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A Complete Quadrilateral in Rectangular Coordinates II

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ABSTRACT

This paper is a continuation of research on a geometry of a complete quadrilateral in the Euclidean plane. We present the well-known facts on the complete quadrilateral but all proved in the same way by using rectangular coordinates, symmetrically on all four sides of the quadrilateral with four parameters a,b,c,d. The properties related to the central circle, orthopolar circle, Hervey's circle, Kantor's point and Plücker's points are observed. During this study, some new results come up.

Key words: complete quadrilateral, central circle, orthopolar circle, Hervey's circle, Kantor's point, Plücker's points

MSC2020: 51N20

Potpuni četverostran u pravokutnim koordinatama II

SAŽETAK

Ovaj rad je nastavak proučavanja geometrije potpunog četverostrana u euklidskoj ravnini. Iznosimo neke poznate činjenice o potpunom četverostranu, ali ih ovdje dokazujemo jedinstvenom metodom koristeći pravokutne koordinate, simetrično s obzirom na četiri stranice četverostrana, odnosno četiri parametra a,b,c,d. Promatramo svojstva vezana za centralnu kružnicu, ortopolarnu kružnicu, Herveyevu kružnicu, Kantorovu točku i Plückerove točke. Tijekom ovog istraživanja dobili smo i neke nove rezultate.

Ključne riječi: potpuni četverostran, centralna kružnica, ortopolarna kružnica, Herveyeva kružnica, Kantorova točka, Plückerove točke

1 Introduction

In this paper we present a continuation of research on a geometry of a complete quadrilateral in the Euclidean plane. The first part is given in [34] where we explained that we presented the well known facts on the complete quadrilateral but all proved in the same way by using rectangular coordinates, symmetrically on all four sides of the quadrilateral with four parameters a,b,c,d. The same method we apply in this paper as well. Hence, the title of the paper is fully justified. During this investigation, some of new results come up and in the text they are written in the form of theorem. We will not give an exhaustive introduction on the points, lines, circles of the quadrilateral but we will refer to the first paper [34]. Nevertheless, some elements of the complete quadrilateral and valid equalities are still mentioned.

Our complete quadrilateral has the lines $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ and the vertices $T_{AB}=\mathcal{A}\cap\mathcal{B}$, $T_{CD}=\mathcal{C}\cap\mathcal{D}$, $T_{AC}=\mathcal{A}\cap\mathcal{C}$, $T_{BD}=\mathcal{B}\cap\mathcal{D}$, $T_{AD}=\mathcal{A}\cap\mathcal{D}$, $T_{BC}=\mathcal{B}\cap\mathcal{C}$. Parabola $\mathcal{P}$ inscribed to the complete quadrilateral is of the form

$$\mathcal{P} \ldots y^2=4x, \tag{1}$$

with the focus

$$S=(1,0), \tag{2}$$

and the directrix $\mathcal{H}$ is $x=-1$. By A,B,C,D we denote the contact points of the lines $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ and its inscribed parabola $\mathcal{P}$

$$A=(a^2,2a),\ \ B=(b^2,2b),\ \ C=(c^2,2c),\ \ D=(d^2,2d). \tag{3}$$

The lines are of the forms

$$\mathcal{A}\ \ldots\ ay=x+a^2,\quad \mathcal{B}\ \ldots\ by=x+b^2,$$
$$\mathcal{C}\ \ldots\ cy=x+c^2,\quad \mathcal{D}\ \ldots\ dy=x+d^2, \tag{4}$$

and vertices

$$T_{AB}=(ab,a+b),\ \ T_{AC}=(ac,a+c),\ \ T_{AD}=(ad,a+d),$$
$$T_{CD}=(cd,c+d),\ \ T_{BD}=(bd,b+d),\ \ T_{BC}=(bc,b+c). \tag{5}$$

Fig. 1 presents the complete quadrangle with its elements. There are denoted the diagonal points and lines as well.

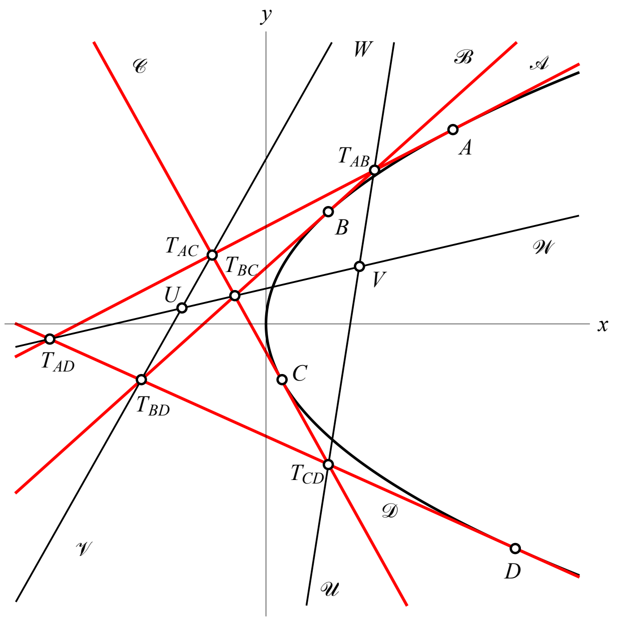

Figure 1: *The complete quadrangle $\mathcal{ABCD}$.*

Further on, the midpoints of pairs of points T_{AB},T_{CD}; T_{AC},T_{BD}; T_{AD},T_{BC} are lying on the median

$$\mathcal{N}\ \ldots\ y=\frac{1}{2}s. \tag{6}$$

Basic symmetric functions will be useful hereby as well:

$$s=a+b+c+d,\quad q=ab+ac+ad+bc+bd+cd,$$
$$r=abc+abd+acd+bcd,\quad p=abcd.$$

We also use abbreviations $\alpha=1+a^2$, $\beta=1+b^2$, $\gamma=1+c^2$ and $\delta=1+d^2$.

Let $\mathcal{S}_a,\mathcal{S}_b,\mathcal{S}_c,\ \mathcal{S}_d$ be circumcircles of trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$, respectively. The circle $\mathcal{S}_d$ has the equation

$$\mathcal{S}_d\ \ldots x^2+y^2-(ab+ac+bc+1)x-(a+b+c-abc)y$$
$$+ab+ac+bc=0, \tag{7}$$

and the center

$$S_d=\left(\frac{1}{2}(ab+ac+bc+1),\frac{1}{2}(a+b+c-abc)\right). \tag{8}$$

Circles $\mathcal{S}_a,\mathcal{S}_b,\mathcal{S}_c$ and their centers S_a,S_b,S_c are obtained analogously because of the symmetry on a,b,c,d.

The central circle $\mathcal{M}$ of the quadrilateral $\mathcal{ABCD}$, the circle passing through the centers of $\mathcal{S}_a,\mathcal{S}_b,\mathcal{S}_c\ \mathcal{S}_d$, has the equation

$$\mathcal{M}\ldots x^2+y^2-\frac{1}{2}(3+q-p)x+\frac{1}{2}(r-s)y-\frac{1}{2}(1+q-p)=0, \tag{9}$$

and center

$$M=\left(\frac{1}{4}(3+q-p),\frac{1}{4}(s-r)\right) \tag{10}$$

that we call a central point of the $\mathcal{ABCD}$, see Fig. 2.

2 Results

The lines $\mathcal{A}$ and ST_{AD} have the slopes $\frac{1}{a}$ and $\frac{a+d}{ad-1}$, so for the tangent of the angle $\angle(\mathcal{A},ST_{AD})$ due to formula (22) from [34] we obtain $\frac{1}{d}$, meaning that this angle is equal to the angle $\angle(\mathcal{N},\mathcal{D})$. Hence, *for the complete quadrilateral $\mathcal{ABCD}$ with the focus S and vertices $T_{AB},T_{AC},T_{AD},T_{BC},T_{BD},T_{CD}$ the equalities $\angle(\mathcal{A},ST_{AD})=\angle(\mathcal{B},ST_{BD})=\angle(\mathcal{C},ST_{CD})=\angle(\mathcal{N},\mathcal{D})$ are valid as well as three more sets of such equalities.* The statement appears in [9].

The tangent to the circle $\mathcal{S}_d$ with the equation (7) in the point T_{BC} has the equation

$$2bcx+2(b+c)y-(ab+ac+bc+1)(x+bc)$$
$$-(a+b+c-abc)(y+b+c)+2(ab+ac+bc)=0,$$

that after some computing obtains the form

$$(bc-ab-ac-1)x+(b+c-a+abc)y=$$
$$=b^2c^2+b^2+c^2+bc-ab-ac$$

and it passes through the point

$$S_A'=\Big(\frac{1}{\alpha}(a^2+bc+bd+cd-ab-ac-ad+abcd),$$
$$\frac{1}{\alpha}(b+c+d-a+abc+abd+acd-bcd)\Big) \tag{11}$$

and because of symmetry on b,c,d the tangents to $\mathcal{S}_b$ and $\mathcal{S}_c$ at the points T_{CD} and T_{BD} are incident to S_A' as well (see Fig. 2). It was proved in [34] that lines $T_{BC}S_d,T_{BD}S_c,T_{CD}S_b$ are intersected in one point $S_A=(\frac{1}{\alpha}[a(abc+abd+acd-bcd+b+c+d)+1],\frac{a}{\alpha}(-abcd+ab+ac+ad-bc-bd-cd+1))$, an intersection of circles $\mathcal{M}$ and $\mathcal{A}$. The points S_A' and S_A have for the midpoint the point S_a analogous to S_d from (8). It means that the point S_A' is diametrically opposite to S_A on the circle $\mathcal{S}_a$. Moreover, *for the complete quadrilateral $\mathcal{ABCD}$, the points S_A',S_B',S_C',S_D' are diametrically opposite to S_A,S_B,S_C,S_D on circles $\mathcal{S}_a,\mathcal{S}_b,\mathcal{S}_c,\mathcal{S}_d$, respectively.* [6] and [26] have this statement.

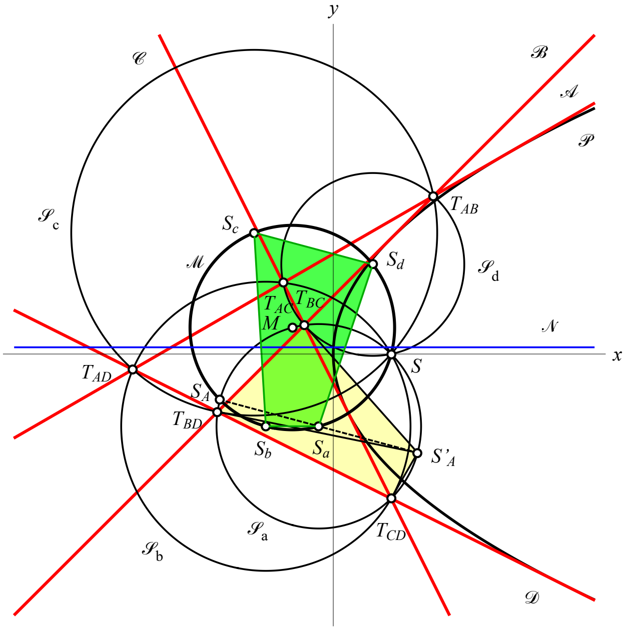

Figure 2: *Circumcenters S_a, S_b, S_c, S_d of trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$ lie on the central circle $\mathcal{M}$ with center M.*

As for the point S'_A from (11) we have $x-1=\frac{1}{\alpha}(bc+bd+cd-ab-ac-ad+abcd-1)$ and

$$(bc+bd+cd-ab-ac-ad+abcd-1)^2+$$
$$+(b+c+d-a+abc+abd+acd-bcd)^2=$$
$$=(a^2+1)(b^2+1)(c^2+1)(d^2+1)=\alpha\beta\gamma\delta$$

the equality $SS_A'^{\,2}=\frac{\beta\gamma\delta}{\alpha}$, and similarly $SS_B'^{\,2}=\frac{\alpha\gamma\delta}{\beta}$, $SS_C'^{\,2}=\frac{\alpha\beta\delta}{\gamma}$, $SS_D'^{\,2}=\frac{\alpha\beta\gamma}{\delta}$ are valid. Obviously, the equalities $T_{AB}S^2=\alpha\beta$, $T_{AC}S^2=\alpha\gamma$, $T_{AD}S^2=\alpha\delta$, $T_{BC}S^2=\beta\gamma$, $T_{BD}S^2=\beta\delta$, $T_{CD}S^2=\gamma\delta$ are valid. For points S and S_d from (2) and (8) we get $SS_d^2=\frac{1}{4}\alpha\beta\gamma$, and analogously $SS_a^2=\frac{1}{4}\beta\gamma\delta$, $SS_b^2=\frac{1}{4}\alpha\gamma\delta$, $SS_c^2=\frac{1}{4}\alpha\beta\delta$ follow. Out of these equalities we obtain

$$SS_a:SS_b:SS_c:SS_d=\frac{1}{\sqrt{\alpha}}:\frac{1}{\sqrt{\beta}}:\frac{1}{\sqrt{\gamma}}:\frac{1}{\sqrt{\delta}}. \tag{12}$$

The lines SS_a and ST_{BC} have the slopes $\frac{b+c+d-bcd}{bc+bd+cd-1}$, $\frac{b+c}{bc-1}$ so for tangent of the angle $\angle(ST_{BC},SS_a)$ due to formula (22) in [34] and after some computing we get the value $-d$. Because of symmetry on a,b,c the angles $\angle(ST_{AC},SS_b)$ and $\angle(ST_{AB},SS_c)$ have the same value for the tangents. Lines SS'_D and SS_d have the slopes

$$\frac{a+b+c-d+abd+acd+bcd-abc}{ab+ac+bc-ad-bd-cd+abcd-1}$$

and

$$\frac{a+b+c-abc}{ab+ac+bc-1},$$

so for the tangent of the angle $\angle(SS'_D,SS_d)$ we get the value $-d$ again. Because of that, there are the equalities among angles

$$\angle(ST_{BC},SS_a)=\angle(ST_{AC},SS_b)=\angle(ST_{AB},SS_c)=\angle(SS'_D,SS_d).$$

Besides that, the equalities

$$SS_a:ST_{BC}=SS_b:ST_{AC}=SS_c:ST_{AB}=SS_d:SS'_D=\frac{1}{2}\sqrt{\delta}$$

are valid. This means that cyclic quadrangles $T_{BC}T_{AC}T_{AB}S'_D$ and $S_aS_bS_cS_d$ are directly similarly and that the factor of similarity is equal to $\frac{1}{2}\sqrt{\delta}$, and the center of similarity is the focus S. These statements can be found in [26]. Similarly as this, the cyclic quadrangles $S'_AT_{CD}T_{BD}T_{BC}$, $T_{CD}S'_BT_{AD}T_{AC}$ and $T_{BD}T_{AD}S'_CT_{AB}$ are similar to quadrangle $S_aS_bS_cS_d$ where the center of similarity is the focus S, and factors of similarity are $\frac{1}{2}\sqrt{\alpha}$, $\frac{1}{2}\sqrt{\beta}$, $\frac{1}{2}\sqrt{\gamma}$, see Fig. 3. The angles of these studied similarities have the tangents $-d,-a,-b,-c$, respectively. However, these are slopes of lines perpendicular to lines $\mathcal{D},\mathcal{A},\mathcal{B},\mathcal{C}$ so the mentioned angles of similarity are the same as the ones formed by median $\mathcal{N}$ and normals to the lines $\mathcal{D},\mathcal{A},\mathcal{B},\mathcal{C}$, respectively. We have just proved our original theorem.

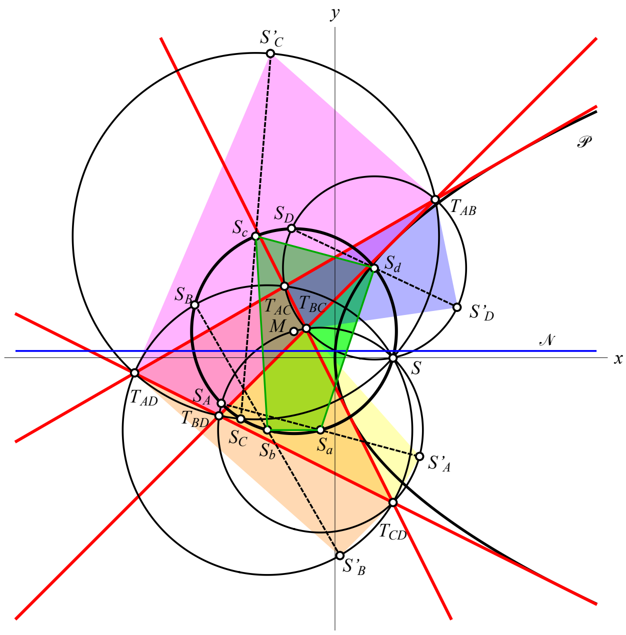

Figure 3: *Visualisation of Theorem 1*

Theorem 1 *Let $\mathcal{ABCD}$ be a complete quadrilateral. The cyclic quadrangles $S'_AT_{CD}T_{BD}T_{BC}$, $T_{CD}S'_BT_{AD}T_{AC}$, $T_{BD}T_{AD}S'_CT_{AB}$ and $T_{BC}T_{AC}T_{AB}S'_D$ are similar to quadrangle $S_aS_bS_cS_d$. Angles of similarity are the same as the ones formed by median $\mathcal{N}$ and normals to the lines $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$, respectively.*

From the latest results the next follows: *triangles $S_aS_bS_c$, $S_aS_bS_d$, $S_aS_cS_d$, $S_bS_cS_d$ are directly similar to trilaterals $\mathcal{ABC}$, $\mathcal{ABD}$, $\mathcal{ACD}$, $\mathcal{BCD}$ where the center of each similarity is the focus S*, and that is the statement in [31].

The circle with the equation

$$x^2+y^2-(ab+cd)x-sy+p+(a+b)(c+d)=0 \qquad (13)$$

is incident to T_{AB} and T_{CD} and the center of the circle is the point $\left(\frac{1}{2}(ab+cd),\frac{1}{2}s\right)$, the midpoint of these two points. Hence, (13) is the circle with diameter $T_{AB}T_{CD}$. Similar, the circle with the diameter $T_{AC}T_{BD}$ has the equation

$$x^2+y^2-(ac+bd)x-sy+p+(a+c)(b+d)=0. \qquad (14)$$

Subtracting these two equations, the radical axis of these circles with the equation $(a-d)(c-b)x+(a-d)(c-b)=0$ is obtained. It is the line with equation $x=-1$, being the directrix $\mathcal{H}$. Because of symmetry, it is the radical axis of the third circle with diameter $T_{AD}T_{BC}$ as well. The statement that *circles with diameters $T_{AB}T_{CD}$, $T_{AC}T_{BD}$ and $T_{AD}T_{BC}$ have the directrix $\mathcal{H}$ for the radical axis,* can be found in [13] and it is attributed to Bodenmiller, and some of authors call it the Plücker's theorem. The elementary proofs can be found in [5], [19] and [23]. The intersection point of the line $\mathcal{H}$ with each of these three circles are the points with ordinates that fulfill the equation

$$y^2-sy+p+q+1=0, \qquad (15)$$

i.e.

$$P_{1,2}=\left(-1,\frac{1}{2}(s\pm\sqrt{s^2-4p-4q-4})\right), \qquad (16)$$

that are so called *Plücker's points* of the quadrilateral $\mathcal{ABCD}$, real ones and imaginary. The pencil of circles incident to these two points is Plücker's pencil of circles.

Let the points A'',B'',C'',D'' be orthogonal projections of any Plücker's point $P_i, i\in 1,2$, to the lines $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$, respectively. Then, *$A'',B''$ are incident to the circle with the diameter P_iT_{AB}, and points C'',D'' are on the circle with the diameter P_iT_{CD}*. However, these two diameters are perpendicular because the points $P_i, i\in 1,2$, are incident to the circle with diameter $T_{AB}T_{CD}$, so then *the circles $P_iA''B''$ and $P_iC''D''$ are orthogonal circles.* This statement appears in [33].

The circle $\mathcal{P}'$ with equation

$$\mathcal{P}'\ldots x^2+y^2-\frac{1}{2}(p+q+1)x-sy+\frac{1}{2}(p+q-1)=0 \qquad (17)$$

is incident to the focus $S=(1,0)$ and Plücker's points (16). Namely, out of $x=-1$ and (17) we obtain the equation (15). It is so called the *orthopolar circle* of the quadrilateral $\mathcal{ABCD}$. The center of this circle is the point

$$P=\left(\frac{1}{4}(p+q+1),\frac{1}{2}s\right), \qquad (18)$$

orthopolar center of the quadrilateral $\mathcal{ABCD}$, incident to the median $\mathcal{N}$, see Fig. 4. For the radius ρ' of the circle $\mathcal{P}'$ we get $16\rho'^2=(p+q-3)^2+4s^2$.

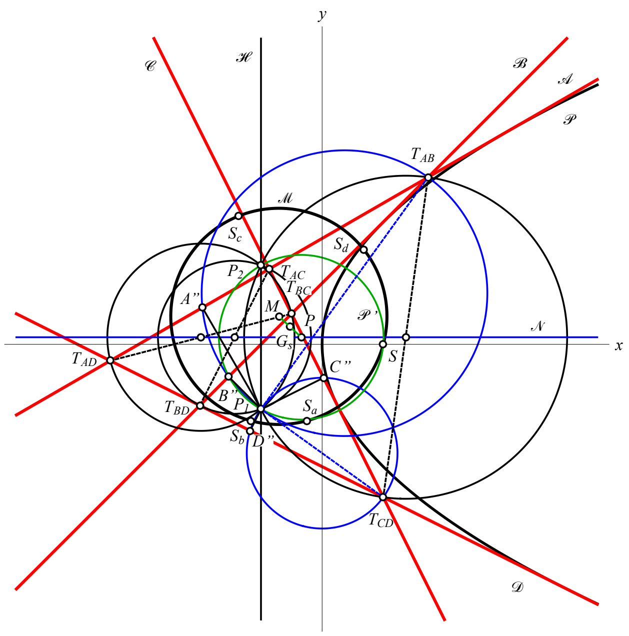

Figure 4: *Plücker's points P_1,P_2, orthopolar circle $\mathcal{P}'$, and orthopolar center P of the quadrilateral $\mathcal{ABCD}$.*

For the point P from (18) and the point S_d from (8) we obtain the equality

$$16PS_d^2=(ad+bd+cd-ab-ac-bc-1+abcd)^2 \\ +4(abc+d)^2,$$

and adding up this equality and three more analogous ones we get

$$16\left(PS_a^2+PS_b^2+PS_c^2+PS_d^2\right)=4(a^2+1)(b^2+1)(c^2+1)(d^2+1)$$

i.e. because of (21) in [34] we show that *for the orthopolar center P and the centers S_a,S_b,S_c,S_d the equality $PS_a^2+PS_b^2+PS_c^2+PS_d^2=4\rho'^2$ is valid.* This appears in [26].

The points S_a,S_b,S_c,S_d from (8) have the centroid $G_s=\left(\frac{1}{4}(q+2),\frac{1}{8}(3s-r)\right)$, the midpoint of P from (18) and M from (10). This proves the statement in [27] that *the point symmetric to the point M with respect to the centroid G_s is incident to the median of $\mathcal{ABCD}$.* The point G_s is called *a dimidium point* of the quadrilateral $\mathcal{ABCD}$. Adding the equations (9) and (17) of the circles $\mathcal{M}$ and $\mathcal{P}'$ we obtain the equation of the so called *dimidium circle* of that quadrilateral with the form

$$x^2+y^2-\frac{1}{2}(2+q)x-\frac{1}{4}(3s-r)y+\frac{1}{2}q=0. \qquad (19)$$

Its center is the dimidium point G_s and it passes through the focus $S=(1,0)$.
We observe the triangle HMP where $H=\left(-1,\frac{1}{4}(3s+r)\right)$ is the centroid of the orthocenters of four trilaterals of the quadrilateral $\mathcal{ABCD}$, M is its central point from (10), and P is its orthopolar center from (18). The triangle HMP has the centroid in the point $T=\left(\frac{1}{6}q,\frac{1}{2}s\right)$ from (10) in [34], and that is the centroid of all six vertices of the quadrilateral $\mathcal{ABCD}$. *The midpoint of the points H and M is the point* $\left(\frac{1}{8}(q-p-1),\frac{1}{2}s\right)$ *and it is incident to the median* $\mathcal{N}$. *The orthopolar center P is incident to the median as well.* These statements are coming from [2].
The circle $\mathcal{P}_d$ with equation

$$x^2+y^2+2x-2(a+b+c+abc)y \\ +a^2+b^2+c^2+ab+ac+bc+abc(a+b+c)=0 \tag{20}$$

has the center in the orthocenter $H_d=(-1,a+b+c+abc)$ of the trilateral $\mathcal{ABC}$ and the radius given by

$$\rho_d^2=1+ab+ac+bc+abc(a+b+c+abc).$$

The point T_{AB} has the polar line $abx+(a+b)y+x+ab-(a+b+c+abc)(y+a+b)+a^2+b^2+c^2+ab+ac+bc+abc(a+b+c)=0$ with respect to the circle $\mathcal{P}_d$, i.e. the line $x-cy+c^2=0$ that is the line $\mathcal{C}$. Similarly, we obtain lines $\mathcal{B}$ and $\mathcal{A}$ for polar lines of points T_{AC} and T_{BC}, respectively. Hence, the circle $\mathcal{P}_d$ is the polar circle of the trilateral $\mathcal{ABC}$. Analogously, the polar circle $\mathcal{P}_c$ of the trilateral $\mathcal{ABD}$ is of the form

$$x^2+y^2+2x-2(a+b+d+abd)y \\ +a^2+b^2+d^2+ab+ad+bd+abd(a+b+d)=0. \tag{21}$$

The radical axis of (20) and (21) is of the final form $2y+s=0$ that makes the equation of the median. The same applies to the polar circles of the other two trilaterals. Hence, from [8] and [24] we have the statement: *The polar circles of $\mathcal{ABC}$, $\mathcal{ABD}$, $\mathcal{ACD}$ and $\mathcal{BCD}$ belong to one pencil of circles whose radical axis is the median of the quadrilateral $\mathcal{ABCD}$.* As it appears in [3] we will show that *this pencil of circles is conjugate to Plücker's pencil of circles, i.e. each circle of one pencil is orthogonal to each circle of another pencil*, see Fig. 5. It is enough to prove that the Plücker's points $P_{1,2}$ are conjugated with respect to the circle (20). Two points (x_i,y_i), $i=1,2$, are conjugated with respect to the circle (20) under the condition that

$$x_1x_2+y_1y_2+x_1+x_2-(a+b+c+abc)(y_1+y_2) \\ +a^2+b^2+c^2+ab+ac+bc+abc(a+b+c)=0$$

holds. For Plücker's points from (16) the equalities $x_1x_2=1$, $y_1y_2=p+q+1$, $x_1+x_2=-2$, $y_1+y_2=s$ are valid, and the next equality

$$p+q-(a+b+c+abc)s+a^2+b^2+c^2 \\ +ab+ac+bc+abc(a+b+c)=0$$

is valid as well.
Putting $y=\frac{1}{2}s$ into (20) we obtain $4x^2+8x+s^2-4p-4q=0$ with solutions $x_{1,2}=-1\pm\frac{1}{2}\sqrt{4(p+q+1)-s^2}$. Hence, the polar circles of four trilaterals of quadrilateral $\mathcal{ABCD}$ have the common points (real or imaginary)

$$P'_{1,2}=\left(1\pm\frac{1}{2}\sqrt{4(p+q+1)-s^2},\frac{1}{2}s\right).$$

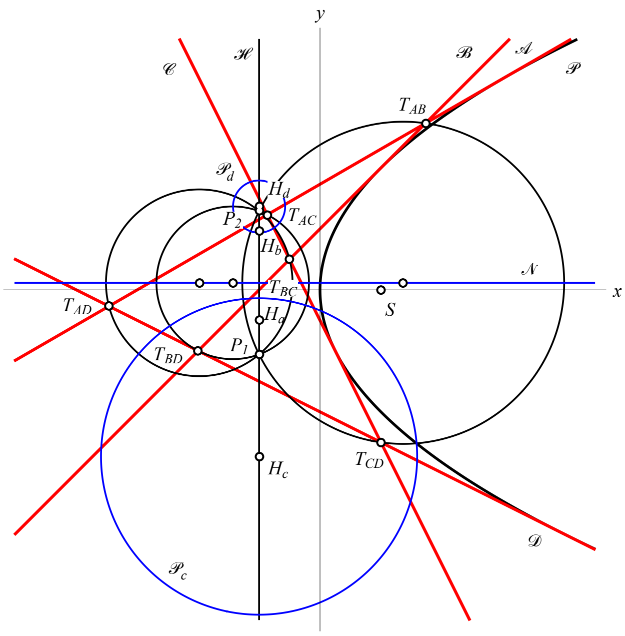

Figure 5: *Two conjugated pencils of circles.*

Let us remind that Plücker's pencil of circle has the common points $P_{1,2}$ from (16). Out of two conjugated pencil of circles, one is always elliptic, and another is hyperbolic, or both of them are parabolic. It depends on the sign of $4(p+q+1)-s^2$. The case, when both of pencils are parabolic, we get $4(p+q+1)=s^2$ i.e. $P_1=P_2=P'_1=P'_2=\left(-1,\frac{1}{2}s\right)$.
The points T_{AB},T_{CD} from (5) are conjugated with respect to the circle with the equation $x^2+y^2+2hx+2jy+k=0$ if and only if the equality $abcd+(a+b)(c+d)+h(ab+cd)+j(a+b+c+d)+k=0$ is valid, i.e.

$$(h-1)(ab+cd)+js+k+p+q=0.$$

The analogous equalities for pairs of points T_{AC},T_{BD} and T_{AD},T_{BC} are

$$(h-1)(ac+bd)+js+k+p+q=0$$
$$(h-1)(ad+bc)+js+k+p+q=0.$$

If two out of these three equalities are valid, than $h=1$ follows, and the third equality is valid as well. Besides,

$k = -js - p - q$ is valid and the observed circle has the equation $x^2 + y^2 + 2x + 2jy - js - p - q = 0$ and the center incident to the directrix of the quadrilateral $\mathcal{ABCD}$. Any two circles of this form have the radical axis with the equation $2y - s = 0$, and it is the median of the quadrilateral $\mathcal{ABCD}$. Hence, *if two pairs of opposite vertices of the quadrilateral are conjugated with respect to some circle, then it is valid for the third pair as well, and the circle belongs to one pencil of circles whose the radical axis is the median and its centers are incident to the directrix of the quadrilateral*, as it is in [22]. This statement is in accordance with the Hesse's theorem: if two pairs of the opposite vertices of the quadrilateral are conjugated with respect to some conic, then this holds for the third pair of its opposite vertices as well.

The previous statement can be refined a little more. Each circle with the center lying on the directrix of the quadrilateral $\mathcal{ABCD}$ has the equation of the form $x^2 + y^2 + 2hx + 2jy + k = 0$, where $h = 1$. Because of that the points T_{AB}, T_{CD} are conjugated with respect to this circle under the condition $js + k + p + q = 0$, so such a circle has the equation $x^2 + y^2 + 2x + 2jy - sj - q - p = 0$ where j is a parameter. The symmetry on a, b, c, d of the previous condition proves our original statement:

Theorem 2 *Let $\mathcal{ABCD}$ is a complete quadrilateral. If one pair of opposite vertices of the quadrilateral is conjugated with respect to the circle with the center on the directrix of the quadrilateral, then it is also valid for other two pairs of opposite vertices.*

We give below the well known statement: If two points are conjugated with respect to the circle then this circle is orthogonal to the circles for which these two points are diametrically opposite. For the sake of completeness in proving all the statements using rectangular coordinates, we will prove this statement. Without loss of generality, we can take that our circle has the center $(0,0)$, the radius equal to 1 and the equation $x^2 + y^2 = 1$, so two points are conjugated with respect to it under the condition that for their coordinates the equality $x_1x_2 + y_1y_2 = 1$. We can take one of these points of the forms $(u, 0)$, and then from previous condition follows that another point is of the form $\left(\frac{1}{u}, v\right)$. The center of another circle is the point $\left(\frac{1}{2}(u + \frac{1}{u}), \frac{1}{2}v\right)$, and the square of its radius is $\frac{1}{4}\left((u - \frac{1}{u})^2 + v^2\right)$, while for the square of distance of these two circles we get the form $\frac{1}{4}\left((u + \frac{1}{u})^2 + v^2\right)$. Our statement follows from the fact that

$$\frac{1}{4}\left(\left(u - \frac{1}{u}\right)^2 + v^2\right) + 1 = \frac{1}{4}\left(\left(u + \frac{1}{u}\right)^2 + v^2\right).$$

From the previous consideration, due to this proven statement, it follows that circles with centers on the directrix, to which the opposite vertices of the quadrilateral $\mathcal{ABCD}$ are conjugate points, form one pencil of circles, which are orthogonal to the circles, of which the diagonals of that quadrilateral are diameters, and the last three circles belong to another pencil of circles (with centers on the median of the quadrilateral).

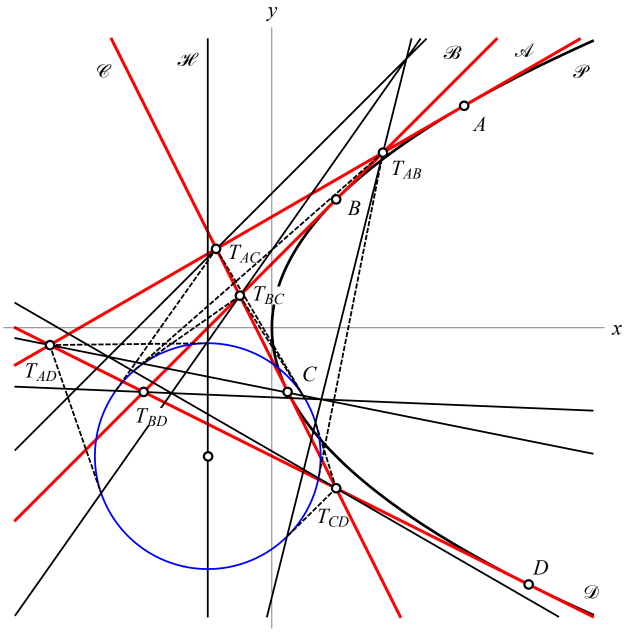

Figure 6: *The complete quadrilateral $\mathcal{ABCD}$ whose opposite vertices are conjugated with respect to the circle with the center on the directrix of the quadrilateral.*

The trilaterals $\mathcal{ABC}$ has the orthocenter $H_d = (-1, a + b + c + abc)$ and circumcenter S_d given by (8), and their midpoint is the Euler's center N_d of that trilateral and it is given by

$$N_d = \left(\frac{1}{4}(ab + ac + bc - 1), \frac{1}{4}(3a + 3b + 3c + abc)\right) \tag{22}$$

Normals to the side $\mathcal{D}$ have the slope $-d$, hence the normal from the point N_d to this line is $4dx + 4y = 3s + r - 4d$. This line is incident to the point

$$H = \left(-1, \frac{1}{4}(3s + r)\right), \tag{23}$$

the same is valid and for the perpendiculars of the Euler's centers of trilaterals $\mathcal{ABD}$, $\mathcal{ACD}$, $\mathcal{BCD}$ to the lines $\mathcal{C}, \mathcal{B}, \mathcal{A}$, respectively. The point H is the *Morley's point* of the quadrilateral $\mathcal{ABCD}$. It is incident to the directrix of this quadrilateral and it is the centroid of points $H_a, H_b.H_c, H_d$, see Fig. 7. The fact that *normals from Euler's centers of four trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$ to lines $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are incident to one point on directrix* we find in [20] and [21].

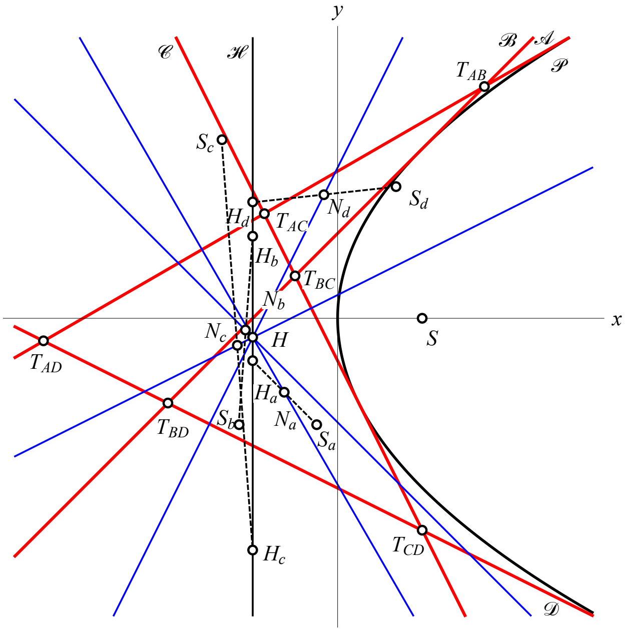

Figure 7: *Morley's point H of $\mathcal{ABCD}$*

The parallel line to $\mathcal{D}$ through the point N_d from (22) has the equation $x - dy = \frac{1}{4}(ab + ac + bc - 1 - 3ad - 3bd - 3cd - abcd)$ and it passes through the point $\left(\frac{1}{4}(ab+ac+bc+ad+bd-3cd-1-abcd), a+b\right)$. The parallel line to $\mathcal{C}$ and through the point N_d is incident to this point as well. The midpoint of it and $T_{CD} = (cd, c+d)$ is the point $\left(\frac{1}{8}(q-p-1), \frac{1}{2}s\right)$. Hence, *the quadrilateral $\mathcal{ABCD}$ is symmetric to the quadrilateral formed by parallels to $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ through the Euler's centers of $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$ with the center of the symmetry $\left(\frac{1}{8}(q-p-1), \frac{1}{2}s\right)$ that is incident to the median $\mathcal{N}$ of the quadrilateral $\mathcal{ABCD}$.* This point is in [32] called nine-point center homothetic center $QL-P22$.

Points S_c, S_b, S_a are analogous to the point S_d from (8). These three points have the centroid

$$G_D = \Big(\frac{1}{6}(ab+ac+2ad+bc+2bd+2cd+3), \quad \frac{1}{6}(2a+2b+2c+3d-abd-acd-bcd)\Big), \qquad (24)$$

i.e. it is the centroid of the triangle $S_aS_bS_c$, and circumcenter of the triangle is the point M from (10). As in any triangle the following is valid: for the circumcenter S, orthocenter H and the centroid G the equality $H + 2S = 3G$ holds. So, for the orthocenter H_D of the triangle $S_aS_bS_c$ the equality $H_D = 3G_D - 2M$ holds and from (10) and (24) we get

$$H_D = \left(\frac{1}{2}(ad+bd+cd+p), \frac{1}{2}(d+abc+s)\right). \qquad (25)$$

The midpoint of the points S_d and H_D is the orthopolar center P from (18). The coordinates of the point H_D given in (25) obviously satisfy the equation (4) of the side $\mathcal{D}$. Therefore, we can conclude that the points H_A, H_B, H_C, H_D lie on the sides $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, respectively. So, the fact found in [7] and [25] is valid: *The orthopolar center P is the midpoint of pairs of points S_a, H_A; S_b, H_B; S_c, H_C; S_d, H_D where H_A, H_B, H_C, H_D are orthocenters of triangles $S_bS_cS_d$, $S_aS_cS_d$, $S_aS_bS_d$, $S_aS_bS_c$, incident to $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, respectively*, see Fig. 10.

The circle $\mathcal{S}_d$ with the equation (7) has the tangent line at $T_{AB} = (ab, a+b)$ of the form

$$2abx + 2(a+b)y - (ab+ac+bc+1)(x+ab) - (a+b+c-abc)(y+a+b) + 2(ab+ac+bc) = 0.$$

This tangent intersects the line $\mathcal{D}$ with $dy = x + d^2$ in the point

$$x = \frac{acd+bcd-abd-a^2d-b^2d+ad^2+bd^2-cd^2-a^2b^2d+abcd^2}{c+d-a-b-abc-abd+acd+bcd}$$
$$y = \frac{ac+bc-a^2-ab-b^2+d^2-a^2b^2-abd^2+acd^2+bcd^2}{c+d-a-b-abc-abd+acd+bcd}. \qquad (26)$$

The equation

$$[c+d-a-b-ab(c+d)+cd(a+b)] \cdot [x^2+y^2-(ab+cd)x-sy+p+(a+b)(c+d)] + [1-ab-cd+(a+b)(c+d)+abcd] \cdot [(a+b-c-d)x+(cd-ab)y+ab(c+d)-cd(a+b)]=0$$

is the linear combination of the equation (13) of the circle with diameter $T_{AB}T_{CD}$ and the equation $(cd-ab)y = (c+d-a-b)x + (a+b)cd - ab(c+d)$ of the line $T_{AB}T_{CD}$, so it is the equation of the circle $ST_{AB}T_{CD}$. The point from (26) is incident to this circle as well. Hence, the following statement from [1] and [3] is proved: *the intersection points of the tangents at the point T_{AB} to the circles S_c and S_d with the sides $\mathcal{C}$ and $\mathcal{D}$, respectively, and the intersections of the tangents at the point T_{CD} to the circles S_a and S_b with the sides $\mathcal{A}$ and $\mathcal{B}$, respectively, lie on the circle $ST_{AB}T_{CD}$. Similarly, two quadruples of analogous intersection points lie on the circles $ST_{AC}T_{BD}$ and $ST_{AD}T_{BC}$.*

It can be checked that the circle with equation

$$(a-b)(x^2+y^2) + (ab^2-a^2b+b^2c-a^2d-a+b+c-d)x - (ab^2c-a^2bd+a^2-b^2+ac-bd)y + a^2b^2c-a^2b^2d+a^2b-ab^2+a^2c-b^2d = 0$$

passes through the points T_{AB}, T_{AD}, T_{BC} and the circle with

$$2(x^2+y^2) - [(a+b)(c+d)+2ab+2]x + [(ab+1)(c+d)-2(a+b+c+d)]y + (a+b)(c+d)+2ab = 0$$

passes through the points S and T_{AB}, and through the midpoints of the line segment $T_{AC}T_{AD}$ and $T_{BC}T_{BD}$. Their radical axis is

$$(a^2+b^2+2)x-(a^2b+ab^2+a+b)y+2a^2b^2+a^2+b^2=0,$$

and due to symmetry on c and d, the circle that is incident to T_{AB},T_{AC},T_{BD} also belongs to the same pencil of circles. All three of circles have the common point T_{AB} and an additional point S_{ab}. However, the circles $T_{AB}T_{AD}T_{BC}$ and $T_{AB}T_{AC}T_{BD}$ are circumcircles of the trilaterals $\mathcal{ABW}$ and $\mathcal{ABV}$ of the quadrilateral $\mathcal{ABVW}$, so the point S_{ab} is the focus of that quadrilateral. We have just proved the statement from [3]: *If S_{ab} is the focus of the quadrilateral $\mathcal{ABVW}$, then the circle $\mathcal{S}_{ab}=ST_{AB}S_{ab}$ passes through the midpoints of line segments $T_{AC}T_{AD}$ and $T_{BC}T_{BD}$. There are five more such circles $\mathcal{S}_{cd}$, $\mathcal{S}_{ac}$, $\mathcal{S}_{bd}$, $\mathcal{S}_{ad}$, $\mathcal{S}_{bc}$ incident to triples of points S,T_{CD},S_{cd}; S,T_{AC},S_{ac}; S,T_{BD},S_{bd}; S,T_{AD},S_{ad}; S,T_{BC},S_{bc} and passing through the two of midpoints of corresponding line segments, where S_{cd}, S_{ac}, S_{bd}, S_{ad}, S_{bc} are foci of quadrilaterals $\mathcal{CDVW}$, $\mathcal{ACUW}$, $\mathcal{BDUV}$, $\mathcal{ADUV}$, $\mathcal{BCUV}$.*

Euler's line of the trilateral $\mathcal{ABC}$, i.e. the line connecting $S_d=\left(\frac{1}{2}(ab+ac+bc+1),\frac{1}{2}(a+b+c-abc)\right)$ and $H_d=(-1,a+b+c+abc)$ has the slope $-\frac{a+b+c+3abc}{ab+ac+bc+3}$, so the perpendicular line from N_d from (22) has the equation

$$\begin{aligned}&4(ab+ac+bc+3)x-4(a+b+c+3abc)y=\\&-3a^2b^2c^2-8abc(a+b+c)+a^2b^2+a^2c^2+b^2c^2\\&-3(a^2+b^2+c^2)d-4(ab+ac+bc)-3.\end{aligned}$$

The point

$$K=\left(\frac{1}{4}(q+3p-1),\frac{1}{4}(3s+r)\right) \qquad (27)$$

fulfill this equation. With analogous statements for other three trilaterals of the quadrilateral $\mathcal{ABCD}$ we have proved statement from [15] and [18]: *The perpendicular lines to Euler's lines of four trilaterals of the quadrilateral at its Euler's centers are incident to one point K.* We will call the point K *Kantor's point* of the quadrilateral $\mathcal{ABCD}$, see Fig. 8. *The points M and K from* (10) *and* (27) *have the point P from* (18) *as their midpoint*, as stated in [2].

The point H and the point K from (23) and (27) have the same ordinate, so *the line HK is parallel to median, i.e. perpendicular to the directrix* that is the result given in [2], [7] and [10].

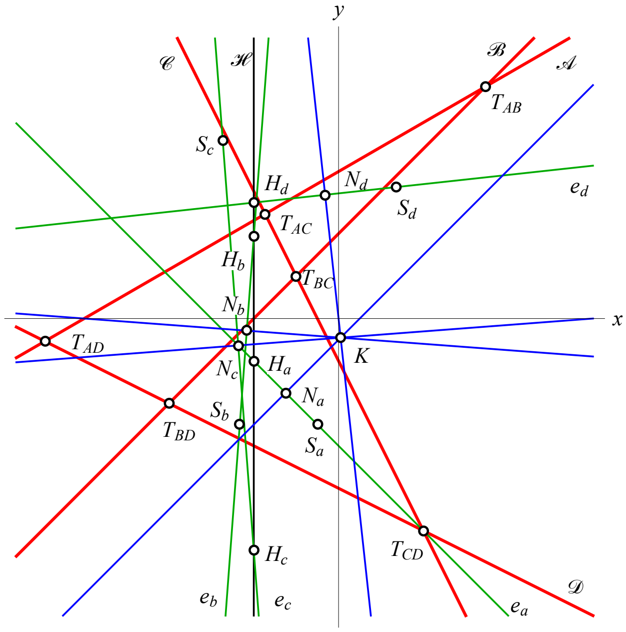

Figure 8: *Euler's lines e_a, e_b, e_c, e_d and Kantor's point K of $\mathcal{ABCD}$*

The center M from (10) and Kantor's point K from (27) have the midpoint P, the orthopolar center from (18), and points M and P have the dimidium center G_s for the midpoint. Thus, for the oriented lengths we have ratios $KP:PG_s:G_sM=2:1:1$. The point $P22=\left(\frac{1}{8}(q-p-1),\frac{1}{2}s\right)$ is the midpoint of the Morley's point H from (23) and central point M from (10), as well as the midpoint of the orthopolar center P from (18) and the point $P20=\left(-\frac{1}{2}(p+1),\frac{1}{2}s\right)$. For the centroid $T=\left(\frac{1}{6}q,\frac{1}{2}s\right)$, dimidium point $G_s=\left(\frac{1}{4}(q+2),\frac{1}{8}(3s-r)\right)$ and Morley's point $H=\left(-1,\frac{1}{4}(3s+r)\right)$, the equality $3T=2G_s+H$ is valid meaning that these points are collinear and that for oriented lengths the equality $TH=-2TG_s$ holds. Out of (23) and (27) the equality $K-H=\left(\frac{1}{4}(q+3p+3),0\right)$ follows, and for the point T and P from $P-T=\left(\frac{1}{12}(q+3p+3),0\right)$ is valid, so the lines HK and TP are parallel, and for the oriented lengths the equality $HK=3TP$ holds. The points $T,P,P20$ and $P22$ are incident to the median $\mathcal{N}$ and we derived the equality $\overrightarrow{TP}=\left(\frac{1}{12}(q+3p+3),0\right)$. Out of the equalities $P20=\left(-\frac{1}{2}(1+p),\frac{1}{2}s\right)$ and $P22=\left(\frac{1}{8}(q-p-1),\frac{1}{2}s\right)$, the forms $\overrightarrow{P20P22}=\left(\frac{1}{8}(q+3p+3),0\right)$, $\overrightarrow{P22T}=\left(\frac{1}{24}(q+3p+3),0\right)$ follow, and because of that $P20P22:P22T:TP=3:1:2$. From the previous consideration points $T,M,P,G_s,H,K,P20$ and $P22$ in each quadrilateral have mutual relationships as it is presented on the Fig. 9. Then the fact from [32] that *lines $MP20,PH$ and G_sP22 are parallel* follows.

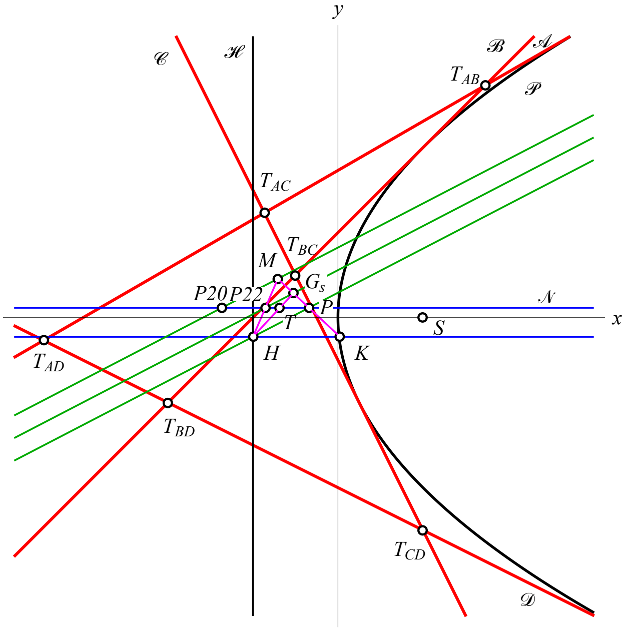

Figure 9: *Centroid T, orthopolar center P, dimidium point G_s, Morley's point H and Kantor's point K of $\mathcal{ABCD}$*

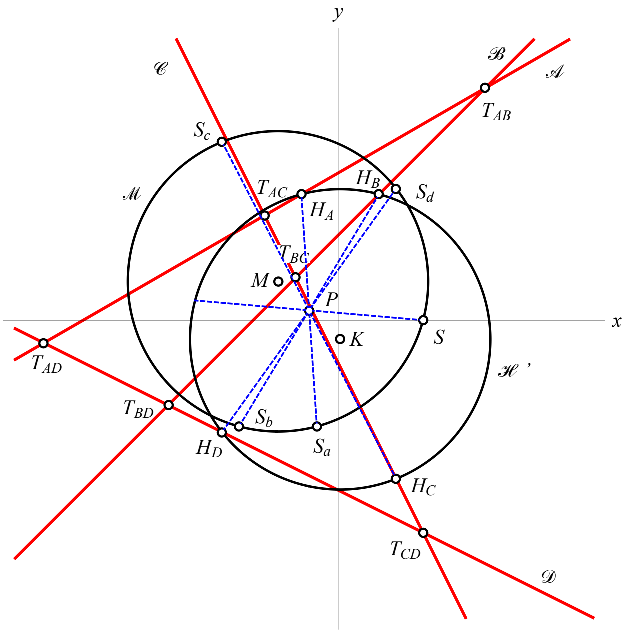

Figure 10: *Hervey's circle $\mathcal{H}'$ of $\mathcal{ABCD}$*

Symmetry with respect to the point P from (18) is done by replacement

$$x \to \frac{1}{2}(1+q+p)-x, \quad y \to s-y. \tag{28}$$

By this symmetry the points S_a, S_b, S_c, S_d are mapped into H_A, H_B, H_C, H_D and because of that the last mentioned points are incident to one circle $\mathcal{H}'$ whose equation is derived from the equation (9) of the central circle $\mathcal{M}$. We obtain the circle $\mathcal{H}'$

$$x^2+y^2-\frac{1}{2}(q+3p-1)x-\frac{1}{2}(3s+r)y$$
$$+\frac{1}{2}(s^2+sr-p+p^2+pq)=0, \tag{29}$$

so called *Hervey's circle* of the quadrilateral $\mathcal{ABCD}$. Its center is Kantor's point K of the quadrilateral $\mathcal{ABCD}$, see Fig. 10. The fact that $H_AH_BH_CH_D$ *lie on one circle with the center* K appears in [17]. The point symmetric to the focus S with respect to the point P obviously is incident to the circle $\mathcal{H}'$.

From (17) and (21) given in [34] the next equations are obtained

$$16(\rho_a^2+\rho_b^2+\rho_c^2+\rho_d^2)=12s^2+8q^2+4r^2-4sr$$
$$-2qp-6q+4p+4,$$
$$16\rho^2=s^2+q^2+r^2+p^2-2sr-2qp-2q+2p+1,$$

and out of (2), (10) and (27) the following equalities can be derived

$$16SK^2=9s^2+q^2+r^2+9p^2+6sr+6pq-10q-30p+25,$$
$$16MK^2=4s^2+4r^2+16p^2+8sr-32p+16.$$

And now, we easily get the equality $\rho_a^2+\rho_b^2+\rho_c^2+\rho_d^2-7\rho^2=SK^2-MK^2$ that is statement from [11]. The solution of Guillotin reaches $\overrightarrow{MS_a}+\overrightarrow{MS_b}+\overrightarrow{MS_c}+\overrightarrow{MS_d}=\overrightarrow{MK}$ that then can be found in [14], while [32] attributes it to Morley. This means that equality $S_a+S_b+S_c+S_d=3M+K$ holds for the points themselves. However, it is easy to see by (8) and analogous formulas, and (10) and (27) that the left and the right sides are equal to $\left(q+2, \frac{1}{2}(3s-r)\right)$.

We can prove famous Zeeman's theorem from [12] saying: *if one side of the quadrilateral is parallel to Euler's line of the trilateral formed by the remaining three sides, then this holds for the rest three analogous combinations*. Indeed, slopes $-\frac{a+b+c+3abc}{ab+ac+bc+3}$ and $\frac{1}{d}$ of the Euler's line of the trilateral $\mathcal{ABC}$ and the line $\mathcal{D}$ are the same under the condition $q+3p+3=0$ that is symmetric on a,b,c,d. The quadrilateral with such properties is called *Zeeman's quadrilateral*.

The midpoint of Kantor's point K from (27) and central point M from (10) is $\left(\frac{1}{4}(1+q+p), \frac{1}{4}(s-r)\right)$. *In the case of Zeeman's quadrilateral* $1+q+p=\frac{2}{3}q$ *holds, hence this midpoint is the centroid* T. The statement comes from [12]. *In the same case Kantor's point* K *from* (27) *and Morley's point* H *from* (23) *coincide*, that is the result in [17].

As explained in [16], if perpendiculars are dropped on any line from the vertices of a triangle, the perpendiculars to the opposite sides from their feet are concurrent at a point called the orthopole of the line. The construction of the orthopol O_d of the line $\mathcal{D}$ with respect to the trilateral $\mathcal{ABC}$ is shown in Fig. 12.

The perpendicular line from the point S to the line $\mathcal{A}$ has the equation $y+ax=a$ and intersects the line $\mathcal{A}$ in the point $A''=(0,a)$. Similarly, ortoghonal projections of the focus S to lines $\mathcal{B},\mathcal{C},\mathcal{D}$ are points $B''=(0,b)$, $C''=(0,c)$, $D''=(0,d)$, respectively. All of four points lie on $y-$axis that is vertex tangent to parabola $\mathcal{P}$. Let $\mathcal{A}'',\mathcal{B}'',\mathcal{C}'',\mathcal{D}''$ be parallel lines to the lines $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ through the points H_a,H_b,H_c,H_d. The lines $\mathcal{A}''$ and $\mathcal{B}''$ have the equations

$$ay=x+1+ab+ac+ad+abcd,$$
$$by=x+1+ab+bc+bd+abcd,$$

and their intersection point is $(-1-ab-abcd,c+d)$. The midpoint of this point and the point $T_{AB}=(ab,a+b)$ is the point $N=(-\frac{1}{2}(p+1),\frac{1}{2}s)$ from (28) in [34]. Similarly, the same is valid for the quadrilaterals $\mathcal{ABCD}$ and $\mathcal{A}''\mathcal{B}''\mathcal{C}''\mathcal{D}''$, hence these two quadrilaterals are symmetric with respect to the point N. The perpendicular from $\mathcal{A}''\cap\mathcal{B}''$ to the line $\mathcal{H}$ has the equation $y=c+d$ and intersects it in the point $(-1,c+d)$, and the perpendicular line from that point to the line $\mathcal{C}''$ has the equation $y+cx=d$ so it passes through the point $D''=(0,d)$. Because of symmetry on a,b,c we conclude that D'' is the orthopole of $\mathcal{H}$ with respect to the trilateral $\mathcal{A}''\mathcal{B}''\mathcal{C}''$. In the same way we obtain three more orthopoles. The statement that *orthogonal projections of the point S to lines $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are orthopoles of the line $\mathcal{H}$ with respect to trilaterals $\mathcal{B}''\mathcal{C}''\mathcal{D}''$, $\mathcal{A}''\mathcal{C}''\mathcal{D}''$, $\mathcal{A}''\mathcal{B}''\mathcal{D}''$, $\mathcal{A}''\mathcal{B}''\mathcal{C}''$* appears in [28] and [29].

Parallel lines to $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ through the points S_a,S_b,S_c,S_d form a quadrilateral symmetric to the quadrilateral $\mathcal{ABCD}$ with respect to the point P, and parallel to lines $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ through points H_a,H_b,H_c,H_d form a quadrilateral symmetric to $\mathcal{ABCD}$ with respect to point N. The second quadrilateral is obtained by using translation for a vector $\mathbf{v}$ from the first quadrilateral. If τ is any real number and if T_a,T_b,T_c,T_d are points such that $S_aT_a=\tau S_aH_a$, $S_bT_b=\tau S_bH_b$, $S_cT_c=\tau S_cH_c$, $S_dT_d=\tau S_dH_d$ hold, then lines through the points T_a,T_b,T_c,T_d parallel to $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ form the quadrilateral that is obtained by using the translation for vector $\tau\mathbf{v}$ from the first new before mentioned quadrilateral. It is symmetric to $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ with respect to the point T_τ that is obtained from P by the translation for the vector $\tau\overrightarrow{PN}$. All of these centers of symmetry T_τ are incident to the common median of all of mentioned quadrilaterals. Particularly, *the quadrilateral made by lines parallel to $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ through centroids G_a,G_b,G_c,G_d of trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$, is symmetric to the quadrilateral $\mathcal{ABCD}$ with respect to* a point T_τ such that $\overrightarrow{PT_\tau}=\frac{1}{3}\overrightarrow{PN}$. It is easy to see that $T_\tau=(\frac{1}{6}q,\frac{1}{2}s)$, and that is *the centroid T of the vertices of $\mathcal{ABCD}$*. This statement appears in [4].

The points S_a,S_b,S_c,S_d are in accordance with (8). The midpoint of e.g. S_c and S_d have coordinates of the form

$$x=\frac{1}{4}(2ab+ac+ad+bc+bd+2),$$
$$y=\frac{1}{4}(2a+2b+c+d-abc-abd),$$

and the line S_aS_b has the slope $\frac{1-cd}{c+d}$ so the perpendicular line to this line through the point P from (18) has the equation

$$(c+d)x+(1-cd)y=\frac{1}{4}(c+d)(p+q+1)+\frac{1}{2}(1-cd)s,$$

that is fulfilled by before mentioned coordinates. Analogously, the perpendicular lines from the point P to any side of the cyclic quadrangle $S_aS_bS_cS_d$ pass through the midpoints of corresponding opposite sides. A well known properties of the cyclic quadrangle is following: the perpendicular lines from the midpoint of the side to the opposite side pass through one point such that Wallace's lines of the vertices of this quadrangle with respect to the triangles formed by remaining three vertices are incident to this point as well as Euler's circles of all four triangles. This point is so called *anticenter* of this cyclic quadrangle. Hence, the orthopolar center P of the quadrilateral $\mathcal{ABCD}$ is the anticenter of the cyclic quadrangle $S_aS_bS_cS_d$. Because of symmetry with respect to the point P it follows that P *is the anticenter for the cyclic quadrangle $H_AH_BH_CH_D$* as well. We find this statement in [21].

The line $\mathcal{D}_0$ through the point S_d from (8) and parallel to $\mathcal{D}$ has the equation

$$dy-x=\frac{1}{2}(ad+bd+cd-bc-bd-cd-abcd-1).$$

This equation is possible to obtain by substitutions (28), so lines $\mathcal{D}$ and $\mathcal{D}_0$ are symmetric with respect to the orthopolar center P. Similarly, this is valid for the remaining sides of $\mathcal{ABCD}$. Hence, we shown our original theorem:

Theorem 3 *Let $\mathcal{ABCD}$ be a complete quadrilateral. Quadrilaterals $\mathcal{ABCD}$ and $\mathcal{A}_0\mathcal{B}_0\mathcal{C}_0\mathcal{D}_0$ are symmetric with respect their join orthopolar center P, where $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathcal{C}_0$, $\mathcal{D}_0$ are lines through the circumcenters S_a, S_b, S_c, S_d of trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$, and parallel to $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ respectively. The circumcenters of trilaterals of one quadrilateral are incident to corresponding sides of the other quadrilateral. The central circle of one quadrilateral is Hervey's circle of the other. The central point of one quadrilateral is Kantor's point of the other.*

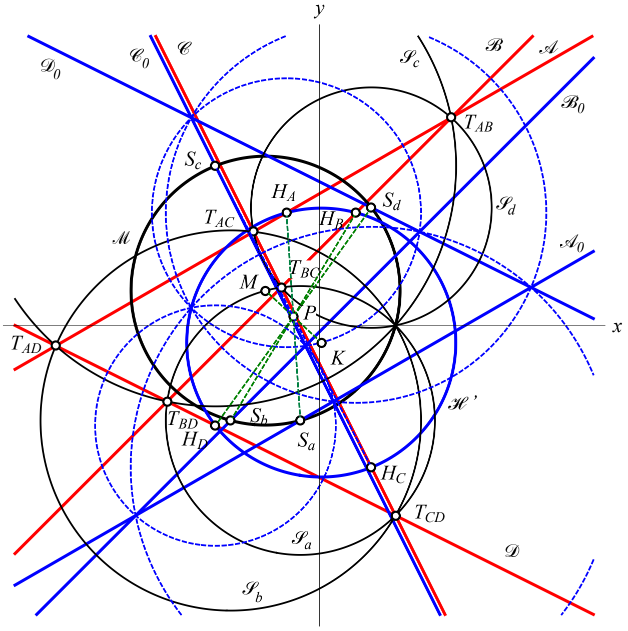

Figure 11: *Hervey's circle $\mathcal{H}'$ of $\mathcal{ABCD}$ is central circle of $\mathcal{A}_0\mathcal{B}_0\mathcal{C}_0\mathcal{D}_0$, and vice versa.*

The line $\mathcal{D}_0$ through the point S_d from (8) parallel to $\mathcal{D}$ is $x-dy=\frac{1}{2}(1+ab+ac+bc-ad-bd-cd+abcd)$, and the normal from the point T_{AB} to the line $\mathcal{D}_0$ has the equation $dx+y=a+b+abd$ and intersects it in the point having coordinates

$$x=\frac{1}{2\delta}(1+ab+ac+ad+bc+bd-cd+abcd+2abd^2),$$
$$y=\frac{1}{2\delta}(2a+2b-d+ad^2+bd^2+cd^2+abd-acd-bcd-abcd^2).$$

The normal from this point to the line $\mathcal{C}$ has the equation

$$2\delta(cx+y)=2a+2b+c-d+abc+abd+ac^2+ad^2 +bc^2+bd^2+cd^2-c^2d+abc^2d+abcd^2$$

and passes through the point O_{d0} with coordinates

$$x=\frac{1}{2\delta}(ab+ac+bc-ad-bd-cd+abcd-1),$$
$$y=\frac{1}{2\delta}(2a+2b+2c-d+abd+acd+bcd +ad^2+bd^2+cd^2+abcd^2).$$

As these coordinates are symmetric on a,b,c this point is incident to two more analogous normals, i.e. the point O_{d0} is the orthopole of the line $\mathcal{D}_0$ with respect to the trilateral $\mathcal{ABC}$. It is easy to prove that the point O_{d0} is incident to orthopolar circle $\mathcal{P}'$ of the quadrilateral $\mathcal{ABCD}$ with equation (17). The same is valid for orthopoles O_{a0},O_{b0},O_{c0} of lines $\mathcal{A}_0,\mathcal{B}_0,\mathcal{C}_0$ with respect to trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$ that justify the name of this circle. The fact that *four orthopoles $O_{a0},O_{b0},O_{c0},O_{d0}$ lie on one circle incident to focus*, we find in [30]. Leemans uses in his proof inscribed parabola with the equation $y^2=4x$ as well, and as the result he obtains the equation (17) using notations different from ours. The point O_{d0} can be written in the form

$$O_{d0}=\left(\frac{1}{2\delta}(q+p-1-2ds+2d^2),\frac{1}{2\delta}(2s-3d+dq+dp)\right). \tag{30}$$

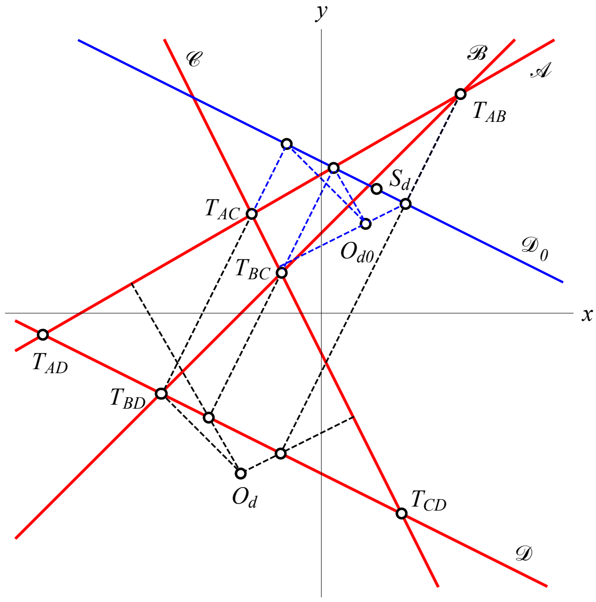

Figure 12: *The orthopoles O_d and O_{d0} of the lines $\mathcal{D}$ and $\mathcal{D}_0$ with respect to the trilateral $\mathcal{ABC}$, respectively.*

The line $\mathcal{D}'''$ symmetric to the line $\mathcal{D}$ with respect to the axis $\mathcal{X}$ of the parabola $\mathcal{P}$ has the slope $-\frac{1}{d}$, so the line through O_{d0} parallel to this line has the equation $x+dy=\frac{1}{2\delta}(q+p-1-d^2+qd^2+pd^2)$ i.e. the equation $x+dy=\frac{1}{2}(q+p-1)$. This line intersects the axis $\mathcal{X}$ in the point $\left(\frac{1}{2}(q+p-1),0\right)$. It is easy to check that this point is incident to orthopolar circle with the equation (17). The line through the point S_d from (8) parallel to the line $\mathcal{D}'''$ has the equation $x+dy=\frac{1}{2}(1+q-p)$ and intersects the axis $\mathcal{X}$ in the point $(\frac{1}{2}(1+q-p),0)$ which lies on the central circle with the equation (9). These results are in [31] attributed to R. Bouvaist-u: *the lines through the circumcenters of four trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$, and parallel to the lines symmetric to $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ with respect to the axis of the inscribed parabola $\mathcal{P}$ intersect in the intersection point of the axis and the central circle different from the focus S.* For the point O_{d0} from (30) we have $x-1=\frac{1}{2\delta}(q+p-3-2ds)$, so together with second coordinate from (30) the equality $4\delta^2 SO_{d0}{}^2=(d^2+1)(p^2+2pq+q^2-6p-6q+4s^2+9)$ is obtained, i.e.

$$SO_{d0}{}^2=\frac{1}{4\delta}(p^2+2pq+q^2-6p-6q+4s^2+9),$$

and formulas for distances of focus S to the points ${O_{a0}}^2, {O_{b0}}^2, {O_{c0}}^2$ look similarly and from them the equality

$$SO_{a0} : SO_{b0} : SO_{c0} : SO_{d0} = \frac{1}{\sqrt{\alpha}} : \frac{1}{\sqrt{\beta}} : \frac{1}{\sqrt{\gamma}} : \frac{1}{\sqrt{\delta}} \qquad (31)$$

follows. From (31) for $x - 1$ and the second coordinate in (30) follows that SO_{d0} has the slope $\frac{2s-3d+dq+dp}{q+p-3-2ds}$ and, analogously, the line SO_{c0} has $\frac{2s-3c+cq+cp}{q+p-3-2cs}$. After applying formula (22) in [34] and leaving out the common factor $p^2+2pq+q^2-6p-6q+4s^2+9$ we obtain $\tan\angle(SO_{d0}, SO_{c0}) = \frac{c-d}{cd+1}$. Lines ST_{AC} and ST_{AD} have the slopes $\frac{a+c}{ac-1}$ and $\frac{a+d}{ad-1}$ and for them we obtain $\tan\angle(ST_{AC}, ST_{AD}) = \frac{c-d}{cd+1}$. Hence, $\angle(SO_{d0}, SO_{c0}) = \angle(ST_{AC}, ST_{AD})$. Earlier we have proved that focus S is the center of similarity of quadrangles $T_{CD}S'_B T_{AD} T_{AC}$ and $S_a S_b S_c S_d$. Because of that $\angle(ST_{AC} ST_{AD}) = \angle(SS_d, SS_c)$. We obtain the equality $\angle(SO_{d0} SO_{c0}) = \angle(SS_d, SS_c)$. With such other equalities with other index pairs, for example, $\angle(SO_{a0} SO_{b0}) = \angle(SS_a, SS_b)$ etc. and from equalities (12) and (31) we conclude that *cyclic quadrangles $S_a S_b S_c S_d$ and $O_{a0} O_{b0} O_{c0} O_{d0}$ are directly similar with the center of similarity S*. This result appears in [26].

According to [34] the orthopole O_d of the line $\mathcal{D}$ with respect to $\mathcal{ABC}$ is given by $O_d = \left(-1, \frac{1}{\delta}\left(a+b+c+(ab+ac+bc)d+abcd^2+d^2\right)\right)$. Its coordinates can be rewritten as $O_d = \left(-1, \frac{1}{\delta}\left(s-d+qd-sd^2+pd+2d^3\right)\right)$. The difference between the ordinates of the points O_d and O_{d0} is equal to $\frac{1}{2\delta}\left(-d-qd-pd+2sd^2-4d^3\right)$, and difference between their abscissas is equal to $\frac{1}{2\delta}\left(1+q+p-2sd+4d^2\right)$. Thus, the slope of the line $O_d O_{d0}$ is $-d$. Therefore, $O_d O_{d0}$ is perpendicular to $\mathcal{D}$. Similarly, the lines $O_a O_{a0}$, $O_b O_{b0}$, $O_c O_{c0}$ are perpendicular to the sides $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ (see Fig. 13). So, the following theorem, which is our original result, is proved:

Theorem 4 *Let $\mathcal{ABCD}$ be a complete quadrilateral. The lines $O_a O_{a0}$, $O_b O_{b0}$, $O_c O_{c0}$, $O_d O_{d0}$ are perpendicular to the sides $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, where O_a, O_b, O_c, O_d are orthopoles of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ with respect to $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$, and O_{a0}, O_{b0}, O_{c0}, O_{d0} are orthopoles of $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathcal{C}_0$, $\mathcal{D}_0$ with respect to $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$, respectively.*

Some long, but elementary calculations show that the midpoint of the point S_d given by (8), and the point O_{d0} given by (30) has the coordinates

$$x = \frac{1}{4\delta}\left(2ab+2ac+2bc-ad-bd-cd+dr+d^2\right)$$
$$y = \frac{1}{4\delta}(3a+3b+3c-d+abd+acd+bcd-abc$$
$$+2(a+b+c)d^2). \qquad (32)$$

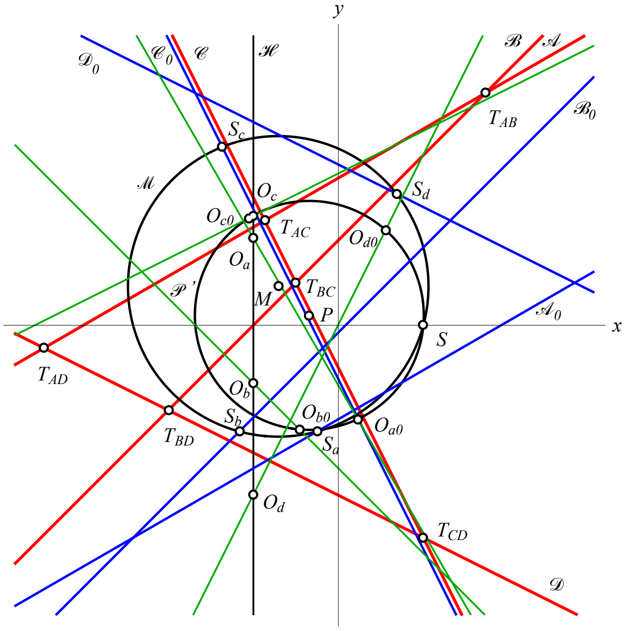

Figure 13: *Lines $O_a O_{a0}$, $O_b O_{b0}$, $O_c O_{c0}$, $O_d O_{d0}$ are perpendicular to $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, where O_a, O_b, O_c, O_d are orthopoles of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ with respect to $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$, and O_{a0}, O_{b0}, O_{c0}, O_{d0} are orthopoles of $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathcal{C}_0$, $\mathcal{D}_0$ with respect to $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$, respectively.*

It follows from (11) that the point S'_D has the coordinates

$$S'_D = \left(\frac{1}{\delta}(d^2+ab+bc+ac-ad-bd-cd+abcd),\right.$$
$$\left.\frac{1}{\delta}(a+b+c-d+abd+acd+bcd-abc)\right),$$

and therefore the point given by (32) is at the same time the centroid of the quadrangle $T_{AB}T_{BC}T_{AC}S'_D$. The fact that *the midpoints of the line segments $S_a O_{a0}$, $S_b O_{b0}$, $S_c O_{c0}$, $S_d O_{d0}$ are the centroids of the quadrangles $T_{BC}T_{CD}T_{BD}S'_A$, $T_{AC}T_{CD}T_{AD}S'_B$, $T_{AB}T_{BD}T_{AA}S'_C$, $T_{AB}T_{BC}T_{AC}S'_D$, respectively*, can be found in [26].

If we write the coordinates of the point S_d in the following form

$$\left(\frac{1}{2}(q+1-sd+d^2), \frac{1}{2}(s-d+d^3-sd^2+qd-r)\right),$$

then we can check that they fulfill the equation

$$2(s+r-2d+2pd)x+2(sd+rd+2-2p)y$$
$$+3sp-rp+r-3s-sq-qr+4d-4pd=0. \qquad (33)$$

The coordinates of the point O_{d0} given by (30) also fulfill (33). So, that equation represents the line $S_d O_{d0}$. Similarly, the line $S_c O_{c0}$ has the equation

$$2(s+r-2c+2pc)x+2(sc+rc+2-2p)y$$
$$+3sp-rp+r-3s-sq-qr+4c-4pc=0. \qquad (34)$$

By subtracting those two equations and dividing the result by the factor $2(d-c)$, we get the equation

$$2(p-1)x+(s+r)y-2(p-1)=0, \qquad (35)$$

the equation of the radical axis of the circles $\mathcal{M}$ and $\mathcal{P}'$ given by (9) and (17). Therefore, lines S_dO_{d0} and S_cO_{c0} intersect each other in a point lying on the radical axis of the central and the orthopolar circle. The lines S_aO_{a0} and S_bO_{b0} pass through the same point as well. From (33) and (34) we obtain its coordinates as

$$x=\frac{1}{2k}\big(pr^2-2qrs-3ps^2+qr^2+2prs+qs^2+8p^2 -r^2+2rs+3s^2-16p+8\big), \qquad (36)$$

$$y=\frac{1}{k}\big(-p^2r+3p^2s-pqr-pqs+4pr-4ps+qr+qs-3r+s\big),$$

where $k=4(p-1)^2+(s+r)^2$. The connection line MP of the centers of the central and the orthopolar circles has the equation

$$4(s+r)x-8(p-1)y+3sp-rp-qr-sq-r-5s=0.$$

Its intersection with the radical axis (35) of $\mathcal{M}$ and $\mathcal{P}'$ has the coordinates

$$x=\frac{1}{4k}\big(pr^2-2prs-3ps^2+qr^2+2qrs+qs^2 +16p^2+r^2+6rs+5s^2-32p+16\big)$$

$$y=\frac{1}{2k}\big(-p^2r+3p^2s-pqr-pqs+4pr-4ps +qr+qs-3r+s\big), \qquad (37)$$

being, therefore, the midpoint of the point (36) and the focus S whose coordinates can be written in the form $S=\big(\frac{1}{2k}(8p^2-16p+8+2s^2+4sr+2r^2),0\big)$. The fact that *the intersection of the lines S_aO_{a0}, S_bO_{b0}, S_cO_{c0}, S_dO_{d0} is the intersection point of the central circle $\mathcal{M}$ and the orthopolar circle $\mathcal{P}'$ (different from S)*, can be found in [26]. It is depicted in Fig. 14.
By using the equalities $d^4-sd^3+qd^2-rd+p=0$ and $abc=r-qd+sd^2-d^3$ the coordinates of the midpoint od S_d and O_{s0} given by (32) get the form

$$\left(\frac{1}{4\delta}(2q-3sd+rd+4d^2),\frac{1}{4\delta}(3s-4d-r+2qd)\right).$$

They satisfy equation (19), so we have proved one more result given in [26]: *the midpoints of the segments S_aO_{a0}, S_bO_{b0}, S_cO_{c0}, S_dO_{d0} lie on the dimidium circle of the quadrilateral $\mathcal{ABCD}$.*

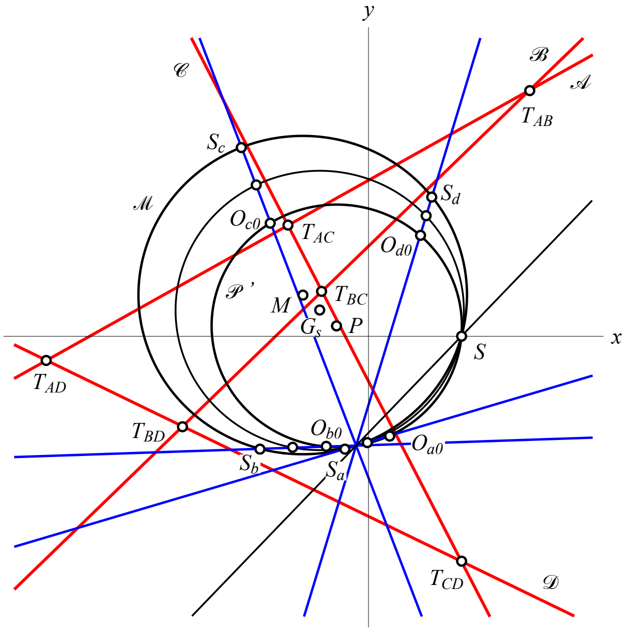

Figure 14: *Lines S_aO_{a0}, S_bO_{b0}, S_cO_{c0}, S_dO_{d0} pass through the intersection point of the orthopolar circle $\mathcal{P}'$ and central circle $\mathcal{M}$ of the quadrilateral $\mathcal{ABCD}$.*

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