.13}$$

when the time dependent functions A(t), B(t) and c(t) satisfy the following system of differential equations

$$\frac{dA}{dt}+\frac{1}{m}B+\frac{b(t)}{mkT}=0\,,$$

$$\frac{1}{m}\frac{dB}{dt}-\omega^2A-\frac{\omega^2a(t)}{kT}=0\,, \tag{3.14}$$

$$\frac{dc}{dt}+aB-bA=0\,.$$

From the first two equation of the above system we get the following solution

$$A(t)=A_0\cos\omega t+A_1\sin\omega t-\frac{1}{mkT}\int_0^t\left[\cos\omega(t-t')b(t')+m\omega\sin\omega(t-t')a(t')\right]dt' \tag{3.15}$$

$$\frac{1}{m} B(t) = A_0 \omega \sin\omega t - A_1 \omega \cos\omega t - \frac{\omega}{kT} \int_0^t \left[\sin\omega(t-t')b(t') - m\omega\cos\omega(t-t')a(t') \right] dt' \tag{3.16}$$

and we obtain the function c(t) by simple integration respect to time.

If the functions a(t) and b(t) are bounded for all values of time, then the solution (3.15) has meaning and the distribution (3.13) is positive.

In the case for which the functions a(t) and b(t) are bounded and also periodic functions of time with period $\frac{2\pi}{\omega_0}$, it is possible to have cases for which for certain relation between ω and ω_0 the solutions of the form (3.13) are both periodic functions of time and integrals of motion according to the meaning mentioned in [3].

For the special case

$$a(t) = a_0 \cos\omega_0 t, \qquad b(t) = a_0 m\omega_0 \sin\omega_0 t \tag{3.17}$$

the solutions (3.15), (3.16) yield

$$A(t) = A_0 \cos\omega t + A_1 \sin\omega t + \frac{a_0}{kT} \frac{\omega_0^2 + \omega^2}{\omega_0^2 - \omega^2} \left[\cos\omega_0 t - \cos\omega t \right] \tag{3.18}$$

$$\frac{1}{m} B(t) = A_0 \omega \sin\omega t - A_1 \omega \sin\omega t + \frac{a_0}{kT(\omega_0^2 - \omega^2)} \left[2\omega_0 \omega \sin\omega_0 t - (\omega_0^2 + \omega^2) \sin\omega t \right] \tag{3.19}$$

and

$$c(t) = a_0 \int \left[m\omega_0 A(t) \sin\omega_0 t - B(t) \cos\omega_0 t \right] dt . \tag{3.20}$$

From the above solutions we obtain that for $\omega = k\omega_0$, $k \neq 1$ and integer we get periodic solutions with period $\frac{2\pi}{\omega_0}$. The case $\omega^2 = \omega_0^2$ has meaning but it does not give periodic solutions.

More details about time fluctuations we can find in Weiner's papers [8] and his contributors, especially for the case of interstitial diffussion as well as for the study of Schrödinger - Langevin equation.

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