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Green's Function in Phase Space

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Green's function is defined in phase space with the help of the eigenfunctions and eigenvalues of the Wigner operator and it can be proved that this function is the Fourier transform of the phase propagator.

As it is known[1], phse space is described with the help of the Wigner operator $W(q,p,\frac{\partial}{\partial q},\frac{\partial}{\partial p})$, of which the eigenfunctions, which coincide to the eigenfunctions in phase spece[2], as well as it's eigenvalues are known. Consequently, fron the usuall definition[3], Green's - function for Wigners operator in the phase space, will be of the form

$$G(\vec{q}\,',\vec{p};\vec{q},\vec{p},E) = \lim_{\eta\to 0} \sum_{n,m} \frac{\varphi^*_{n,m}(\vec{q}\,',\vec{p}\,')\varphi_{n,m}(\vec{q},\vec{p})}{E - E_m + E_n + i\eta}, \tag{1}$$

where $\varphi_{n,m}(\vec{q},\vec{p})$ are the eigenfunctions in phase space and are defined with the help of the eigenfunctions of Schrödinger $\Psi_n(\vec{q})$, in the following way,

$$\varphi_{n,m}(\vec{q},\vec{p}) = \left(\frac{1}{2\pi}\right)^3 \int e^{i\frac{\vec{p}\cdot\vec{\tau}}{h}} \Psi^*_n(\vec{q} + \frac{\vec{\tau}}{2})\Psi_m(\vec{q} - \frac{\vec{\tau}}{2})\, d\vec{\tau}\,. \tag{2}$$

The relation (1) can be written also as

$$G(\vec{q}\,',\vec{p}\,';\vec{q},\vec{p},E) = \lim_{\eta\to 0} \frac{1}{i\hbar} \sum_{n,m} \int_0^\infty e^{\frac{it}{\hbar}[E-E_m+E_n+i\eta]} \varphi^*_{n,m}(\vec{q}\,',\vec{p}\,')\varphi_{n,m}(\vec{q},\vec{p})\, dt\,. \tag{3}$$

The above equation, from the definition of the Wigner propagator in phase space takes form[4,5]

$$G(\vec{q}\,',\vec{p}\,';\vec{q},\vec{p},E) = \lim_{\eta\to 0} \frac{1}{i\hbar} \int_0^\infty e^{\frac{it}{\hbar}[E+i\eta]} f(\vec{q}\,',\vec{p}\,';\vec{q},\vec{p},t)\,dt\,, \tag{4}$$

where the Wigner propagator is given by the relation

$$f(\vec{q}\,',\vec{p}\,';\vec{q},\vec{p},t) =$$

$$= \left(\frac{1}{2\pi}\right)^6 \int\int e^{i\frac{\vec{p}\cdot\vec{\tau}}{\hbar} - \frac{\vec{p}\,'\cdot\vec{\tau}\,'}{\hbar}} \Phi^*(\vec{q}\,' + \frac{\vec{\tau}\,'}{2};\vec{q} + \frac{\vec{\tau}}{2},t)\,\Phi(\vec{q}\,' - \frac{\vec{\tau}\,'}{2};\vec{q} - \frac{\vec{\tau}}{2},t)\,d\vec{\tau}\,d\vec{\tau}\,' \tag{5}$$

and $\Phi(\vec{q}\,',\vec{p},t)$ represents the corresponding Schrödinger propagator.

A simple is the case of the free particles, where the propagator of the Wigner operator is given by the relation [3]

$$f(\vec{q}\,',\vec{p}\,';\vec{q},\vec{p},t) = \delta(\vec{q} - \vec{q}\,' - \frac{\vec{p}\,'}{m}t)\,\delta(\vec{p} - \vec{p}\,') \tag{6}$$

and then Green's function of the free particles in the phase space, because of (4), will be of the form

$$\begin{aligned} G(\vec{q}\,',\vec{q}\,';\vec{q},\vec{p},E) &= \lim_{\eta\to 0}\frac{1}{i\hbar}\int_0^\infty e^{\frac{it}{\hbar}[E+i\eta]}\,\delta(\vec{q}-\vec{q}\,'-\frac{\vec{p}\,'}{m}t)\,\delta(\vec{p}-\vec{p}\,')\,dt \\ &\sim \delta(p-p')\,e^{\frac{iEm}{\hbar p_1'}(q_1-q_1')}\,\delta\left(\frac{q_1-q_1'}{p_1'} - \frac{q_2-q_2'}{p_2'}\right)\delta\left(\frac{q_1-q_1'}{p_1'} - \frac{q_3-q_3'}{p_3'}\right). \end{aligned} \tag{7}$$

We get same result, if use the eigenfunctions and eigenvalues of the Wigner operator for the free particle, which are

$$\varphi_{\vec{k},\vec{k}'}(\vec{q},\vec{p}) = e^{i(\vec{k}-\vec{k}\,')\vec{q}}\,\delta(\vec{k}+\vec{k}\,'+\frac{2\vec{p}}{\hbar}), \tag{8}$$

$$w = \frac{\hbar^2}{2m}(k'^2 - k^2). \tag{9}$$

Because of the definition of Green's function (1) we will have

$$G(\vec{q}\,',\vec{p}\,';\vec{q},\vec{p},E) = \lim_{\eta\to 0}\int\int \frac{e^{i(\vec{k}-\vec{k}\,')(\vec{q}-\vec{q}\,')}\,\delta(\vec{k}+\vec{k}\,'+\frac{2\vec{p}}{\hbar})\,\delta(\vec{k}+\vec{k}\,'+\frac{2\vec{p}\,'}{\hbar})}{E-\frac{\hbar^2}{2m}(\vec{k}\,'-\vec{k}) + i\eta}\,d\vec{k}\,d\vec{k}\,'$$

$$\sim \lim_{\eta\to 0}\delta(\vec{p}-\vec{p}\,')\int \frac{e^{-2i(\vec{k}+\frac{\vec{p}\,'}{\hbar})(\vec{q}-\vec{q}\,')}}{E+\frac{2\hbar}{m}\vec{p}\,'\cdot(\vec{k}+\frac{\vec{p}\,'}{\hbar})+i\eta}\,d\vec{k} \tag{10}$$

or

$$G(\vec{q}\,',\vec{p}\,';\vec{q}.\vec{p}.E) \sim \lim_{\eta\to 0}\delta(p-p')\int\frac{e^{i\vec{u}(\vec{q}-\vec{q}\,')}\,d\vec{u}}{E+\frac{\hbar}{m}\vec{p}\,'\cdot\vec{u}+i\eta}, \qquad \vec{u} = 2(\vec{k}+\frac{\vec{p}\,'}{\hbar}). \tag{11}$$

After integration arround the pole

$$E + \frac{\hbar}{m}(p_1'u_1 + p_2'u_2 + p_3'u_3) = 0 \tag{12}$$

we get the result (7).

Another example is the case of all quadratic forms of the Hamilton operator, for which the Propagator of the Wigner operator has the following form:

$$f(\vec{q}',\vec{p}';\vec{q},\vec{p},E) = \delta[\vec{q}-\vec{q}(t)]\,.\,\delta[\vec{p}-\vec{p}(t)]\ , \tag{13}$$

where $\vec{q}(t)$ and $\vec{p}(t)$ are the solutions of the canonical Hamilton equations; and $\vec{q}',\vec{p}'$, are constants of integration.

For these cases Green's function (6) is written

$$G(\vec{q}',\vec{p}',\vec{q},\vec{p},E) \sim \frac{1}{i\hbar} \int e^{i\frac{Et}{\hbar}}\, \delta[\vec{q}-\vec{q}(t)]\,\delta[\vec{p}-\vec{p}(t)]\ dt\ . \tag{14}$$

In the case of free particles, we have the solutions

$$\vec{q}(t) = \vec{q}' + \frac{t}{m}\vec{p}', \qquad \vec{p}(t) = \vec{p}' \tag{15}$$

and Green's function coincides with (7).

The same occurs when a uniform electric field is present; we have then the solutions

$$\vec{q}(t) = \vec{q}' + \frac{t}{m}\vec{p}' + \frac{e\vec{F}}{2m}t^2, \qquad \vec{p}(t) = \vec{p}' + e\vec{F}\,t \tag{16}$$

and Green's function takes the form:

$$G(\vec{q}',\vec{p}',\vec{q},\vec{p},E) \sim e^{i\frac{E(p_1-p_1')}{e\hbar F_1}} \cdot \delta[F_1(q_2-q_2')-F_2(p_1-p_1')]\,\delta[F_1(q_3-q_3')-F_3(p_1-p_1')]$$
$$\cdot\ \delta(\vec{q}-\vec{q}' - \frac{t_1}{m}\vec{p}' - \frac{et_1^2}{2m}\vec{F}) \tag{17}$$

for $t_1 = \dfrac{p_1-p_1'}{eF_1}$.

Also, in the case of the spherical harmonic oscillator, the solutions of the Hamiltons equations are of the form:

$$\vec{q}(t) = \vec{q}'\ \cos\omega t + \vec{p}'\,\frac{\sin\omega t}{m\,\omega}$$
$$\vec{p}(t) = -\vec{q}'\ m\omega\sin\omega t + \vec{p}'\cos\omega t \tag{18}$$

and Green's function takes the form:

$$G(\vec{q}',\vec{p}';\vec{q},\vec{p},E) \sim \frac{1}{i\hbar}\int_0^\infty e^{\frac{i}{\hbar}Et}\,\delta(\vec{q}-\vec{q}'\cos\omega t-\vec{p}'\frac{\sin\omega t}{m\omega})\,\delta(\vec{p}-\vec{p}'\cos\omega t+\vec{q}'m\omega\sin\omega t)\,dt. \tag{19}$$

In the same way we can study the case of free electron in a uniform magnetic and electric field and other quadratic forms of Hamilton operator, for which the Wigner operator coincides the Liouville operator[7]

References

1) A. Jannussis and N. Patargias: Phys. Lett. 53 A, 357 (1975);

2) L. Cohen: J. Math. Physics 7, 781 (1966);

3) J. Eisele: Modern Quantum Mechanics with Applications to Elementary Particle Physics, Wiley (1969);

4) A. Jannussis and N. Patargias: Wigner Operator and Wigner Distribution (Preprint) 1976;

5) J. Vlieger P. Mazur and S. de Groot: Physics 27, 353 (1961);

6) H. Mori, I. Oppenheim and J. Ross: Studies in Stat. Mechs. Vol. I, North Holland (1962);

7) F. Bopp: Werner Heisenberg und die Physik unserer Zeit (Verl. Vieweg und Sohn, Braunschweig) 128, (1961).