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INVESTIGATION OF SOLUTIONS OF THE DYSON-SCHWINGER EQUATION FOR ELECTRON PROPAGATOR IN QUANTUM ELECTRODYNAMICS

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Abstract: The Dyson-Schwinger equation for the electron propagator in the "finite" quantum electrodynamics of Johnson et al. is considered. The solutions of the corresponding nonlinear second-order differential boundary value problem in momentum representation are investigated in detail. It is shown that the electron propagator has singular points in the complex p^2 plane. The problem of generating nonvanishing physical electron mass in the theory with vanishing bare electron mass is discussed.

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1. Introduction

It is well known that the standard perturbation approach to local quantum electrodynamics leads to the logarithmic divergences for the electron self-energy δm as well as for the renormalization constants $Z_1 = Z_2$ and Z_3[1]. The renormalization theory successfully removes these singularities, but the question of the exact solutions structure still remains open. The possibility of a finite charge renormalization in an exact theory seems to be the most attractive one. An interesting way to develop such "finite" quantum electrodynamics has been proposed by Johnson et al. [2,3]. In the basis of their approach is the idea of the application of a nonstandard perturbation theory with a zero bare electron mass m_0. In that case, the bare electron charge e_0 has to be determined from the conditions of the finiteness of vacuum polarization effects. In an exact theory that would correspond to the vanishing of the Gell-Mann - Low function $\psi(z)$ for the finite value of $z = \frac{e_0^2}{4\pi}$.

The first step of this programme appears to be the solving of the nonlinear homogeneous equation for the electron propagator (2.6). The corresponding nonlinear differential boundary value problem has been discussed several times in literature[2-5], but it has not been completely examined as yet. The behaviour of the solutions in the region of large space-like momenta and small bare coupling constant has been discussed in Ref.[2] In Refs.[2,4] investigation of the nonlinear equation has been reduced to an examination of the linearized problem by replacing the square value of the unknown function by a constant. Of course, such a method does not allow one to explain completely the nature of the exact nonlinear problem solutions.

In the recent paper of R.Fukuda and T.Kugo[5] the nonlinear second-order differential boundary value problem has been reduced to the first-order problem, solving then the latter by phase trajectories method. On the basis of numerical analysis the authors have come to the conclusion that the electron propagator has no singularities for $-\infty < p^2 < \infty$. In fact, this conclusion in Ref.[5] has not been proved, but rather implicitely assumed. It has virtually been shown only that the propagator cannot have poles. From the investigations performed in the above-mentioned paper, it does not necessarily follow that the propagator cannot have a branch point on the real axis; that very point could determine the finite electron mass.

The present work is devoted to the investigation of the nonlinear integral equation solutions (2.6) as well as of the corresponding differential boundary value problem (2.7), (2.8). In Section 2 the derivation of these equations based on the Dyson-Schwinger equation for the electron propagator in the first approximation is demonstrated. In Section 3 some general properties of the solutions have been obtained either directly from the one-dimensional integral equation, or by means of an analysis of the nonlinear differential equation. Section 4 contains the proof of convergence of power series representing the solutions of the investigated equations in a circle of a finite radius with the centre at $p^2 = 0$. As a result of the numerical analysis dependence of the radius of convergence of the above - mentioned power series on the coupling constant is given. The basic results of the present work are summarized and discussed in a short conclusion (Section 5).

2. Nonlinear equations for the electron propagator in the first approximation

The Dyson-Schwinger equation for the complete electron propagator in quantum electrodynamics has the following form:

$$S^{-1}(p) = S_0^{-1}(p) - \frac{i e_0^2}{(2\pi)^4} \int d\tilde{q}\, \gamma^\mu S(q) \Gamma^\nu(p,q;k) D_{\mu\nu}(k) \qquad (2.1)$$

where S and S_0 are complete and free Green's functions respectively, $D_{\mu\nu}$ is a complete photon Green's function, Γ^ν — complete vertex part, and $k = p - q$. In order that (2.1) be an equation for one unknown function $S(p)$, it is necessary to substitute in it some explicit expressions for Γ^ν and $D_{\mu\nu}$, which may be obtained by the perturbation method. In "finite" electrodynamics the first nontrivial approximation is obtained by replacing complete photon Green's function $D_{\mu\nu}$ and the complete vertex part Γ^ν by their free parts $D^0_{\mu\nu}$ and γ^ν respectively. In this approximation, taking into account that

$$S^{-1}(p) = m_0 + \Sigma(p) - \hat{p} \equiv \alpha(-p^2) - \hat{p}\beta(-p^2)$$

$$S_0^{-1}(p) = m_0 - \hat{p}$$

$$D^0_{\mu\nu}(k) = \frac{-1}{k^2}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}(1-d_l)\right] \tag{2.2}$$

where d_l determines the gauge, we obtain a system of two nonlinear integral equations for $\alpha(-p^2)$ and $\beta(-p^2)$:

$$\alpha(-p^2) = m_0 + \frac{i e_0^2}{(2\pi)^4}\int d^4q \frac{\alpha(-q^2)}{\alpha^2(-q^2) - q^2\beta^2(-q^2)} \cdot \frac{3+d_l}{k^2} \tag{2.3a}$$

$$\beta(-p^2) = 1 + \frac{i e_0^2}{(2\pi)^4 p^2}\int d^4q \frac{\beta(-q^2)}{\alpha^2(-q^2) - q^2\beta^2(-q^2)}\left[\frac{2(pq)}{(p-q)^2} + \right.$$

$$\left. + \frac{(p^2+q^2)(pq) - 2p^2q^2}{(p-q)^4}(1-d_l)\right] \tag{2.3b}$$

The system (2.3) can be transformed into another system of two one-dimensional equations after going to the Euclidean metric ($q_0 \to iq_4$, $-q^2 = \vec{q}^{\,2} - q_0^2 \to q^2 = \vec{q}^{\,2} + q_4^2$ and in an analogous manner for p):

$$\alpha(p^2) = m_0 + \frac{e_0^2(3+d_l)}{2(2\pi)^4}\int_0^\infty q^2dq^2 \frac{\alpha(q^2)}{\alpha^2(q^2) + q^2\beta(q^2)}\int \frac{d^3\Omega}{p^2+q^2-2(pq)} \tag{2.4a}$$

$$\beta(p^2) = 1 + \frac{e_0^2}{2(2\pi)^4 p^2}\int_0^\infty q^2dq^2 \frac{\beta(q^2)}{\alpha^2(q^2) + q^2\beta^2(q^2)}\int d^3\Omega\left[\frac{2(pq)}{(p-q)^2} + \right.$$

$$\left. + \frac{(p^2+q^2)(pq) - 2p^2q^2}{(p-q)^4}(1-d_l)\right] \tag{2.4b}$$

After the angular integration has been performed and the Landau gauge $d_l = 0$ (see[1,2]) has been fixed, the system (2.4) becomes maximally simplified because $\beta(p^2) \equiv 1$ and only one equation for $\alpha(p^2)$ remains:

$$\alpha(p^2) = m_0 + \frac{3e_0^2}{(4\pi)^2}\int_0^\infty dq^2 \frac{q^2\alpha(q^2)}{\alpha^2(q^2)+q^2}\left[\frac{\theta(p^2-q^2)}{p^2} + \frac{\theta(q^2-p^2)}{q^2}\right] \tag{2.5}$$

which can be transformed into

$$\alpha(x) = m_0 + g^2\left[\frac{1}{x}\int_0^x \frac{y\,\alpha(y)\,dy}{\alpha^2(y)+y} + \int_x^\infty \frac{\alpha(y)\,dy}{\alpha^2(y)+y}\right] \tag{2.6}$$

where $x = p^2$, $y = q^2$, $g^2 = \frac{3}{4\pi}\cdot\frac{e_0^2}{4\pi} \equiv \frac{3}{4\pi}\alpha_0$

(α_0 is the bare fine structure constant).

It can be easily shown that the nonlinear integral equation (2.6) is equivalent to the following nonlinear second-order differential boundary value problem:

$$\frac{d^2}{dx^2}\left[x\alpha(x)\right] = -g^2\frac{\alpha(x)}{\alpha^2(x)+x} \tag{2.7}$$

$$x^2\left[\alpha(x)\right]' \xrightarrow[x\to 0]{} 0\ , \quad \left[x\alpha(x)\right]' \xrightarrow[x\to\infty]{} m_0 \tag{2.8}$$

We have arrived to the question on the existence of a finite solution of the inhomogeneous equation (2.6). Assuming that the solution of Equ. (2.6) can be represented as a power series in g^2 ,

$$\mathcal{L}(x) = m_0 + \sum_{n=1}^{\infty} a_n(x) g^{2n} \tag{2.9}$$

from (2.6) we obtain

$$a_1(x) = \frac{m_0}{x}\left[x - m_0^2 \ln \frac{m_0^2 + x}{m_0^2}\right] + m_0 \lim_{y\to\infty} \ln \frac{m_0^2 + y}{m_0^2 + x} \tag{2.10}$$

A logarithmic divergence which is present here was treated by the authors of Ref.[2] as an indication that it is impossible to construct a finite quantum electrodynamics for $m_0 \neq 0$.

We shall further discuss only the homogeneous integral equation (2.6) ($m_0 = 0$) and the corresponding homogeneous differential boundary value problem (2.7), (2.8). Such a problem can give nontrivial finite Green's functions[2,3] the singularities of which in the time-like region would determine the electron mass. We notice that the presence of infrared divergences "smears out" the pole of the propagator in quantum electrodynamics, and that the electron mass must be determined by the branch point of the propagator, closest to zero.

3. General properties of solutions of the nonlinear integral and differential equations

In this section we pay attention to some solution properties which can be obtained either directly from the integral equation, or from an analysis of the differential equation (2.7).

If $\mathcal{L}(x)$ is a solution of the integral equation

$$\mathcal{L}(x)=g^2\left[\frac{1}{x}\int_0^x \frac{y\,\mathcal{L}(y)\,dy}{\mathcal{L}^2(y)+y} + \int_x^\infty \frac{\mathcal{L}(y)\,dy}{\mathcal{L}^2(y)+y}\right] \tag{3.1}$$

then $-\mathcal{L}(x)$ is also its solution. Equation (3.1) has evidently always a trivial solution $\mathcal{L}(x)=1$. It is not difficult to show that for an arbitrary behaviour of $\mathcal{L}(x)$ for $x\to 0$, there exists a limit

$$\lim_{x\to 0}\frac{1}{x}\int_0^x \frac{y\,\mathcal{L}(y)\,dy}{\mathcal{L}^2(y)+y} = 0 \tag{3.2}$$

Tending to the limit $x\to 0$ in Equ. (3.1), we obtain

$$\mathcal{L}(0) = g^2\int_0^\infty \frac{\mathcal{L}(y)\,dy}{\mathcal{L}^2(y)+y} \tag{3.3}$$

On the basis of the equality (3.3) the integral equation (3.1) can be rewritten in a more suitable form:

$$\mathcal{L}(x) = \mathcal{L}(0) + g^2\int_0^x \frac{\mathcal{L}(y)}{\mathcal{L}^2(y)+y}\left(\frac{y}{x}-1\right)dy \tag{3.4}$$

If $\mathcal{L}(0)=0$, then Equ. (3.4) has only the trivial solution. In fact, let us assume that $\mathcal{L}(0)=0$ and that there exists an interval $(0,\epsilon]$ such that $\mathcal{L}(x)>0$, $x\in(0,\epsilon]$. Then (from (3.4)) it follows

$$\mathcal{L}(\varepsilon) = g^2 \int_0^{\varepsilon} \frac{\mathcal{L}(y)}{\mathcal{L}^2(y)+y} \left(\frac{y}{\varepsilon} - 1 \right) dy < 0 \tag{3.5}$$

and this is in a contradiction with the assumption $\mathcal{L}(\varepsilon) > 0$ (of course, there is an essential assumption here on a monotonous behaviour of the solution in a certain neighbourhood of the point $X = 0$, see below). For an arbitrary $\mathcal{L}(0) \neq C$

Equ. (3.4) can be solved by the perturbation method giving expressions of the types (2.9), (2.10), but in the present case it is finite. In an arbitrary order in g^2, $\mathcal{L}(x)$ has a branch point for $x = -\mathcal{L}^2(0)$. We notice that for $x \approx -\mathcal{L}^2(0)$ the ordinary perturbation theory is not applicable and we shall show below, that the exact solution at that point actually has no singularity.

Let us consider the behaviour of the solution of the differential equation (2.7) in the neighbourhood of the points $X = 0, \infty$. For small values of X we shall look for the solution in the form of a power series

$$\mathcal{L}(x) = X^{\lambda} \sum_{n=0}^{\infty} a_n X^n \tag{3.6}$$

Substituting (3.6) in (2.7) we find that λ can have the following values: 1) $\lambda_1 = 0$, $\lambda_2 = -1$, a_n - an arbitrary constant; 2) $\lambda_3 = \frac{1}{2}$, $a_0^2 = -(1 + \frac{4}{3} g^2)$. Here coefficients a_n satisfy certain nonlinear recurrence relations. Solutions of the differential equation, corresponding to λ_2 and λ_3 do not satisfy the boundary conditions (2.8) therefore, they do not appear as solutions of the boundary problem (2.7), (2.8) and consequently, of the integral equation (3.1). Let us turn to the solution for $\lambda_1 = 0$ which evidently satisfies the boundary condition (2.8) and can be represented in a form

$$\alpha(x) = \sum_{n=0}^{\infty} a_n(g^2, a_0)\, x^n = a_0 - \frac{g^2}{2a_0} x + \frac{g^2(2-g^2)}{12\, a_0^3} x^2 - \frac{g^2(3-5g^2+g^4)}{36\, a_0^5} x^3 + \dots$$
$$= a_0 \sum_{n=0}^{\infty} a_n(g^2, 1) \left(\frac{x}{a_0^2}\right)^n \qquad (3.7)$$

In such a way, the solution we consider depends on an arbitrary constant $a_0 = \alpha(0)$, where if $\alpha(x, g^2)$ is a solution of the equation, then $a_0\, \alpha\left(\frac{x}{a_0^2}; g^2\right)$ is also a solution (this conclusion can also be obtained directly from the differential equation (2.7)). Therefore, it will be enough to further consider only power series with $a_0 = 1$. For $x \to \infty$ the solutions can have the following behaviour: $\alpha(x) \underset{x\to\infty}{\sim} B x^{\delta}$ where δ can take the following values: 1) $\delta_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - g^2}$ where B is an arbitrary constant; 2) $\delta_3 = \frac{1}{2}$, $B^2 = -(1 + \frac{4}{3} g^2)$. The solution corresponding to δ_3 evidently does not satisfy the boundary condition at infinity (2.8) (let us mention that $B x^{\frac{1}{2}}$ is an exact solution of Equ. (2.7)).

Concluding this section let us notice that from the convergence of the power series (3.7) (see Sec. 4) it follows that the equality (3.3)

(3.8)

is identically satisfied for all values of g^2 . In order that the identity (3.8) be satisfied for $g^2 \to 0$, it is necessary that in an asymptotic limit $\alpha(x)$ contains the term $x^{-\frac{1}{2}+\sqrt{\frac{1}{4}-g^2}}$.

4. Investigation of the power series convergence and analytical properties of the solutions

The principal question in the further study of the solutions of the boundary value problem (2.7), (2.8) appears to be investigation of convergence of power series constructed in Section 3 around the point $x=0$. It yields the following possibilities:

1. To obtain an important information concerning analytical properties of the solutions (finite or infinite radius of convergence);

2. To numerically determine $\alpha(x, g^2)$ with an arbitrary degree of accuracy;

3. To correctly define the task of the analytical continuation of the solution onto the whole real axis (time-like momentum region is specially interesting), and into the complex plane.

Substituting the power series (3.7) into Equ. (2.7) one can easily get the following nonlinear recurrence relations for the coefficients:

$$a_0 = 1$$

$$a_{n+1} = -\frac{\sum_{i=1}^{n} c_i b_{n-i} + \{h(n+1) + g^2\} a_n}{(n+1)(n+2)} \tag{4.1}$$

where $b_n = (n+1)(n+2)a_{n+1}$, $c_n = \sum_{i=0}^{n} a_i a_{n-i}$. The complicated structure of these nonlinear recurrence equations does neither admit the obtainment of their solutions in an explicit form nor asymptotic behaviour of a_n for $n \to \infty$ what would in principle be sufficient, for the answer to the question on the power series convergence (3.7). Thus, we use a slightly different analytical method that enables us to find the upper limit for $|a_n|$ and to utilize completely the recurrence relation (4.1) even without solving them. After that, we pursue numerical investigation of the nonlinear equations (4.1), which allows us to estimate the radius of convergence of the power series (3.7) and to determine its dependence on g^2.

We shall carry out the proof of convergence of the power series (3.7) by the method of mathematical induction. It is evident that $|a_0| = 1 = \rho^0$. Supposing that

$$|a_n| \leqslant \frac{\rho^n}{(n+1)^q} \quad , \qquad q \geqslant 3 \tag{4.2}$$

let us prove that $|a_{n+1}| \leqslant \rho^{n+1}(n+2)^{-q}$. Using the recurrence relations (4.1) one can easily learn that the inequality

$$|a_{n+1}| \leqslant \frac{\sum_{i=1}^{n} |c_i||b_{n-i}| + \{n(n+1)+g^2\}|a_n|}{(n+1)(n+2)} \leqslant \frac{\rho^{n+1}}{(n+2)^q} \tag{4.3}$$

is satisfied under the condition that

$$\rho \geqslant \frac{\mathcal{K}^{(q)}}{1-\chi^{(q)}} \tag{4.4}$$

where $\mathcal{K}^{(q)}$ and $\mathcal{X}^{(q)}$ are the upper limits of the functions

$$\mathcal{H}_n^{(q)} = \left(\frac{n+2}{n+1}\right)^q \frac{n(n+1) + g^2}{(n+1)(n+2)} \tag{4.5}$$

$$\mathcal{X}_n^{(q)} = \frac{(n+2)^{q-1}}{(n+1)} \sum_{k=1}^{n} \frac{n-k+1}{(n-k+2)^{q-1}} \sum_{i=0}^{k} \frac{1}{(i+1)^q (k-i+1)^q} \tag{4.6}$$

respectively, i.e. $\mathcal{H}_n^{(q)} \leq \mathcal{K}^{(q)}$, $\mathcal{X}_n^{(q)} \leq \mathcal{X}^{(q)}$. If it is possible to find the upper limit for $\mathcal{X}_n^{(q)}$ such that $\mathcal{X}^{(q)} < 1$, then we automatically get a nontrivial inequality for ρ : $\rho \geqslant \varphi(g^2)$. For all $\rho(g^2)$, satisfying the latter inequality, an estimate for coefficients (4.2) will be proved inductively, i.e. $|a_n| \leqslant \varphi^n(g^2)(n+1)^{-q}$. Hence, we will get absolute and uniform convergence of the power series (3.7) in the circle with the radius $R = \varphi^{-1}(g^2)$. In such a way, the proof by induction is possible only provided that the condition

$$\mathcal{X}^{(q)} \equiv \max_{0 \leq n < \infty} \mathcal{X}_n^{(q)} < 1 \tag{4.7}$$

is fulfilled. Then the best (lower) estimate of the radius of convergence is obtained choosing $\mathcal{X}^{(q)} = \mathcal{X}^{(q)}$ and $\mathcal{K}^{(q)} = \mathcal{H}^{(q)} = \max\limits_{0 \leq n < \infty} \mathcal{H}_n^{(q)}$. Without difficulties one estimates the maximum of $\mathcal{H}_n^{(q)}$:

$$\mathcal{H}^{(q)} < \mathcal{K}_0^{(q)} = 2^q \begin{cases} \frac{g^2}{2} \ , & g^2 \geqslant 2 \\ 1 \ , & g^2 < 2 \end{cases} \tag{4.8}$$

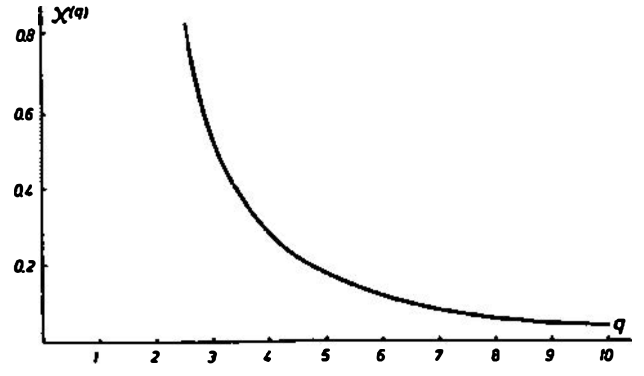

Fig. 1. Results of the numerical investigation of $\chi^{(q)}$ (see def.(4.7)) in the region $2.5 \leqslant q \leqslant 10$.

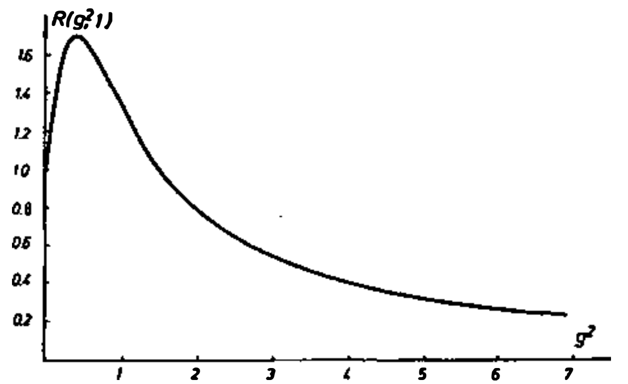

Fig. 2. Results of the numerical investigation of the radius of convergence of the power series (3.7) for $a_o = 1$.

It is much more difficult to estimate the maximum of $\mathcal{X}_n^{(q)}$ It could be achieved analytically by means of rigorous inequalities for double sums contained in the relation (4.6) and by using the properties of Riemann's ξ -functions. This rather complicated calculation with $q=3$ leads to the estimation

$$\mathcal{X}^{(3)} < 0.9198 \tag{4.9}$$

It is more convennient to investigate numerically the maximum of $\mathcal{X}_n^{(q)}$. The results of such investigation for $2.5 \leq q \leq 10$ are illustrated in Fig. 1. It is obvious that in the given region the inequality (4.7) could be satisfied, what is necessary for the proof. In such a way the estimate obtained for g allows one to conclude that the power series (3.7) absolutely and uniformly converge in the circle with the radius

$$R(q^2) = \frac{1 - \mathcal{X}^{(q)}}{\mathcal{K}_0^{(q)}} \tag{4.10}$$

where $\mathcal{K}_0^{(q)}$ is determined from Equ. (4.8). For $\mathcal{X}^{(q)}$ one can utilize either analytical upper limit estimate of the type (4.9), or numerical results Fig.1).

The theoretical estimate of the radius of convergence appears to be very crude and represents only the lower limit estimate for the actual radius of convergence. Hence, the main goal of the above considerations is not to express the most rigorous estimate for the radius, but to express the rigorous proof of the principal fact that the power series

(3.7) can represent the solution of Equ. (2.7) in a circle with a non-zero radius. Starting from the recurrence relations (4.1), for arbitrary g^2 a numerical analysis of $|a_{n+1}|/|a_n|$ and of $\sqrt[n]{|a_n|}$ has been performed for an exact determination of the radius of convergence. Due to that it is very suitable to use Cauchy's[6] convergence criterion, by means of which the value of the radius of convergence

$$R(g^2,1)=[|a_n(g^2,1)|]^{-\frac{1}{n}}, \quad n\gg 1 \tag{4.11}$$

has been obtained. This value changes very slightly with the increase of n. As the final value of the radius of convergence the numerical value of (4.11) corresponding to $n\sim 10^3$ has been taken. We notice that as n changes from $5\cdot 10^2$ to 10^3 three decimal places remain stable. The calculations have been performed for $a_0=1$ and $g^2\in(0,10]$ and results are illustrated in Fig.2, , showing the dependence $R(g^2,1)$ on g^2. The radius of convergence of the power series for an arbitrary a_0 can be easily expressed by the already obtained function $R(g^2,1)$:

$$R(g^2,a_0) = a_0^2\, R(g^2,1) \tag{4.12}$$

The finite radius of convergence of the power series (3.7) points out to the existence of singular points for the function $\alpha(x,g^2)$ at the circle $|x|=R$. The questions on positions of these singular points in the complex plane, as well as the questions concerning their nature (branch points or poles), naturally arise. Inspite of not having the definite answers to these questions let us consider now some interesting results, which are obtained analysing the development $\mathcal{L}(x, g^2)$ for small values of g^2.

For small values of g^2 one can inductively obtain a stronger estimate for the coefficients of the power series (3.7):

$$|a_n| \leq g^2 \frac{\rho^n}{n(n+1)} \quad , \quad n = 1, 2, 3, \cdots \tag{4.13}$$

In order that the inequality (4.13) be satisfied for a few first coefficients (see (3.7), for example $n = n_0 = 1, 2, 3, \ldots$) it is necessary that the corresponding ρ satisfies an inequality

$$\rho > \rho_{n_0} = 1 - \mathcal{A}_{n_0} g^2 + O(g^4) \tag{4.14}$$

where $\mathcal{A}_{n_0}$ can be easily obtained from the solutions of the corresponding inequalities (see (4.13) and explicite expressions for a few first coefficients in (3.7)): $\mathcal{A}_1 = 0, \mathcal{A}_2 = \frac{1}{4}, \mathcal{A}_3 = \frac{5}{9}, \ldots$ It can easily be shown that if the inequality (4.14) is satisfied for some n_0, it will be satisfied for all $n < n_0$. Using the **recurrence** relations (4.1) it is possible to demonstrate, through the estimate of the upper limits of the double sums contained in the relations (4.1), that from the validity of the inequality (4.13) for arbitrary n $(n \geq n_0)$ follows its validity for $n+1$ under the following condition

$$\rho > \tilde{\rho}_{n_0} = 1 - \mathcal{B}_{n_0} g^2 + O(g^4) \tag{4.15}$$

where $\mathcal{B}_{n_0}$ can be easily obtained from the corresponding upper limit estimates and it depends only on n_0, i.e. on the initial step of induction. In such a way, the estimate (4.13) is proved, provided the inequalities (4.14) and (4.15) are both satisfied, i.e. under the condition that

$$\varrho > 1 - \gamma_{n_0} g^2 + O(g^4) \tag{4.16}$$

where $\gamma_{n_0} = \min\{\mathcal{A}_{n_0}, \mathcal{B}_{n_0}\}$. From the inequality (4.16) we obtain the lower limit estimate for the radius of convergence of the power series (3.7) for small values of g^2 :

$$R(g^2;1) \geqslant 1 + \gamma_{n_0} g^2 + O(g^4) \tag{4.17}$$

This estimate appears to be very realistic because it proves the previously numerically obtained fact $\lim\limits_{g^2 \to 0} R(g^2;1) = 1$ (Fig.2.). With the increase of n_0, γ_{n_0} also increases and for $n_0 \leqslant 3$ it follows that $\gamma_{n_0} < 2$. Therefore, we do not write them because we have obtained a stronger estimate starting from other considerations, as shown below (see (4.21)).

The result we have obtained (4.17) means that the solution $\alpha(x,g^2)$ is analytic in a unit circle and that singularities can reach the unit circle for $g^2 \to 0$. Let us try to determine more precisely positions of these singular points in the complex plane. The most natural way is to solve Equ. (3.4) by a perturbation method, i.e. by looking for the representation of $\alpha(x,g^2)$ in g^2 powers by an iteration method. The first iteration step leads to the function

$$\alpha(x,g^2) \simeq 1 + g^2\left[1 - \frac{1+x}{x}\ln(1+x)\right] \tag{4.18}$$

having a branch point at x = -1. In connection with this fact, it is interesting to investigate the behaviour of the solutions around this point. This can be done by means of the double inequalities for $\alpha(x,g^2)$:

$$\alpha(x,g^2) \leq \left[1 - g^2\left\{1 - \frac{1+x}{x}\ln(1+x)\right\}\right]^{-1} \tag{4.19}$$

$$\alpha(x,g^2) \geq 1 + g^2\left[1 + \frac{x+\alpha^2(x,g^2)}{x}\ln\left(1 - \frac{x}{\alpha^2(x,g^2)}\right)\right] \tag{4.20}$$

which one obtains from the integral equation (3.4). They are valid for $x \in (-R(g^2;1), 0]$ and appear to be the consequence of analyticity of $\alpha(x,g^2)$ within the circle of the radius (4.17), i.e. they are valid for $g^2 \to 0$. For x = -1 from the inequalities (4.19) and (4.20), we get $\alpha(-1,g^2) = 1 + g^2 + O(g^4)$ whose substitution in the initial nonlinear equation leads to the conclusion on the existence of a logarithmic branch point on the real axis: $x_0 = -1 - 2g^2 + O(g^4)$ (for small values of g^2). Hence, the double inequalities estimate allows: first, to describe more exactly the behaviour of the radius of convergence (see (4.17))

$$R(g^2;1) = 1 + 2g^2 + O(g^4) \tag{4.21}$$

and second, to predict the positions and character of at least one singular point (the question of the existence and positions of other singular points on the circle remains an open one). Taking into account the inequalities (4.19) and (4.20) it is also possible to obtain interesting inequalities for $\alpha'_x(-1,g^2)$, $(g^2 \to 0)$:

$$\alpha'_x(-1,g^2) \geq g^2\left(\alpha^2(-1,g^2)\left|\ln\{2g^2(1-\tfrac{g^2}{2})\}\right|-1\right) \tag{4.22}$$

$$\alpha'_x(-1,g^2) \leq g^2\left(\alpha^2(-1,g^2)\left|\ln\{2g^2(1-\tfrac{g^2}{2})\}\right|+\mathrm{const}\right) \tag{4.23}$$

from which it follows that $\alpha'_x(-1,g^2) \underset{g^2\to 0}{\sim} g^2 \ln g^2$. The latter fact means that the standard perturbation theory (expansion into powers in g^2) is not applicable as a method of solving Equ. (3.4) and that it is necessary to use a modified variant[7]. Such a situation turns out to be customary for unrenormalized field theories but in the case of quantum electrodynamics this result seems to be somehow unexpected.

5. Conclusion

In the present paper a successive investigation of the solutions of the electron propagator equation in the "finite" electrodynamics of Johnson et al.[2] is initiated. The final goal of such investigation is an explanation of the electron mass generating problem as well as of the possibilities for setting up approximate solutions in quantum electrodynamics without divergences. The first step in this programme appears to be a complete investigation of the nonlinear equation (2.6) for the electron propagator. If a solution has singularities only on the real axis of p^2 ; then the closest to zero singularity determines the unrenormalized electron mass. If there are singularities only for the complex p^2 values then the approach in Ref.[2] to the construction of finite quantum electrodynamics is perspectiveless because in such a case the basic requirements for a "good" quantum field theory are not fulfilled.

We have shown that the solution of the nonlinear boundary value problem (if it exists) is necessarily a holomorfic one in a certain circle of the complex plane p^2, and we have found the radius of this circle numerically for various values of the bare electron charge e_0^2. We notice the existence of two essentially different regions of e_0^2 : for the small enough values of e_0^2 the radius of the circle increases with the increase of e_0^2, but for greater values of e_0^2 this radius decreases. If the singularity determining this radius is on the real axis it means that in the former case the physical mass increases with the increase of e_0^2, and that in the latter case it decreases. It is possible to call the corresponding solutions "the different phases": a normal phase for small e_0^2 and an anomalous one for larger e_0^2. The point where $\frac{dR(e_0^2)}{de_0^2}=0$ is a point of "phase transition". This fact deserves further investigation. Still, it is necessary to check once more whether the singularities lie on the real axis. For small values of e_0^2 approximate solutions have a branch point for real values of p^2, but we have not as yet succeeded to prove that this branch point is a singularity of the exact solution. The difficulties in investigating the analytical properties of the solutions are stressed by the obtained singularity of the exact solution in e_0^2: an exact solution cannot be expanded into a power series in e_0^2.

In the subsequent papers we wish to pursue the investigation initiated here, first of all considering the behaviour of the solutions at the infinity ($p^2 \to \infty$) and the positions of singularities in the complex p^2 plane. That investigation should give an answer on the existence, uniqueness and analytical properties of the solutions of the problem (2.6), (2.7), (2.8).

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R e f e r e n c e s

1) N.N.Bogoliubov and D.V.Shirkov, Introduction to the Theory of Quantized Fields (in Russian) Nauka, Moscow (1973)

2) K.Johnson, M.Baker and R.Willey, Phys.Rev. 136 (1964) B1111; K.Johnson, R.Willey and M.Baker, Phys.Rev. 163 (1967) 1699; M.Baker and K.Johnson, Phys.Rev. 183 (1969) 1292; ibid D3 (1971) 2516; ibid D3 (1971) 2541; K.Johnson and M.Baker. Phys.Rev. D8 (1973) 1110;

3) S.Adler. Phys.Rev. D5 (1972) 3021.

4) R.Haag and Th.A.J.Maris. Phys.Rev. 132 (1963) 2325; Th.A.J.Maris, V.E.Herscovitz and G.Jacob, Phys.Rev.Lett., 12 (1964) 313;

5) R.Fukuda and T.Kugo, Nucl.Phys. B117 (1976) 250;

6) W.Rudin. Principles of Mathematical Analysis, McGrow-Hill Book Company, New York, 1964.

7) V.Sh. Gogohia, A.T.Filippov, Yad.Fiz. 15 (1972) 1294.

ISPITIVANJE REŠENJA DYSON-SCHWINGEROVE JEDNAČINE ZA ELEKTRONSKI PROPAGATOR U KVANTNOJ ELEKTRODINAMICI

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S a d r ž a j

U ovom radu razmatra se Dyson-Schwingerova jednačina za elektronski propagator u "konačnoj" kvantnoj elektrodinamici Johnson-Baker-Willeya. Problem je formulisan u formi 1-dimenzione integralne jednačine i u ekvivalentnoj formi graničnog problema sa diferencijalnom jednačinom drugoga reda. Opšta svojstva rešenja dobijena su analizom integralne jednačine. Granični problem je ispitivan detaljno. Rešenja su predstavljena u obliku stepenih redova po p^2, za koje je analitički pokazano da su konvergentni unutar nekog konačnog radijusa konvergencije. Iz konačnosti radijusa konvergencije sledi da elektronski propagator ima singularitete u kompleksnoj p^2 ravni, pa je u vezi sa tim razmatrano pitanje mogućnosti elektromagnetnog porekla mase elektrona.