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On Permutations of Desarguesian Sextuples

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ABSTRACT

Desargues's theorem plays an essential role at the axiomatic foundations of Projective Geometry. The configuration behind this theorem contains ten lines, the sides of two triangles, three lines through the center and the axis. We focus on the ordered sextuple of intersection points with the axis and call it Desarguesian. A permutation of this sextuple is called admissible if it preserves the property of being Desarguesian. Some permutations are admissible only if Pappus's theorem holds in the plane. Under this assumption we can prove that for each permutation there exist particular Desarguesian sextuples which remain Desarguesian under the permutation.

Key words: Desargues's theorem, Pappus's theorem, Desarguesian sextuple, involution

MSC2020: 51A30

O permutacijama Desarguesovih šestorki

SAŽETAK

Desarguesov teorem ima bitnu ulogu u aksiomatskim temeljima projektivne geometrije. Konfiguracija iz ovog teorema sastoji se od deset pravaca: šest stranica dvaju trokuta, triju pravaca kroz središte i od osi. Usredotočujemo se na uređenu šestorku sjecišta s osi koju nazivamo Desarguesovom. Permutacija ove šestorke se naziva dopustivom ako čuva svojstvo "biti Desarguesov". Neke su permutacije dopustive samo ako u ravnini vrijedi Pappusov teorem. Pod ovom pretpostavkom dokazujemo da za svaku permutaciju postoje određene Desarguesove šestorke koje ostaju Desarguesove nakon djelovanja te permutacije.

Ključne riječi: Desarguesov teorem, Pappusov teorem, Desarguesova šestorka, involucija

1 Introduction

At the beginning of the twentieth century, Gerhard Hessenberg proved in a synthetic way that Pappus's theorem implies Desargues's theorem [11]. A number of relevant papers has been created since then, all in the context of using Pappus's Theorem for proving Desargues's Theorem (note [6], [15, p. 35], and in particular [18] and [13] with comments on Hessenberg's proofs). The converse is not true. The main result in this direction dates already back to David Hilbert's 'Grundlagen der Geometrie' [12] in 1899 (see [20, p. 78–168]). Hilbert defined an addition and a multiplication of points on a line and proved that each Desarguesian projective plane is isomorphic to a projective coordinate plane over a (not necessarily commutative) field. Moreover, the Desarguesian plane is Pappian if and only if this field is commutative. More about the algebraization of Desarguesian projective planes can be found, e.g., in [16, 10, 2, 4, 13], the latter even with many details on Pappus's theorem. An extensive survey on results obtained during the last about hundred years around the theorems of Desargues and Pappus together with a wealth of references has been provided in [14].

Our work focusses on the Desargues configuration and the ordered sextuple of points on the axis (Fig. 1). We call this sextuple 'Desarguesian' and study permutations that preserve the property of being Desarguesian. We confirm that a Desarguesian projective plane is Pappian if and only if Desarguesian sextuples remain Desarguesian under a certain transposition (Theorem 1) thus rephrasing a result from [9]. Our main result (Theorem 3) states that in Pappian planes for each permutation there exist Desarguesian sextuples which remain Desarguesian under the permutation. However, in the majority of cases the sextuple has to satisfy one particular condition in terms of cross ratios. Our paper concludes with analytic characterizations of Desarguesian sextuples in Desarguesian and in Pappian planes.

The included figures shall illustrate the underlying ideas. In general they cannot not serve as proofs since the real projective plane is Pappian.

2 Desarguesian sextuples

Definition 1 *In a Desarguesian projective plane, a sextuple* $(S_1,S_2,S_3,T_1,T_2,T_3)$ *of mutually different collinear points is called Desarguesian if there exists a Desargues configuration with center Z, axis a and two Z-perspective triangles* $P_1P_2P_3$ *and* $Q_1Q_2Q_3$ *such that for* $i=1,2,3$ *the point* T_i *is the intersection between a and the line passing through Z,* P_i *and* Q_i*, while for each permutation* (i,j,k) *of* $(1,2,3)$ *the point* $S_i \in a$ *is common to the sides* $[P_j,P_k]$ *and* $[Q_j,Q_k]$.[1]

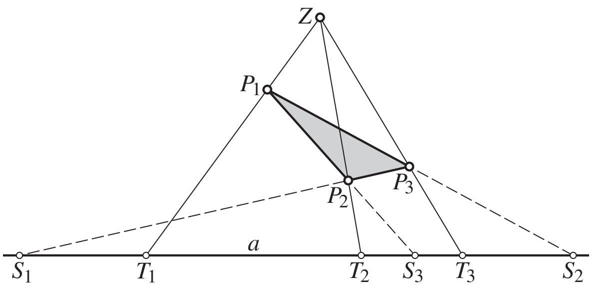

Figure 1: *Desarguesian sextuple* $(S_1,\ldots,T_3)$ *on the line a together with the center Z and one triangle* $P_1P_2P_3$ *of a Desargues configuration. Any Z-perspective triangle* $Q_1Q_2Q_3$ *is not shown.*

Below we only pay attention to one of the two Z-perspective triangles, namely to $P_1P_2P_3$ (see Fig. 1). The existence of six mutually different points of a Desarguesian sextuple implies that $ZP_1P_2P_3$ is a quadrangle and for each i the points S_i and T_i belong to opposite sides of this quadrangle. Moreover, the axis a does not contain any vertex of this quadrangle. In finite projective planes Desarguesian sextuples exist only if the order is at least five. It is wellknown that the axiom of Desargues guarantees the existence of perspective collineations for any given center, axis and pair of corresponding points.

Lemma 1 *In Desarguesian projective planes, a Desarguesian sextuple* $(S_1,\ldots,T_3)$ *on the axis a of a Desargues configuration does not depend on the choice of the center* $Z \notin a$ *and of the vertex* $P_1 \in [Z,T_1]\setminus\{Z,T_1\}$.

Proof. Beside a configuration with Z,P_1,P_2,P_3 let another center $Z' \notin a$ and a first vertex $P'_1 \in [Z',T_1]\setminus\{Z',T_1\}$ of another triangle $P'_1P'_2P'_3$ be given. Then we obtain the remaining vertices as $P'_2 = [Z',T_2]\cap[P'_1S_3]$ and $P'_3 = [Z',T_3]\cap[P'_1S_2]$. If Z,Z',P_1,P'_1 is a quadrangle, then exists in the Desarguesian plane a unique perspective collineation κ with the axis a and the center $C := [Z,Z']\cap[P_1,P'_1]$ which sends P_1 to P'_1 (Fig. 2). Since κ maps also P_2 to P'_2 and P_3 to P'_3, the side $[P'_2,P'_3]$ must pass through S_1.

If three of the four points Z,Z',P_1,P'_1 are collinear, then we choose other points $Z'' \notin a$ and $P''_1 \in [Z'',T_1]\setminus\{Z'',T_1\}$ such that Z,Z'',P_1,P''_1 as well as Z',Z'',P'_1,P''_1 form quadrangles. Now we can conclude as before using two collineations, one with $Z \mapsto Z''$ and the other with $Z'' \mapsto Z'$. This confirms the claim. □

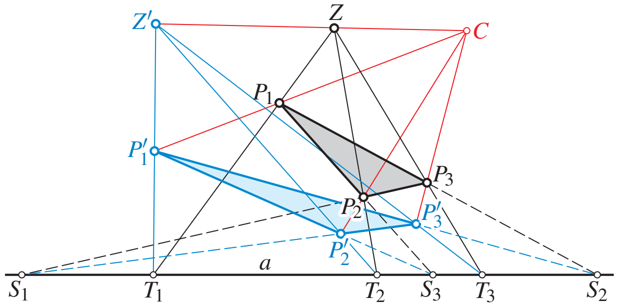

Figure 2: *There is a perspective collineation* κ *with center C and axis a that sends* Z,P_1,P_2,P_3 *respectively to* Z',P'_1,P'_2,P'_3.

Lemma 2 *If the sextuple* $(S_1,S_2,S_3,T_1,T_2,T_3)$ *is Desarguesian, then for each permutation* (i,j,k) *of the indices* $(1,2,3)$ *the sextuple* $(S_i,S_j,S_k,T_i,T_j,T_k)$ *is Desarguesian, too.*

Proof. This is trivial since we only need to renumber the vertices in the triangle $P_1P_2P_3$. □

Lemma 3 *If the sextuple* $(S_1,S_2,S_3,T_1,T_2,T_3)$ *is Desarguesian, then the sextuple* $(S_1,T_2,T_3,T_1,S_2,S_3)$ *is Desarguesian, too.*

Proof. Exchange in a corresponding configuration the points Z and P_1, while P_2 and P_3 and the axis a remain unchanged. This implies $S_2 \mapsto T_3$, $S_3 \mapsto T_2$, $T_2 \mapsto S_3$, and $T_3 \mapsto S_2$ and means that $(S_1,T_3,T_2,T_1,S_3,S_2)$ is Desarguesian. Finally we exchange the indices 2 and 3 in the sense of Lemma 2. Similarly, we can replace Z with P_2 or P_3 and obtain two other Desarguesian sextuples. □

Remark 1 *The statements of the Lemmas 2 and 3 can already be found in [16, p. 128]. The triples* (S_1,S_2,S_3) *and* (T_1,T_2,T_3) *are respectively called triangle triple and point triple (German: Dreieckstripel and Sterntripel), and the sextuple* $(S_1,\ldots,T_3)$ *has the name quadrangular set of points (German: Viereckschnitt). The English notation dates back to [19, p. 49], where the symbol* $Q(T_1T_2T_3;S_1,S_2,S_3)$ *is used for Desarguesian sextuples. In [5, p. vii], this symbol is replaced by* $(S_1T_1)(S_2T_2)(S_3T_3)$*. In [17, p. 129], the name quadrilateral set stands for a Desarguesian sextuple. In [8], a 'quadrangular section' is axiomatically introduced as a ternary relation on the pairs*

[1] Throughout the paper we use the symbol $[X,Y]$ for the line connecting the two points X and Y.

of a set called 'line', and the line is called Desarguian if the statements of Lemma 2 and Lemma 3 hold. The Lemmas 2 and 3 are also subject of [9, Lemma 5.6], where the symbol $(S_1,T_1;S_2,T_2;S_3,T_3)$ stands for the Desarguesian sextuple $(S_1,\dots,T_3)$.

Definition 2 *A permutation of the six points of a Desarguesian sextuple is called admissible if the permuted sextupel is still Desarguesian.*

In view of Desarguesian sextuples, we call the pair consisting of the first and the fourth point of a sextuple the *first pair*. Similarly, the *second pair* consist of the second and the fifth point, and the *third pair* of the third and sixth point of the sextuple.

2.1 Characterizations of Pappian planes

In order to illustrate the role of admissible permutations, we present and prove two pertinent results that can already be found in the literature. The first is hidden in the statement of [9, Theorem 5.12].

Theorem 1 *A Desarguesian projective plane is Pappian if and only if for each Desarguesian sextuple $(S_1,S_2,S_3,T_1,T_2,T_3)$ the exchange of (S_1,T_1) in the first pair is admissible.*

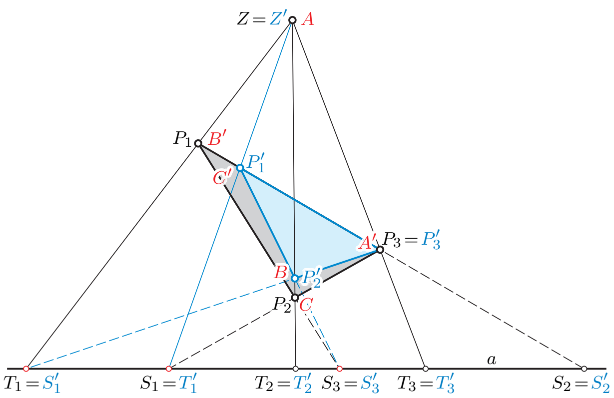

Figure 3: *Illustrating the statement of Theorem 1.*

Proof. Let $Z,\dots,P_3$ and $Z',\dots,P'_3$ be two Desargues configurations which share the axis a and on a the points $S_i = S'_i$ and $T_i = T'_i$ for $i = 2,3$. In both cases, the six points $S_1,\dots,T_3$ and $S'_1,\dots,T'_3$ are supposed to be mutually different. Then, without loss of generality, we can apply Lemma 1 and replace Z' by Z and P'_3 by P_3 while the Desarguesian sextuples on a remain unchanged.
Suppose that for the second configuration holds $S'_1 = T_1$ and $T'_1 = S_1$ (see Fig. 3). Then a Pappus configuration is included: there are two triples of collinear points $(A,B,C) := (Z,P'_2,P_2)$ and $(A',B',C') := (P_3,P_1,P'_1)$, and the intersection points $T_1 = [A,B'] \cap [A',B]$, $S_1 = [A,C'] \cap [A',C]$ and $S_3 = [B,C'] \cap [B',C]$ are collinear, too. This is the basis for confirming the stated characterization of Pappian projective planes.

(i) If the projective plane is Pappian, then $T'_1 = S_1$ implies that $a = [S_1,S_3]$ is the Pappus axis. Consequently, the line $[A',B] = [P'_3,P'_2]$ must intersect the axis a at the point $S'_1 = T_1$ on $[A,B']$. This means that both sextuples $(S_1,S_2,S_3,T_1,T_2,T_3)$ and $(S'_1,\dots,T'_3) = (T_1,S_2,S_3,S_1,T_2,T_3)$ are Desarguesian, i.e., the switch $S_1 \leftrightarrow T_1$ preserves Desarguesian sextuples.
If in the Pappian plane the Fano axiom holds, then this is an immediate consequence of Desargues's involution theorem: pairs of opposite sides of the quadrangle $ZP_1P_2P_3$ intersect the line a in pairs (S_i,T_i) of an involution (note, e.g., [1, 5] or [7, Sect. 7.4]).

(ii) Conversely, let two collinear triples of points (A,B,C) and (A',B',C') on different lines be given such that no point coincides with the intersection of the two lines. Then we recognize two triangles $(P_1,P_2,P_3) := (B',C,A')$ and $(P'_1,P'_2,P'_3) := (C',B,A')$ in the figure as well as the center $Z = Z' := A$ of two Desargues configurations. We define the axis a as the connection of the two points $[A,C'] \cap [A',C]$ and $[B,C'] \cap [B',C]$. Thus, these points coincide respectively with $S_1 = T'_1$ and $S_3 = S'_3$. Moreover holds $S_2 = S'_2$, $T_2 = T'_2$ and $T_3 = T'_3$ per definition. Now we can state: If the switch $S_1 \leftrightarrow T_1$ preserves a sextuple of being Desarguesian, then follows $S'_1 = T_1$, which means that also the point $[A,B'] \cap [A',B]$ lies on the axis a. This confirms the validity of Pappus's theorem for the given points $A,\dots,C'$. □

According to Theorem 1, only in Pappian planes all Desarguesian sextuples $(S_1,\dots,T_3)$ remain Desarguesian if we commute the points in one pair (S_i,T_i). Note that by Lemma 2 a simultaneous switch in two pairs is already possible in Desarguesian planes. The corollary below addresses the simultaneous switch in all three pairs. The respective statement can already be found in [16, p. 140, no. 5].

Corollary 1 *A Desarguesian projective plane is Pappian if and only if each Desarguesian sextuple $(S_1,S_2,S_3,T_1,T_2,T_3)$ remains Desarguesian after exchanging S_i with T_i for all $i = 1,2,3$.*

Proof. We apply at first Lemma 3 and replace $(S_1,\dots,T_3)$ by $(S_1,T_2,T_3,T_1,S_2,S_3)$. Then the claim (note Fig. 4) follows directly from Theorem 1. □

Another characterization of Pappian planes among the Desarguesian projective planes can, e.g., be found in [3].

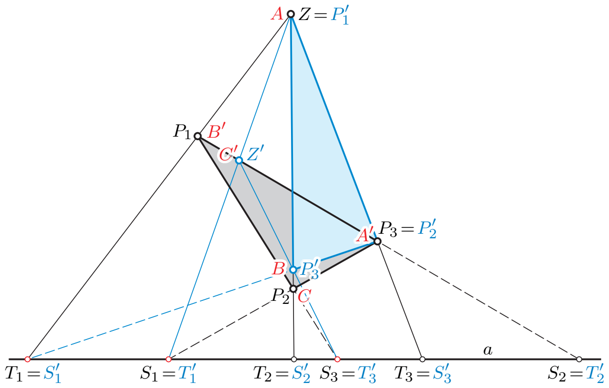

Figure 4: *Illustrating the statement of Corollary 1.*

3 Permutations in Desarguesian and in Pappian projective planes

The goal of this section is to determine for a given Desarguesian sextuple all permutations which are admissible, i.e., which preserve the property of being Desarguesian.

3.1 Non-Pappian case

Theorem 2 *If in a Desarguesian projective plane the sextuple* $(S_1,\dots,T_3)$ *is Desarguesian, then there exist at least* 24 *admissible permutations. In Pappian projective planes there exist at least* 48 *admissible permutations.*

Proof. In Desarguesian planes we can combine each permutation of the indices $1,2,3$ according to Lemm 2 with each of the permutations mentioned in Lemma 3 or with the identity.

In Pappian planes we proceed in the following way: The first three points in the sextuple define the last three since the sequence of subscripts is the same but the symbols S and T are exchanged. There are six possibilities for the sequence of indices. By virtue of Theorem 1 follows that for each symbol in the first triple we have the choice between S and T. This results in $6\cdot 2^3 = 48$ admissible permutations. In non-Pappian planes only those permutations are admissible where the first triple contains either no or exactly two T-points. □

3.2 Three instructive examples

As demonstrated in the following examples, under special conditions the number of admissible permutations can be higher than those mentioned in Theorem 2. These examples will be of fundamental importance for the proof of Theorem 3.

Example 1 Let $(S_1,\dots,T_3)$ be a Desarguesian sextuple in a Pappian Fano-plane where S_3 and T_2 are harmonic with respect to (w.r.t., for short) S_1 and T_1. What are the consequences?

The Desarguesian involution $\iota_1 : S_i \mapsto T_i$ for $i = 1,2,3$ transforms the harmonic quadruple (S_1T_1, S_3T_2) into (T_1S_1, T_3S_2). This means that the pairs (S_2,T_3) and (S_3,T_2) are corresponding under the involution ι_2 with the fixed points S_1 and T_1. The product $\iota_2\circ\iota_1$ commutes the points in the pairs (S_1,T_1), (S_2,S_3), and (T_2,T_3). Therefore, $\iota_3 := \iota_2\circ\iota_1 = \iota_1\circ\iota_2$ is again an involution and shows that the sextuple $(S_1,S_2,T_2,T_1,S_3,T_3)$ is again Desarguesian as well as $(S'_1,\dots,T'_3) = (S_1,T_2,S_3,T_1,T_3,S_2)$ (Fig. 5). These permutations are different from the 48 as mentioned in Theorem 2, since the resulting sextuples contain a pair with two S-points.

By exchanging the elements in the first pair or in the second pair, as well as by exchanging the subscripts 2 and 3, we obtain eight additional admissible permutations. On the other hand, the given harmonic position allows a simultaneous change $S_2 \leftrightarrow T_3$ and $S_3 \leftrightarrow T_2$. Thus, we end up with 16 additional admissible permutations, where (S_1,T_1) or (T_1,S_1) remains as the first pair:

$(S_1,S_2,T_2,T_1,S_3,T_3)$, $(S_1,S_3,T_3,T_1,S_2,T_2)$, $(T_1,S_2,T_2,S_1,S_3,T_3)$, $(T_1,S_3,T_3,S_1,S_2,T_2)$, $(S_1,S_3,T_2,T_1,S_2,T_3)$, $(S_1,S_2,T_3,T_1,S_3,T_2)$, $(T_1,S_3,T_2,S_1,S_2,T_3)$, $(T_1,S_2,T_3,S_1,S_3,T_2)$, $(S_1,T_3,S_3,T_1,T_2,S_2)$, $(S_1,T_2,S_2,T_1,T_3,S_3)$, $(T_1,T_3,S_3,S_1,T_2,S_2)$, $(T_1,T_2,S_2,S_1,T_3,S_3)$, $(S_1,T_2,S_3,T_1,T_3,S_2)$, $(S_1,T_3,S_2,T_1,T_2,S_3)$, $(T_1,T_2,S_3,S_1,T_3,S_2)$, $(T_1,T_3,S_2,S_1,T_2,S_3)$.

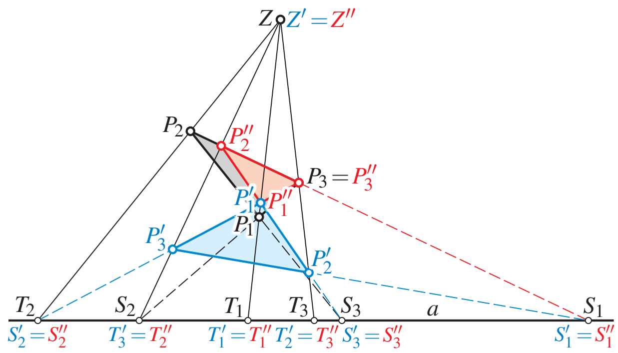

Figure 5: *A Desarguesian sextuple according to Example 1 with* $(S'_1,\dots,T'_3) = (S_1,T_2,S_3,T_1,T_3,S_2)$ *and the harmonic quadruples* (S_1T_1, S_2T_3) *and* (S_1T_1, S_3T_2) *(Type Ib in Theorem 3). After exchanging* T'_2 *and* T'_3*, we obtain a Desarguesian sextuple according to Example 2 with* $(S''_1,\dots,T''_3) = (S'_1,S'_2,S'_3,T'_1,T'_3,T'_2)$ *(Type Ia).*

Example 2 Suppose that with the Desarguesian sextuple $(S_1,\dots,T_3)$ also $(S_1,S_2,S_3,T_1,T_3,T_2)$ is Desarguesian.

Then in a Pappian Fano-plane there are two involutions, the involution ι_1 with $S_i \leftrightarrow T_i$ and ι_2 with $S_1 \leftrightarrow T_1$, $S_2 \leftrightarrow T_3$, and $S_3 \leftrightarrow T_2$. The product of these involutions sends, independent of the order, $S_1 \mapsto S_1$, $S_2 \leftrightarrow S_3$, $T_1 \mapsto T_1$, and $T_2 \leftrightarrow T_3$. Therefore this product is the involution ι_3 which keeps the points S_1 and T_1 fixed. Consequently, the pairs

(S_2,S_3) and (T_2,T_3) are in harmonic position w.r.t. (S_1,T_1) (note $(S_1',\dots,T_3')$ and $(S_1'',\dots,T_3'')$ in Fig. 5). This is sufficient, because ι_2 is the product of ι_1 and ι_3.

We have again three mutually commuting involutions like in Example 1. Indeed, we obtain the harmonic quadruples here from those in the Example 1 by exchanging the points S_2 and T_2. This can also be used to transfer the 16 additional admissible permutations from Example 1 to those of Example 2.

Example 3 Can it happen that with $(S_1,\dots,T_3)$ also $(S_1,S_2,S_3,T_3,T_1,T_2)$[2] is Desarguesian?

If both sextuples are Desarguesian, then there are two involutions involved, $\iota_1 : S_i \leftrightarrow T_i$ and $\iota_2 : S_1 \leftrightarrow T_3$, $S_2 \leftrightarrow T_1$, $S_3 \leftrightarrow T_2$. The product $\iota_1\circ\iota_2$ with $S_i \mapsto S_{i-1}$ and $T_j \mapsto T_{j+1}$ (subscripts modulo 3) is a cyclic projectivity with $(\iota_1\circ\iota_2)^3 = \mathrm{id}$ and $\iota_2\circ\iota_1 = (\iota_1\circ\iota_2)^{-1}$. On the other hand, there is a third involution involved: $\iota_3 := \iota_1\circ\iota_2\circ\iota_1 = \iota_2\circ\iota_1\circ\iota_2$ with $S_1 \leftrightarrow T_2$, $S_2 \leftrightarrow T_3$, $S_3 \leftrightarrow T_1$ satisfies $\iota_3^2 = (\iota_1\circ\iota_2)^3 = \mathrm{id}$. These three involutions are symmetric in the sense that for each permutation (i,j,k) of $(1,2,3)$ the transformation of ι_i by ι_j equals ι_k, i.e., $\iota_k = \iota_j\circ\iota_i\circ\iota_j$.

Necessary and sufficient for such Desarguesian sextuples in a Pappian Fano-plane is, for example, that the cross ratios $\mathrm{cr}(S_1S_2S_3T_1)$ and $\mathrm{cr}(S_3S_1S_2T_2)$ are equal[3] (see Fig. 7). The necessity follows from the projectivity $\iota_1\circ\iota_2$. Conversely, in the case of equal cross ratios there exists the projectivity π: $S_i \mapsto S_{i-1}$, $T_1 \mapsto T_2$. If $\iota_1 : S_i \leftrightarrow T_i$ is an involution, then $\iota_1\circ\pi$ maps $S_1 \mapsto T_3$, $S_2 \mapsto T_1$, $S_3 \mapsto T_2$, and $T_1 \mapsto S_2$. Therefore $\iota_1\circ\pi$ is an involution and equal to ι_2 which guarantees the requested Desarguesian sextuple.

Due to $\iota_3 = \iota_1\circ\iota_2\circ\iota_1$, also $(S_1,S_2,S_3,T_2,T_3,T_1)$ is Desarguesian (Fig. 7). By virtue of Lemma 2 and Theorem 1, there exist at least 96 additional admissible permutations.

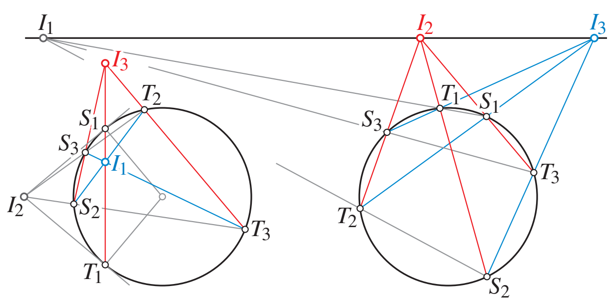

Figure 6: *Suppose that the axis a as carrier of the Desarguesian sextuple $(S_1,\dots,T_3)$ is stereographically projected on a conic. Then the points I_1,I_2,I_3 are the respective centers of the involutions ι_1,ι_2,ι_3 as used in Example 1 (left, Type Ib) and in Example 3 (right, Type IIa).*

At the examples above, the involutions ι_1, ι_2 and ι_3 act on the points $S_1,\dots,T_3$ of the axis a. If the involutions are shown in an isomorphic model on a conic, then each involution ι_j is represented by a center of involution I_j, the common point of the chords connecting corresponding points (see, e.g., [7, p. 251]). Figure 6 shows the involved centers I_1,I_2,I_3 in the Examples 1 and 2 on the left and that of Example 3 on the right.

3.3 Conditional cases in Pappian planes

Theorem 3 *Given a Pappian Fano-plane, then for each permutation of six elements there exist Desarguesian sextuples which remain Desarguesian under the permutation. Beside the 48 admissible permutations of the Desarguesian sextuple $(S_1,\dots,T_3)$ according to Theorem 2 we distinguish two types of permutations which are admissible under special conditions.*

Type I, *the permutation fixes one unordered pair of the given Desarguesian sextuple $(S_1,\dots,T_3)$: Then, in the permuted sextuple the remaining pairs are either (a) of the form (S_i,T_j) in any order, or (b) they contain a pair (S_i,S_j). After appropriate relabeling this yields either*
Type Ia: $(S_1,S_2,S_3,T_1,T_3,T_2)$ *(note Example 2) or*
Type Ib: $(S_1,T_2,S_3,T_1,T_3,S_2)$ *(Example 1, Fig. 6).*
Referring to the first case Ia, $(S_1,S_2,S_3,T_1,T_3,T_2)$ is Desarguesian if and only if (S_1,T_1) separates the pairs (S_2,S_3) and (T_2,T_3) harmonically. On the other hand, the sextuple Ib $(S_1,T_2,S_3,T_1,T_3,S_2)$ is Desarguesian if and only if (S_1,T_1) separates (S_2,T_3) and (S_3,T_2) harmonically. The latter sextuple can be converted into the first one by exchanging S_2 with T_2.

Type II, *no unordered pair of the given Desarguesian sextuple $(S_1,\dots,T_3)$ remains fixed under the permutation: Then, in the permuted sextuple the pairs are either (a) of the form (S_i,T_j), or (b) they contain a pair (S_i,S_j). There is a numbering such that we obtain either*
Type IIa: $(S_1,S_2,S_3,T_3,T_1,T_2)$ *(Example 3) or*
Type IIb: $(S_1,T_2,S_3,T_3,T_1,S_2)$.
Necessary and sufficient for the first sextuple IIa being Desarguesian is that there are equal cross ratios $\mathrm{cr}(S_1S_2S_3T_1) = \mathrm{cr}(S_3S_1S_2T_2)$. In the second case IIb the permutation is admissible if and only if $\mathrm{cr}(S_1T_2S_3T_1) = \mathrm{cr}(S_3S_1T_2S_2)$. Again, the second sextuple arises from the first one by exchanging S_2 with T_2.

Proof. Due to Lemma 1 and Theorem 1, in Pappian planes the Desarguesian sextuples remain Desarguesian if the order of the included pairs or the order in the pairs changes. For each permutation of the six points $(S_1,\dots,T_3)$ which keeps one pair fixed we can assume that this pair is (S_1,T_1).

[2]Note that here the permutation $(3,1,2)$ of the subscripts at the T-points is even. In comparison, the permutation $(1,3,2)$ in Example 2 is odd.

[3]This condition is not unique. It is easy to find other point quadruples with the property that the equality of their cross ratios guarantees that the requested permutation is admissible.

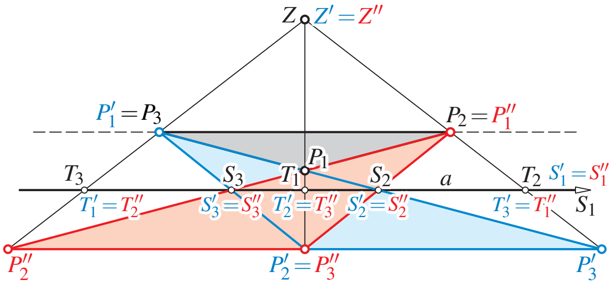

Figure 7: *Particular case according to Example 3 with* $(S'_1,\ldots,T'_3) = (S_1,S_2,S_3,T_3,T_1,T_2)$ *(Type IIa) and* $(S''_1,\ldots,T''_3) = (S_1,S_2,S_3,T_2,T_3,T_1)$ *where* S_1 *is a point at infinity. The depicted sextuple* $(S_1,\ldots,T_3)$ *satisfies also the conditions for Type Ia from Theorem 3, for example with the triangle* $P'_2P'_1P_2$.

If the other pairs in the permuted sextuple are of the form (S_i,T_j) with $i \neq j$ (Type Ia), then they can be assumed as (S_2,T_3) and (S_3,T_2). The sextuple $(S_1,S_2,S_3,T_1,T_3,T_2)$ was analyzed in Example 2 under the condition that the Fano axiom holds. If one pair in the permuted sextuple is of the form (S_i,S_j) (Type Ib), then there is a particular numbering which yields $(S_1,T_2,S_3,T_1,T_3,S_2)$ as treated in Example 1.

The permutations of Type II are characterized by the condition that no unordered pair (S_i,T_i) is preserved. In the case (a) all pairs are supposed to be of the form (S_i,T_j) with $i \neq j$. Then there is a particular numbering such that the permuted sextuple is $(S_1,S_2,S_3,T_3,T_1,T_2)$ as analyzed in Example 3. If the permuted sextuple contains a pair of two S-points, then it contains also a pair of two T-points and a mixed pair (Type IIb). Hence, we can assume a numbering such that the permuted sextuple is $(S_1,T_2,S_3,T_3,T_1,S_2)$, which arises from Type IIa by exchanging S_2 and T_2. □

Remark 2 *A count in Pappian Fano-planes reveals: For sextuples of Type I there exist at least 64 additional admissible permutations. Sextuples of Type II have at least 144 additional admissible permutations.*

Beside the 48 admissible permutations for generic Desarguesian sextuples, there exist 144 admissible permutations for Desarguesian sextuples which satisfy the harmonic positions of the Type Ia and as many for the Type Ib. There are 96 permutations corresponding to Desarguesian sextuples of Type IIa and 288 of Type IIb.

The sextuple depicted in Fig. 7 satisfies simultaneously the conditions for Type IIa (Example 3) and Ia (Example 2). This means that $\mathrm{cr}(S_1S_2S_3T_1) = \mathrm{cr}(S_3S_1S_2T_2)$ and (S_1T_1,S_2S_3) as well as (S_1T_1,T_2T_3) are harmonic quadruples.

For sextuples which are simultaneously of the Types IIa and Ib, the cross ratio $\delta := \mathrm{cr}(S_1S_2S_3T_1) = \mathrm{cr}(S_3S_1S_2T_2)$ has to satisfy $\mathrm{cr}(S_1T_1S_3T_2) = -1$, hence $\mathrm{cr}(S_1S_3T_1T_2) = 2$ and $\mathrm{cr}(S_1S_3S_2T_1) = 1-\delta$, and as the product $\mathrm{cr}(S_1S_3S_2T_2) = 2(1-\delta) = \frac{1}{\delta}$. This results in the quadratic condition $2\delta^2 - 2\delta + 1 = 0$ which has no solution in the real projective plane.

Since exchanging S_2 and T_2 means a switch between the Types Ia and Ib as well as between IIa and IIb, Fig. 7 can easily be converted into an example which satisfies the conditions of the Types Ib and IIb. On the other hand, no real sextuple can simultaneously satisfy the conditions of the Types Ia and IIb.

4 Analytic characterizations of Desarguesian sextuples

4.1 Non-Pappian plane

In Desarguesian projective coordinate planes the points X are one-dimensional subspaces $\mathbf{x}\mathbb{F}$ of a three-dimensional right vector space over a (not necessarily commutative) field $\mathbb{F}$. The question arises what replaces the Desarguesian involution in non-Pappian planes?

Theorem 4 *The sextuple* $(\mathbf{s}_1\mathbb{F},\ldots,\mathbf{t}_3\mathbb{F})$ *of mutually different points in a Desarguesian projective plane is Desarguesian if and only if there is a representation which satisfies* $\mathbf{s}_1+\mathbf{s}_2+\mathbf{s}_3 = \mathbf{0}$ *and*

$$\mathbf{t}_1 = \mathbf{s}_2\lambda_3 - \mathbf{s}_3\lambda_2,\ \mathbf{t}_2 = \mathbf{s}_3\lambda_1 - \mathbf{s}_1\lambda_3,\ \mathbf{t}_3 = \mathbf{s}_1\lambda_2 - \mathbf{s}_2\lambda_1 \quad (1)$$

for some $\lambda_1,\lambda_2,\lambda_2 \in \mathbb{F}\setminus\{0\}$.

Proof. (i) We begin with an analytic standard proof of Desargues's theorem: If the triangles $\mathbf{p}_1\mathbb{F}\mathbf{p}_2\mathbb{F}\mathbf{p}_3\mathbb{F}$ and $\mathbf{q}_1\mathbb{F}\mathbf{q}_2\mathbb{F}\mathbf{q}_3\mathbb{F}$ are perspective w.r.t. the point $\mathbf{z}\mathbb{F}$, then we can assume

$$\mathbf{p}_1+\mathbf{q}_1 = \mathbf{p}_2+\mathbf{q}_2 = \mathbf{p}_3+\mathbf{q}_3 = \mathbf{z}.$$

Hence, the intersection points between corresponding sides have coordinate vectors

$$\mathbf{s}_1 = \mathbf{p}_2-\mathbf{p}_3 = \mathbf{q}_3-\mathbf{q}_2,\ \mathbf{s}_2 = \mathbf{p}_3-\mathbf{p}_1 = \mathbf{q}_1-\mathbf{q}_3,\\ \mathbf{s}_3 = \mathbf{p}_1-\mathbf{p}_2 = \mathbf{q}_2-\mathbf{q}_1.$$

They satisfy $\mathbf{s}_1+\mathbf{s}_2+\mathbf{s}_3 = \mathbf{0}$, which expresses their collinearity.

The three vectors $(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$ form a basis of our vector space. Hence, there are scalars $\lambda_1,\lambda_2,\lambda_3 \in \mathbb{F}$ such that

$$\mathbf{z} = \mathbf{p}_1\lambda_1+\mathbf{p}_2\lambda_2+\mathbf{p}_3\lambda_3.$$

Since $\mathbf{z}\mathbb{F}\,\mathbf{p}_1\mathbb{F}\mathbf{p}_2\mathbb{F}\mathbf{p}_3\mathbb{F}$ is supposed to form a quadrangle, we have $\lambda_1\lambda_2\lambda_3 \neq 0$. This representation of $\mathbf{z}$ implies for the intersection point $\mathbf{t}_1\mathbb{F}$ between the lines $[Z,P_1]$ and $[S_2,S_3]$ a solution

$$\mathbf{t}_1 = \mathbf{s}_2\lambda_3 - \mathbf{s}_3\lambda_2,$$

since $\mathbf{t}_1$ is a linear combination of $\mathbf{s}_2$ and $\mathbf{s}_3$, and on the other hand

$$\begin{aligned}\mathbf{t}_1 &= \mathbf{z} - \mathbf{p}_1(\lambda_1+\lambda_2+\lambda_3)\\ &= (\mathbf{p}_2-\mathbf{p}_1)\lambda_2 + (\mathbf{p}_3-\mathbf{p}_1)\lambda_3 = \mathbf{s}_2\lambda_3 - \mathbf{s}_3\lambda_2.\end{aligned}$$

(ii) For proving the converse, we follow the computation in the reverse order: Let the vectors $\mathbf{s}_1,\dots,\mathbf{t}_3$ with $\mathbf{s}_1+\mathbf{s}_2+\mathbf{s}_3=\mathbf{0}$ be given according to (1). We specify any point $\mathbf{p}_1\mathbb{F}$ outside the line $[S_2,S_3]$ and define $\mathbf{p}_2 := \mathbf{p}_1 - \mathbf{s}_3$ and $\mathbf{p}_3 := \mathbf{s}_2 + \mathbf{p}_1$. This implies $\mathbf{p}_2-\mathbf{p}_3 = -\mathbf{s}_3-\mathbf{s}_2 = \mathbf{s}_1$ and defines a triangle $\mathbf{p}_1\mathbb{F}\ \mathbf{p}_2\mathbb{F}\ \mathbf{p}_3\mathbb{F}$ with sides passing through the respective points $\mathbf{s}_i\mathbb{F}$ for $i=1,2,3$. We can verify that the point $\mathbf{z}\mathbb{F}$ with $\mathbf{z} := \mathbf{p}_1\lambda_1+\mathbf{p}_2\lambda_2+\mathbf{p}_3\lambda_3$ lies aligned with $\mathbf{t}_i\mathbb{F}$ and $\mathbf{p}_i\mathbb{F}$ for each i, thus confirming that the sextuple $(\mathbf{s}_1\mathbb{F},\dots,\mathbf{t}_3\mathbb{F})$ is Desarguesian. □

4.2 Pappian plane

In Pappian planes the underlying field $\mathbb{F}$ is commutative. Then, in the representations (1) of the three vectors $\mathbf{t}_1$, $\mathbf{t}_2$ and $\mathbf{t}_3$ as linear combinations of respectively two $\mathbf{s}$-vectors, the product $\lambda_3\lambda_1\lambda_2$ of the coefficients of the first terms equals the negative product of those of the second terms. This can be generalized.

Lemma 4 *In a Pappian projective plane, let $(\mathbf{s}_1\mathbb{F},\dots,\mathbf{t}_3\mathbb{F})$ be a sextuple of mutually different collinear points. Then this sextuple is Desarguesian if and only if for the representation*

$$\begin{aligned}\mathbf{t}_1 = \mathbf{s}_2\mu_{12}+\mathbf{s}_3\mu_{13},\quad \mathbf{t}_2 = \mathbf{s}_3\mu_{23}+\mathbf{s}_1\mu_{21},\\ \mathbf{t}_3 = \mathbf{s}_1\mu_{31}+\mathbf{s}_2\mu_{32}\end{aligned} \tag{2}$$

holds

$$\mu_{12}\mu_{23}\mu_{31} = -\mu_{13}\mu_{21}\mu_{32}. \tag{3}$$

Proof. The equality (3) of the products of coefficients remains valid when the vectors $\mathbf{s}_i$ or $\mathbf{t}_j$ are replaced by multiples $\mathbf{s}_i^* := \mathbf{s}_i\alpha_i$ or $\mathbf{t}_j^* := \mathbf{t}_j\beta_j$ with $\alpha_i,\beta_j \in \mathbb{F}\setminus\{0\}$. Hence, it means no restriction to assume a representation with $\mathbf{s}_1+\mathbf{s}_2+\mathbf{s}_3=\mathbf{0}$. Then, the representation (1) in Theorem 4 implies the equation (3) for Desarguesian sextuples. Since the six points are supposed to be mutually different, the products in (3) cannot vanish.
Conversely, we replace in (2) $\mathbf{t}_2$ by $\mathbf{t}_2^* := -\mathbf{t}_2\mu_{21}^{-1}\mu_{12}$ and $\mathbf{t}_3$ by $\mathbf{t}_3^* := -\mathbf{t}_3\mu_{31}^{-1}\mu_{13}$ while $\mathbf{t}_1^* := \mathbf{t}_1$. Moreover, let μ_{ij}^* denote the coefficients in the representations of $\mathbf{t}_1^*$, $\mathbf{t}_2^*$ and $\mathbf{t}_3^*$ analogue to (2). Then holds $\mu_{21}^*+\mu_{12}^* = \mu_{31}^*+\mu_{13}^* = 0$ and therefore, as a consequence of $\mu_{12}^*\mu_{23}^*\mu_{31}^* = -\mu_{13}^*\mu_{21}^*\mu_{32}^*$, also $\mu_{32}^*+\mu_{23}^* = 0$. Thus, the representation of the $\mathbf{t}_i^*$ satisfies the conditions in (1), and Theorem 4 confirms that the sextuple is Desarguesian. □

How can Desarguesian sextuples be characterized in terms of coordinates on the axis a? For this purpose we introduce homogeneous coordinates on a in the form $\mathbf{t}_i := (\tau_{i0},\tau_{i1})^T$ and $\mathbf{s}_j := (\sigma_{j0},\sigma_{j1})^T$. Note that from now on $\mathbf{t}_i$ and $\mathbf{s}_j$ stand for two-dimensional vectors over $\mathbb{F}$. In order to obtain a representation $\mathbf{t}_i = \mathbf{s}_j\mu_{ij}+\mathbf{s}_k\mu_{ik}$ for all cyclic permutations (i,j,k) of $(1,2,3)$, we use Cramer's rule and solve the system

$$\mu_{ij}\begin{pmatrix}\sigma_{j0}\\ \sigma_{j1}\end{pmatrix} + \mu_{ik}\begin{pmatrix}\sigma_{k0}\\ \sigma_{k1}\end{pmatrix} = \begin{pmatrix}\tau_{i0}\\ \tau_{i1}\end{pmatrix}$$

by

$$\mu_{ij} = \frac{\det(\mathbf{t}_i\,\mathbf{s}_k)}{\det(\mathbf{s}_j\,\mathbf{s}_k)},\ \mu_{ik} = \frac{\det(\mathbf{s}_j\,\mathbf{t}_i)}{\det(\mathbf{s}_j\,\mathbf{s}_k)}.$$

From (3) follows the characterization

$$\begin{aligned}&\det(\mathbf{t}_1\,\mathbf{s}_2)\det(\mathbf{t}_2\,\mathbf{s}_3)\det(\mathbf{t}_3\,\mathbf{s}_1)\\ &= \det(\mathbf{t}_1\,\mathbf{s}_3)\det(\mathbf{t}_2\,\mathbf{s}_1)\det(\mathbf{t}_3\,\mathbf{s}_2).\end{aligned} \tag{4}$$

For the sake of simplicity, let us assume that neither any τ_{i1} nor any σ_{j1} vanishes. Then we can use inhomogeneous coordinates $t_i = \tau_{i0}/\tau_{i1}$ for T_i and $s_j := \sigma_{j0}/\sigma_{j1}$ for S_j so that $\det(\mathbf{t}_i\,\mathbf{s}_j) = t_i - s_j$. Thus we can rewrite (4) as a product of affine ratios

$$\mathrm{ar}(S_2,S_3,T_1)\cdot\mathrm{ar}(S_3,S_1,T_2)\cdot\mathrm{ar}(S_1,S_2,T_3) = 1, \tag{5}$$

where $\mathrm{ar}(S_i,S_j,T_k) = (s_i-t_k)/(s_j-t_k)$. The equation (5) reminds on Menelaos's theorem characterizing the collinearity of three points T_1,T_2,T_3 on the respective side lines of the triangle $S_1S_2S_3$.
Finally, equation (5) is equivalent to[4]

$$\begin{aligned}(s_1+t_1)(s_2t_2-s_3t_3)+(s_2+t_2)(s_3t_3-s_1t_1)\\ +(s_3+t_3)(s_1t_1-s_2t_2) = 0.\end{aligned} \tag{6}$$

By the way, in (5) we may exchange S_i with T_i for some $i\in\{1,2,3\}$. This yields three other equivalent equations

$$\begin{aligned}&\mathrm{ar}(S_2,S_3,S_1)\cdot\mathrm{ar}(S_3,T_1,T_2)\cdot\mathrm{ar}(T_1,S_2,T_3)\\ &= \mathrm{ar}(T_2,S_3,T_1)\cdot\mathrm{ar}(S_3,S_1,S_2)\cdot\mathrm{ar}(S_1,T_2,T_3)\\ &= \mathrm{ar}(S_2,T_3,T_1)\cdot\mathrm{ar}(T_3,S_1,T_2)\cdot\mathrm{ar}(S_1,S_2,S_3) = 1.\end{aligned}$$

If the underlying plane is a Fano-plane, then the existence of an involution with $S_i\mapsto T_i$ for $i=1,2,3$ and a corresponding nontrivial symmetric bilinear form are equivalent to the condition

$$\det\begin{pmatrix}1 & s_1+t_1 & s_1t_1\\ 1 & s_2+t_2 & s_2t_2\\ 1 & s_3+t_3 & s_3t_3\end{pmatrix} = 0. \tag{7}$$

At the same token, the characterizations in (4), (5) or (6) could also be used to deduce the particular cases with additional admissible permutations as listed in Theorem 3.

[4]Desarguesian sextuples are a refinement of quadrangular sets (note Remark 1) due to the distinction between S- and T-points. By virtue of Theorem 1, this distinction becomes obsolete in Pappian planes. Therefore the equations (4), (5) and (6) can already be found in the literature as characterizations of quadrangular sets in Pappian Fano-planes, e.g., in [17, Sect. 8]. However, our approach is different.

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