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Cramer-Castillon on a Triangle's Incircle and Excircles

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ABSTRACT

The Cramer-Castillon problem (CCP) consists in finding one or more polygons inscribed in a circle such that their sides pass cyclically through a list of N points. We study this problem where the points are the vertices of a triangle and the circle is either the incircle or one of the excircles. We find that (i) in each case there is always a pair of solutions (total of 8 new triangles and 24 vertices); (ii) each pair shares all Brocard geometry objects, (iii) barycentric coordinates are laden with the golden ratio; and (iv) simple operations on the barycentrics of a single vertex out of the 24 yield all other 23.

Key words: Golden ratio, triangle, Brocard, symmedian

MSC2020: 51M04, 51N20, 51N35

Cramer-Castillonov problem u slučaju trokutu upisane i pripisanih kružnica

SAŽETAK

Cramer-Castillonov problem (CCP) se sastoji u traženju jednog ili više mnogokuta upisanih u kružnicu tako da njihove stranice ciklički prolaze kroz N točaka. Ovaj problem proučavamo u slučaju kada su točke vrhovi trokuta, a kružnica ili njegova upisana ili jedna od pripisanih kružnica. Dokazujemo da (i) u svakom slučaju uvijek postoji par rješenja (ukupno 8 novih trokuta i 24 vrha); (ii) svaki par dijeli sve objekte Brocardove geometrije; (iii) u baricentričnim koordinatama se pojavljuje zlatni rez; i (iv) jednostavnim operacijama na baricentričnim koordinatama se iz jednog od 24 vrha mogu dobiti preostala 23.

Ključne riječi: zlatni rez, trokut, Brocard, simedijana

1 Introduction

The Cramer-Castillon problem (CCP) consists in finding one or more N-gons inscribed in a circle $\mathcal{C}$ such that their sides pass cyclically through a set of points P_i, $i = 1 \cdots N$. In Figure 1 this is illustrated for the $N = 3$ case. The solutions to CCP are given[1] by the roots of a quadratic equation (see [7, Section 6.9], [10]), i.e., there can be 0, 1, or 2 real solutions. Geometric conditions for solution existence, though not germane to this article, are described in the aforementioned references.

Referring to Figure 2, we investigate the CCP for the case where the P_i are the vertices A, B, C of a reference triangle and $\mathcal{C}$ is the incircle or one of the excircles. Our findings include:

- For any triangle, the CCP on either the incircle or an excircle has exactly two solutions (total of 8 new triangles and 24 new vertices).

- We derive barycentric coordinates for the 4 pairs of solutions and notice they are laden with the golden ratio $\phi = (\sqrt{5} + 1)/2$.

- Each pair shares circumcenter, symmedian point, and all "Brocard geometry" objects [4], e.g., the Brocard points, Brocard circle and inellipse, Lemoine and Brocard axis, isodynamic points, etc.

- The four distinct Brocard axes shared by each pair concur on the de Longchamps point [11] of the reference triangle.

- Given barycentrics for a single vertex out of the 24 newly generated, all other 23 can be obtained with simple cyclic substitutions.

- Solving the CCP for a triangle's arbitrary inconic is equivalent to solving it (via a projectivity) for the incircle case.
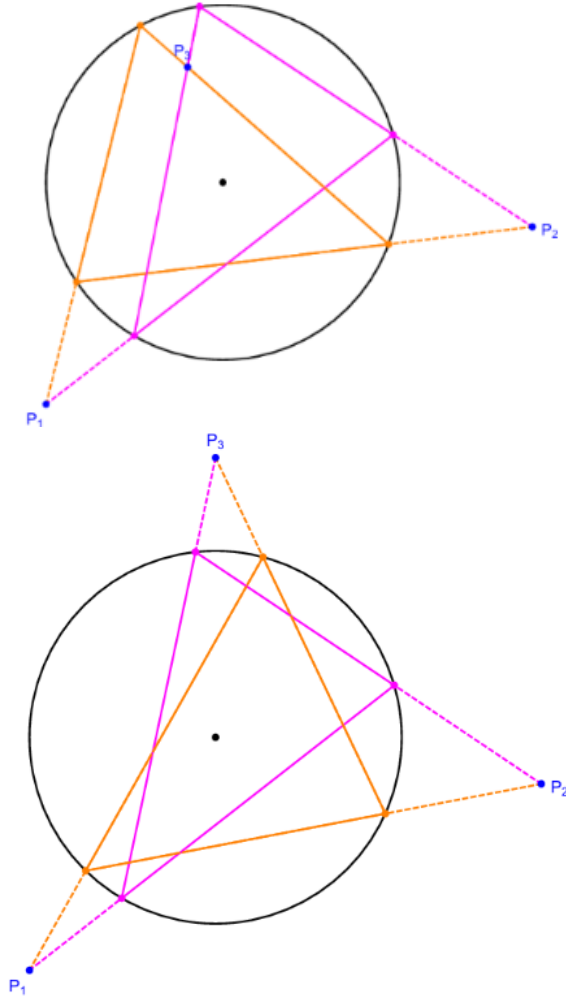
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[1]In the hyperbolic plane, corresponding sides of the two solutions are polar-orthogonal with respect to the ideal circle [1].

Figure 1: *The Cramer-Castillon problem (CCP) in the N = 3 case. In the top (resp. bottom) picture, two points are exterior and one is interior (resp. all exterior) to the target circle. In each case, two solutions are shown (magenta and orange).*

Article organization

The Cramer-Castillon Problem (CCP) on the incircle is covered in Section 2. Its shared Brocard objects are examined in Section 3. The CCP on the excircles are analyzed in Section 4. In Appendix A we provide a list of correspondences between triangle centers in the incircle CCP solutions and the reference triangle.

Notation

We shall use barycentric coordinates [11] and refer to triangle centers using Kimberling's notation X_k [5].

2   Cramer-Castillon on the Incircle

Referring to Figure 2, consider a triangle $T = ABC$ with the sidelengths a, b, c and $s = (a + b + c)/2$ its semiperimeter. Below we use ϕ to denote the golden ratio, $\phi = (1 + \sqrt{5})/2$.
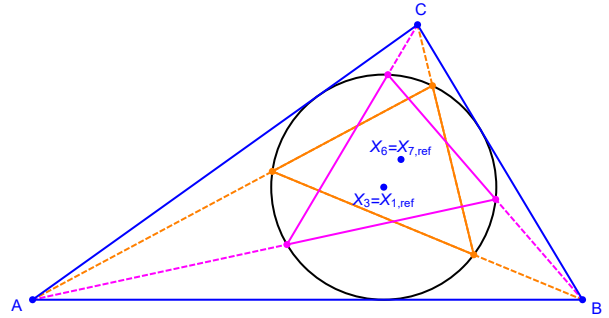

Figure 2: *The two solutions of CCP (orange, magenta) on the incircle $\mathcal{C}$ of a triangle $T = ABC$. Since both are inscribed in $\mathcal{C}$, they share their circumcenter X_3 (at the incenter $X_{1,ref}$ of the reference). Also shared is the symmedian point X_6, which coincides with the Gergonne $X_{7,ref}$ of the reference.*

Proposition 1 *The CCP on a triangle T and its incircle $\mathcal{C}$ admits exactly two solutions T_1 and T_2, whose barycentric vertex matrices with respect to T are given by:*

$$T_1 = \begin{bmatrix} (1-\phi)^2 vw & uw & (2-\phi)^2 uv \\ (2-\phi)^2 vw & (1-\phi)^2 uw & uv \\ vw & (2-\phi)^2 uw & (1-\phi)^2 uv \end{bmatrix}$$

$$T_2 = \begin{bmatrix} (1-\phi)^2 vw & (2-\phi)^2 uw & uv \\ vw & (1-\phi)^2 uw & (2-\phi)^2 uv \\ (2-\phi)^2 vw & uw & (1-\phi)^2 uv \end{bmatrix}$$

where $u = (s-a)$, $v = (s-b)$, $w = (s-c)$.

Proof. While barycentric entries in T_1, T_2 can be obtained as roots of a quadratic equation [7, Section 6.9], we also provide a synthetic construction for the vertices.

Referring to Figure 3, define three paths (A_1, A_2, A_3, A_4), (B_1, B_2, B_3, B_4) and (C_1, C_2, C_3, C_4). Cramer-Castillon requires that $A_i A_{i+1}$ (or $B_i B_{i+1}$ or $C_i C_{i+1}$) be a circle chord.

The points B_1, C_1 are tangent points of $\mathcal{C}$ with AC and AB while A_1 is the reflection of a tangent point with BC with respect to X_1.

The three paths are non-closing: A_4, B_4 and C_4 are respectively different from A_1, B_1 and C_1.

Intersect the two pairs of segments $(A_1 B_4, A_4 B_1)$ and $(A_1 C_4, A_4 C_1)$ to get points H_1 and H_2 on their perspectrix.

Intersecting the homography line with $\mathcal{C}$ gives M_1 and M_4, two vertices of the solution triangles $M_1 M_2 M_3$ and $M_4 M_5 M_6$.   □
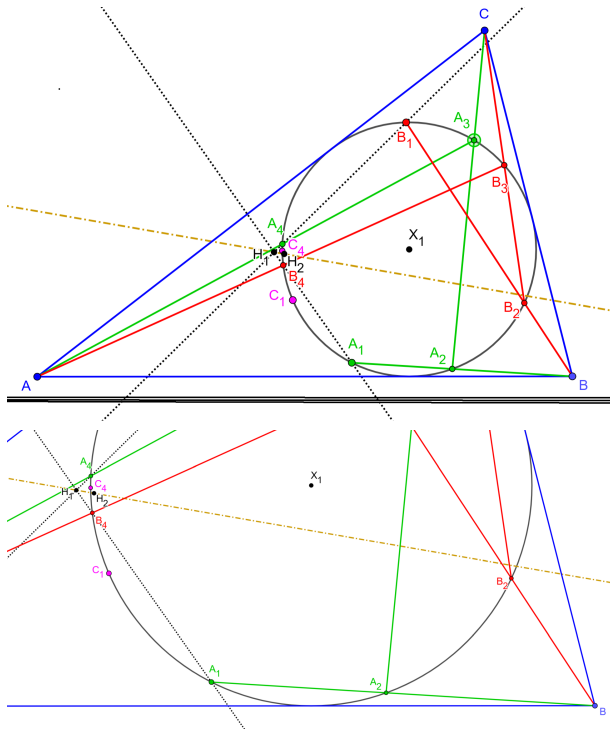
Figure 3: **top**: *The construction for the two points* H_1 *and* H_2 *which define the homography axis used in Proposition 1. The sequence* $C_1,...C_4$ *is not shown, but is as* $A_1...A_4$ *and* $B_1...B_4$. **bottom**: *a slight zoom-in on the region where* H_1 *and* H_2 *are.*

By definition, both solutions share their circumcenter X_3, located at the incenter X_1 of the reference. Interestingly:

Proposition 2 *The two solutions* T_1 *and* T_2 *have a common symmedian point* X_6 *which coincides with the Gergonne point* X_7 *of* T.

Proof. Using a CAS, we obtain $[1/(s-a) : 1/(s-b) : 1/(s-c)]$ as barycentrics for the symmedian point of T_1, T_2 with respect to T, which are precisely those of the Gergonne X_7 of T [5]. $\qquad\square$

3 Shared Brocard objects

Recall (i) the Brocard circle of a triangle has segment $X_3 X_6$ as diameter, and (ii) the Brocard axis is the line that passes through said diameter [11].

Since by definition T_1, T_2 share their circumcenter X_3, and per Proposition 2 their symmedian as well:

Corollary 1 T_1 *and* T_2 *share their Brocard axis and Brocard circles.*

Let $\delta = |X_6 - X_3|$. The Brocard angle ω to a triangle is given by [2, Prop. 3, p. 209]:

$$\tan \omega = \frac{\sqrt{3}}{3} \sqrt{1 - \left(\frac{\delta}{R}\right)^2}$$

Corollary 2 T_1 *and* T_2 *have the same Brocard angle* ω.

Referring to Figure 4, recall the two Brocard points of a triangle lie on the Brocard circle and the line joining them is perpendicular to the Brocard axis $X_3 X_6$. In [9] the following formula is given for the distance between the two Brocard points Ω_1, Ω_2 which only depends on the circumradius R and the Brocard angle ω:

$$|\Omega_1 - \Omega_2|^2 = 4c^2 = 4R^2 \sin^2 \omega (1 - 4\sin^2 \omega)$$

Corollary 3 T_1 *and* T_2 *share their Brocard points* Ω_1 *and* Ω_2.

Proposition 3 *The shared Brocard points* Ω_1 *and* Ω_2 *of* T_1 *and* T_2 *are triangle centers of* T *given by the following barycentric coordinates:*

$$\Omega_1 = [\alpha/u : \beta/v : \gamma/w], \quad \Omega_2 = [\gamma/u : \alpha/v : \beta/w]$$

where:

$$\alpha = (a-b)^2 - (a+b)c$$
$$\beta = (b-c)^2 - (b+c)a$$
$$\gamma = (c-a)^2 - (c+a)b$$

and u, v, w *are as in Proposition 1.*

Let a, b be the semi-axes of the Brocard inellipse of a triangle (whose foci are the Brocard points). The following relation was derived in [8, Lemma 2]:

$$[a, b] = R \left[\sin \omega, 2\sin^2 \omega\right]$$

Corollary 4 T_1, T_2 *share their Brocard inellipses.*

Recall the two isodynamic points X_{15} and X_{16} are the limiting points of the *Schoute pencil* [3], defined by the circumcircle and the Brocard circle (and orthogonal to the Apollonius circles), whose radical axis is the Lemoine axis.

Corollary 5 T_1 *and* T_2 *share their isodynamic points* X_{15} *and* X_{16} *and Lemoine axis.*

The intersection of the Lemoine axis with the Brocard axis is X_{187} [5].
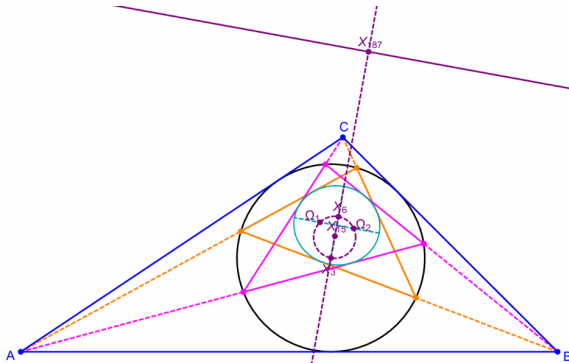
Corollary 6 T_1 and T_2 share their X_{187}.

Figure 4: *The two solutions (orange, magenta) of CCP on a triangle's incircle (black) share all Brocard geometry objects, to be sure: their Brocard points Ω_1, Ω_2, Brocard circle and axis (dashed purple), Brocard inellipse (light blue) whose foci are the Brocard points, the two isodynamic points X_{15}, and X_{16} (not shown) and the Lemoine axis (solid purple).*

Generalizing the CCP to any inconic

Referring to Figure 5 (top), let I be some inconic of a triangle $T = ABC$.

Proposition 4 *The CCP of A, B, C on I has two solutions which circumscribe a single inconic $\mathcal{B}$.*

Proof. The CCP is projectively-invariant since only incidences are involved. Let T' be the image of T under a projectivity Π that sends I to a circle $\mathcal{C}'$, see Figure 5 (bottom). Clearly, $\mathcal{C}'$ is the incircle or an excircle of T'. Per Proposition 1 and Corollary 4, the CCP on $(T', \mathcal{C}')$ has two solutions with a common Brocard inellipse $\mathcal{B}'$. Thus $\mathcal{B}$ is the latter's pre-image under Π, with all tangencies preserved. $\qquad\square$

Proposition 5 *Consider a general inconic I with perspector $[p, q, r]$. The two solutions T_1 and T_2 of the CCP on I have A-vertices given by:*

$$T_1(A\text{-vertex}) = [(2 - \phi)p, q, (1 + \phi)r]$$
$$T_2(A\text{-vertex}) = [(2 - \phi)p, (1 + \phi)r, q]$$

Corollary 7 *If I is the Steiner inellipse (perspector X_2), the two A-vertices are given by:*

$$T_1(A\text{-vertex}) = [2 - \phi, 1, 1 + \phi]$$
$$T_2(A\text{-vertex}) = [2 - \phi, 1 + \phi, 1]$$

Notice that if I is the incircle, whose perspector is the Gergonne point $X_7 : [1/(s-a) : 1/(s-b) : 1/(s-c)]$, we recover Proposition 1.
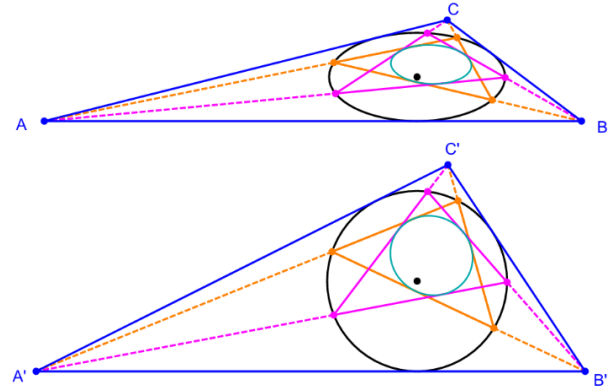

Figure 5: *Since the CCP is projectively-invariant, its solution on a ABC (top) with respect to a generic inconic(black) can be regarded as the pre-image of a perspectivity Π which sends said inconic to a circle (black, bottom). Clearly, this circle is an incircle of the new triangle $A'B'C'$. This implies that the two solutions in the original case will envelop a conic (light blue, top) which is the pre-image of the Brocard inellipse (light blue, bottom) under Π.*

4   Cramer-Castillon on the Excircles

In this section we extend the CCP to the excircles of a given triangle T. Referring to Figure 6:
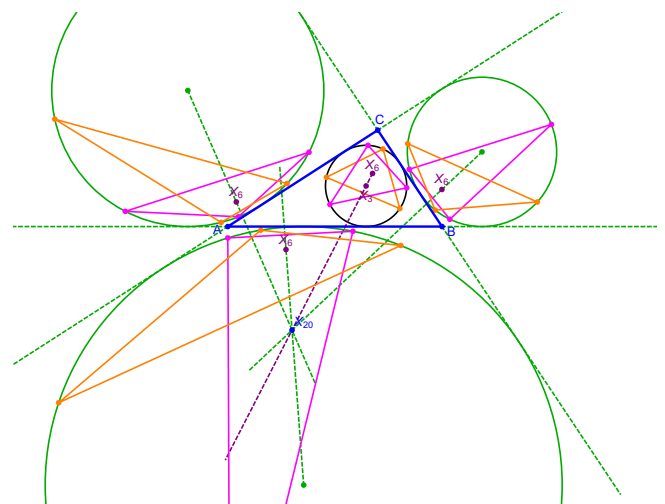

Figure 6: *The CCP applied both to the incircle and the excircles (green). In each of the four circles the symmedian (and all Brocard objects) are shared. The four Brocard axes concur on the de Longchamps point X_{20} of the reference triangle.*

Proposition 6 *The first and second solutions $T_{1,exc}$ and $T_{2,exc}$ of the CCP on the A-excircle are given by the following barycentric vertex matrices:*

$$T_{1,exc} = \begin{bmatrix} -vw(\phi-1)^2 & sv & sw(\phi-2)^2 \\ -vw(\phi-2)^2 & sv(\phi-1)^2 & sw \\ -vw & sv(\phi-2)^2 & sw(\phi-1)^2 \end{bmatrix}$$

$$T_{2,exc} = \begin{bmatrix} -vw(\phi-1)^2 & sv(\phi-2)^2 & sw \\ -vw & sv(\phi-1)^2 & sw(\phi-2)^2 \\ -vw(\phi-2)^2 & sv & sw(\phi-1)^2 \end{bmatrix}$$

where u,v,w are as in Proposition 1.

Proposition 7 *The symmedian X_6 of the A-excircle is shared by the two solution and is the A-exversion of the reference's Gergonne X_7, i.e., its barycentrics are given by $[-vw, sv, sw]$, coinciding with the Gergonne point X_7 of the "outer" contact triangle inscribed in the A-excircle.*

Corollary 8 *Each pair of solutions of the CCP on an excircle shares all of their Brocard objects.*

Proposition 8 *The Brocard axis of the incircle-CCP as well as the 3 Brocard axes of the excircle-CCP solutions concur on the de Longchamps point X_{20} of the reference.*

Proof. The shared Brocard axis of the incircle-CCP solutions contains, by definition, X_3 and X_6 of either solution triangle. We saw above these correspond to X_1 and X_7 of the reference, i.e., it is the Soddy line of the reference [6], which is known to pass through X_{20}. The Brocard axis shared by the A-excircle solutions are the A-exversion ($a \to -a$) of the incircle Brocard axis, and similarly for the B- and C-excircles. It can be shown these 4 lines meet at X_{20}. □

Twenty-three from one

There are a total of 4 pairs of triangles which are solutions to the CCP on both incircle and excircle, i.e., there are eight triangles and a total of 24 vertices.

Proposition 9 *Twenty-three of said vertices can be directly derived from a single vertex of a solution triangle of the CCP in the incircle.*

Proof. As seen in Proposition 1, the A-vertex of the first solution T_1 on the incircle is given by:

$$\left[(1-\phi)^2(s-b)(s-c), (s-a)(s-c), (2-\phi)^2(s-a)(s-b)\right]$$

Perform a bicentric substitution, i.e., $b \to c$ and $c \to b$, and swap postions 2 and 3 to arrive at:

$$\left[(\phi-1)^2(s-b)(s-c), (\phi-2)^2(s-a)(s-c), (s-a)(s-b)\right]$$

i.e., the A-vertex of T_2. The other vertices can be computed cyclically. Now, derive the A-excircle solution from the incircle one by performing an "A-exversion", i.e., changing every $a \to -a$, obtaining the T_1 A-vertex of the A-excircle:

$$\left[(\phi-1)^2(b-s)(s-c), (s-b)s, (\phi-2)^2(s-c)s\right].$$

For the B-vertex of the A-excircle, we first perform a cyclic substitution, then the exversion $a \to -a$, obtaining:

$$\left[(\phi-1)^2(s-b)(s-c), (s-a)(s-c), (\phi-2)^2(s-a)(s-b)\right]$$

which leads to:

$$\left[(\phi-2)^2(s-b)(s-c), (\phi-1)^2(s-a)(s-c), (s-a)(s-b)\right]$$

and then:

$$\left[(\phi-2)^2(s-b)(s-c), (\phi-1)^2s(b-s), s(c-s)\right]$$

The remaining vertices can be obtained similarly. □

Acknowledgements

5 Appendix A. Center Correspondences

Let $[i,k]$ indicate that X_i of either solution of the incircle CCP coincides with X_j of the reference triangle. The following is a list of corresponding pairs: $[3,1]$, $[6,7]$, $[15,3638]$, $[16,3639]$, $[32,10481]$, $[182,5542]$, $[187,1323]$, $[371,482]$, $[372,481]$, $[511,516]$, $[512,514]$, $[575,43180]$, $[576,30424]$, $[1151,176]$, $[1152,175]$, $[1350,390]$, $[1351,4312]$, $[1384,21314]$, $[2076,14189]$, $[3053,279]$, $[3098,30331]$, $[3311,1373]$, $[3312,1374]$, $[3592,21169]$, $[5017,42309]$, $[5023,3160]$, $[5085,11038]$, $[5585,31721]$, $[5611,10652]$, $[5615,10651]$, $[6200,31538]$, $[6221,1371]$, $[6396,31539]$, $[6398,1372]$, $[6409,17805]$, $[6410,17802]$, $[6419,21171]$, $[6425,31601]$, $[6426,31602]$, $[6431,21170]$, $[6437,17804]$, $[6438,17801]$, $[6449,17806]$, $[6450,17803]$, $[11824,31567]$, $[11825,31568]$, $[12305,30333]$, $[12306,30334]$, $[14810,43179]$, $[15815,5543]$, $[21309,20121]$, $[31884,8236]$, $[43118,30342]$, $[43119,30341]$, $[43120,31570]$, $[43121,31569]$.

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