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DIOPHANTINE $D(n)$-QUADRUPLES IN $\mathbb{Z}[\sqrt{4k+2}]$

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Abstract. Let d be a square-free integer and $\mathbb{Z}[\sqrt{d}]$ a quadratic ring of integers. For a given $n \in \mathbb{Z}[\sqrt{d}]$, a set of m non-zero distinct elements in $\mathbb{Z}[\sqrt{d}]$ is called a Diophantine $D(n)$-m-tuple (or simply $D(n)$-m-tuple) in $\mathbb{Z}[\sqrt{d}]$ if product of any two of them plus n is a square in $\mathbb{Z}[\sqrt{d}]$. Assume that $d \equiv 2 \pmod 4$ is a positive integer such that $x^2 - dy^2 = -1$ and $x^2 - dy^2 = 6$ are solvable in integers. In this paper, we prove the existence of infinitely many $D(n)$-quadruples in $\mathbb{Z}[\sqrt{d}]$ for $n = 4m + 4k\sqrt{d}$ with $m, k \in \mathbb{Z}$ satisfying $m \not\equiv 5 \pmod 6$ and $k \not\equiv 3 \pmod 6$. Moreover, we prove the same for $n = (4m+2) + 4k\sqrt{d}$ when either $m \not\equiv 9 \pmod{12}$ and $k \not\equiv 3 \pmod 6$, or $m \not\equiv 0 \pmod{12}$ and $k \not\equiv 0 \pmod 6$. At the end, some examples supporting the existence of quadruples in $\mathbb{Z}[\sqrt{d}]$ with the property $D(n)$ for the above exceptional n's are provided for $d = 10$.

1. Introduction

A set $\{a_1, a_2, \ldots, a_m\}$ of m distinct positive integers is called a Diophantine m-tuple with the property $D(n)$ (or simply $D(n)$-m-tuple) for a given non-zero integer n, if $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$. For $n = 1$, such an m-tuple is called Diophantine m-tuple instead of Diophantine m-tuple with the property $D(1)$. The question of constructing such tuples was first studied by Diophantus of Alexandria, who found a Diophantine quadruple of rationals $\{1/16, 33/16, 17/4, 105/16\}$ with the property $D(1)$. However, it was Fermat who first found a Diophantine quadruple $\{1, 3, 8, 120\}$ in integers. Later, Baker and Davenport [3] proved that Fermat's quadruple can not be extended to Diophantine quintuple. Dujella [12] proved the non-existence of Diophantine sextuple and that there are at most finitely many integer Diophantine quintuples. Recently, He, Togbé and Ziegler [24] proved

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the non-existence of integer Diophantine quintuples, and in this way, they solved a long-standing open problem. On the other hand, Bonciocat, Cipu and Mignotte [5] proved a conjecture of Dujella [9], which states that there are no $D(-1)$-quadruples. It is also known due to Trebješanin and Filipin [4] that there do not exist $D(4)$-quintuples. A brief survey on this topic can be found in [15]. We also refer [6, 8, 13, 14, 16] to the reader for more information about $D(n)$-m-tuples.

Let $\mathcal{R}$ be a commutative ring with unity. For a given $n \in \mathcal{R}$, a set $\{a_1, a_2, \ldots, a_m\} \subset \mathcal{R} \setminus \{0\}$ is called a Diophantine m-tuple with the property $D(n)$ in $\mathcal{R}$ (or simply $D(n)$-m-tuple in $\mathcal{R}$), if $a_i a_j + n$ is a perfect square in $\mathcal{R}$ for all $1 \leq i < j \leq m$. Let K be an imaginary quadratic number field and $\mathcal{O}_K$ be its ring of integers. In 2019, Adžaga [2] proved that there are no $D(1)$-m-tuples in $\mathcal{O}_K$ when $m \geq 42$. Recently, Gupta [23] proved that there do not exist $D(-1)$-m-tuple for $m \geq 37$. It is interesting to note that $D(n)$-quadruples are related to the representations of n by the binary quadratic form $x^2 - y^2$. In particular, Dujella [9] proved that a $D(n)$-quadruple in integers exists if and only if n can be written as a difference of two squares, up to finitely many exceptions. Later, Dujella [11] proved the above fact in Gaussian integers. Further, the above fact also holds for the ring of integers of $\mathbb{Q}(\sqrt{d})$ for certain $d \in \mathbb{Z}$ (see, [1, 17–19, 21, 26]). These results motivated Franušić and Jadrijević to post the following conjecture.

Conjecture 1.1 ([22, Conjecture 1]). *Let $\mathcal{R}$ be a commutative ring with unity 1 and $n \in \mathcal{R} \setminus \{0\}$. Then a $D(n)$-quadruple exists if and only if n can be written as a difference of two squares in $\mathcal{R}$, up to finitely many exceptions of n.*

This conjecture was verified for rings of integers of certain number fields (cf. [1, 17–22, 25, 26]).

The following notation will be followed throughout the paper.

- $(a, b) = a + b\sqrt{d}$,
- $k(a, b) = (ka, kb)$ for $k \in \mathbb{Z}$,
- Let $\alpha = (a, b)$. The norm Nm of α is given by
$$\mathrm{Nm}(\alpha) := (a, b)(a, -b),$$
- $(x, y) \equiv (a, b) \pmod{(c, e)}$ means that $x \equiv a \pmod{c}$ and $y \equiv b \pmod{e}$.

In the rest of paper, we fix $d \equiv 2 \pmod 4$ to be a square-free positive integer. We set $\mathcal{S}$ and $\mathcal{T}$ in $\mathbb{Z}[\sqrt{d}]$ as follows:

$$\begin{aligned}\mathcal{S} := \{&(4m, 4k+1), (4m, 4k+2), (4m, 4k+3), (4m+1, 4k+1),\\ &(4m+1, 4k+3), (4m+2, 4k+1), (4m+2, 4k+3), (4m+3, 4k+1),\\ &(4m+3, 4k+3)\},\end{aligned}$$

$$\begin{aligned}\mathcal{T} := \{&(4m,4k),(4m+1,4k),(4m+1,4k+2),(4m+2,4k),\\ &(4m+2,4k+2),(4m+3,4k),(4m+3,4k+2)\},\end{aligned}$$

where $m,k\in\mathbb{Z}$. It is easy to check that if $n\in\mathbb{Z}[\sqrt{d}]$ then $n\in\mathcal{S}\cup\mathcal{T}$. In [17], Franušić proved that there does not exist any $D(n)$-quadruple in $\mathbb{Z}[\sqrt{d}]$ for $n\in\mathcal{S}$.

Thus, it is natural to ask '*whether there exists any Diophantine quadruple in $\mathbb{Z}[\sqrt{d}]$ for $n\in\mathcal{T}$*'. Very recently, in [7] the present authors answered this question for $n\in\mathcal{T}\setminus\{(4m,4k),(4m+2,4k)\}$. More precisely, the authors proved the following result.

Theorem A ([7, Theorem 1.1]). *Assume that $d\equiv 2 \pmod 4$ is a square-free positive integer and the equations* (1.1) *and* (1.2) *are solvable. Then there exist infinity many quadruples in $\mathbb{Z}[\sqrt{d}]$ with the property $D(n)$ when $n\in\{(4m+1)+4k\sqrt{d},(4m+1)+(4k+2)\sqrt{d},(4m+3)+4k\sqrt{d},(4m+3)+(4k+2)\sqrt{d},(4m+2)+(4k+2)\sqrt{d}\}$ with $m,k\in\mathbb{Z}$.*

As a consequence of Theorem A, we were able to construct some counter examples of Conjecture 1.1. Namely, if $d=10$ and $n=26+6\sqrt{10}$ or $d=58$ and $n=18+2\sqrt{58}$, one can easily see that n can not be represented as a difference of two squares in $\mathbb{Z}[\sqrt{d}]$, but there exists a $D(n)$-quadruple in $\mathbb{Z}[\sqrt{d}]$.

In this paper, we consider the above mentioned problem for the remaining values of n. Let $d\equiv 2 \pmod 4$ be a square-free positive integer such that

$$x^2-dy^2=-1 \tag{1.1}$$

and

$$x^2-dy^2=6 \tag{1.2}$$

are solvable in integers. We prove the following results.

Theorem 1.2. *Let $d\equiv 2 \pmod 4$ be a square-free positive integer such that* (1.1) *and* (1.2) *are solvable in integers. Let $n=(4m,4k)$ with $m,k\in\mathbb{Z}$ such that $(m,k)\not\equiv(5,3) \pmod{(6,6)}$. Then there exist infinitely many $D(n)$-quadruples in $\mathbb{Z}[\sqrt{d}]$.*

Theorem 1.3. *Let d be as in Theorem 1.2. Then for $n=(4m+2,4k)$ with $m,k\in\mathbb{Z}$, there exist infinitely many $D(n)$-quadruples in $\mathbb{Z}[\sqrt{d}]$ such that $(m,k)\not\equiv(9,3),(0,0) \pmod{(12,6)}$.*

In 1996, Dujella [10] obtained several two-parameter polynomial families for quadruples with the property $D(n)$. Our proofs use the technique presented in [10].

2. Preliminaries

We begin this section with the following lemma that follows from the definition of $D(n)$-quadruples in $\mathbb{Z}[\sqrt{d}]$.

Lemma 2.1. *Let $\{a_1, a_2, a_3, a_4\}$ be a $D(n)$-quadruple. Then for any non-zero $w \in \mathbb{Z}[\sqrt{d}]$, with a square-free integer d, the set $\{wa_1, wa_2, wa_3, wa_4\}$ is a $D(w^2n)$-quadruple in $\mathbb{Z}[\sqrt{d}]$.*

The next lemma helps us to find the conditions under which the set $\{a, b, a+b+2r, a+4b+4r\}$ forms a $D(n)$-quadruple in $\mathbb{Z}[\sqrt{d}]$ for any $n \in \mathbb{Z}[\sqrt{d}]$.

Lemma 2.2 ([7, Lemma 2.5]). *The set $\{a, b, a+b+2r, a+4b+4r\}$ of non-zero and distinct elements is a $D(n)$-quadruple in $\mathbb{Z}[\sqrt{d}]$ for any $n \in \mathbb{Z}[\sqrt{d}]$, if $ab + n = r^2$ and $3n = \alpha_1\alpha_2$ with $\alpha_1 = a + 2r + \alpha$ and $\alpha_2 = a + 2r - \alpha$, for some $a, b, r, \alpha \in \mathbb{Z}[\sqrt{d}]$.*

The next two lemmas help us to apply Lemma 2.2 in the proofs of Theorems 1.2 and 1.3. Lemma 2.3 is useful for the factorization of $3n$ in $\mathbb{Z}[\sqrt{d}]$, while Lemma 2.4 is useful to verify that the elements thus found are distinct and non-zero.

Lemma 2.3 ([7, Lemma 3.1]). *Let $d \equiv 2 \pmod 4$ be a square-free integer such that (1.1) and (1.2) are solvable in integers. Then in $\mathbb{Z}[\sqrt{d}]$, the following statements hold:*

(i) elements of norm 1 have the form $(6a_1 \pm 1, 6b_1)$ and there are infinitely many of them;
(ii) elements of norm -1 have the form $(6a_1 \pm 3, 6b_1 \pm 1)$ and there are infinitely many such elements;
(iii) $d \equiv 10 \pmod{48}$;
(iv) elements of norm 6 have the form $(12M \pm 4, 6N \pm 1)$ and there are infinitely many such elements;
(v) elements of norm -6 have the form $(12M \pm 2, 6N \pm 1)$ and there are infinitely many such elements;

where a_1, b_1, M and $N \in \mathbb{Z}$.

Lemma 2.4 ([7, Lemma 2.4]). *Assume that $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1 \in \mathbb{Z}$ with $a_1, a_2, b_1 \neq 0$. Then the following system of simultaneous equations*

$$\begin{cases} a_1x^2 + b_1y^2 + c_1x + d_1y + e_1 = 0, \\ a_2xy + b_2x + c_2y + d_2 = 0 \end{cases}$$

has only finitely many solutions in integers.

3. Proof of Theorem 1.2

We first factorize $3n$ by using Lemmas 2.2 and 2.3. We then use this factorization together with Lemma 2.2 to construct Diophantine quadruples of certain forms with the property $D(n)$ under the condition of non-zero and distinctness. Finally these conditions are verified by using Lemma 2.4.

Here, $n = (4m, 4k)$ with $m, k \in \mathbb{Z}$. Thus $3n = 3(4m, 4k) = 6(2m, 2k)$ and we choose $\alpha_1 = 6$ and $\alpha_2 = (2m, 2k)$ (α_1 and α_2 as in Lemma 2.2). Now Lemma 2.2 entails,

$$a + 2r = (m + 3, k). \tag{3.1}$$

We divide the proof into four cases based on the parity of m and k.

Case I: Both m and k are even. Let $a = (6a_1 + 1, 6b_1)$ with $a_1, b_1 \in \mathbb{Z}$ such that $\mathrm{Nm}(a) = 1$. Then by (i) of Lemma 2.3, there exist infinitely many such a's, and (3.1) can be written as

$$r = (m/2 + 1 - 3a_1, k/2 - 3b_1).$$

As both m and k are even, so $r \in \mathbb{Z}[\sqrt{d}]$. We employ these a and r in the equation $ab + n = r^2$ (as in Lemma 2.2) to get:

$$\begin{aligned} b = \big(&(m/2 + 1 - 3a_1)^2 + d(k/2 - 3b_1)^2 - 4m, \\ &2(m/2 + 1 - 3a_1)(k/2 - 3b_1) - 4k\big) \times (6a_1 + 1, -6b_1). \end{aligned}$$

These choices of a, b and r give us infinitely many $D(n)$-quadruples $\{a, b, a + b + 2r, a + 4b + 4r\}$ in $\mathbb{Z}[\sqrt{d}]$. Non-zero and distinctness of these elements can easily be verified by Lemma 2.4.

Case II: m is odd and k is even. As in Case I, we choose $a = 2(6a_1+1, 6b_1)$ with $a_1, b_1 \in \mathbb{Z}$ and $\mathrm{Nm}(a) = 4$. Then (3.1) gives,

$$2r = (m + 1 - 12a_1, k - 12b_1).$$

We write $m = 2m_1 + 1$ and $k = 2k_1$ for some $m_1, k_1 \in \mathbb{Z}$. Then

$$r = (m_1 + 1 - 6a_1, k_1 - 6b_1),$$

which gives

$$\begin{aligned} b = \frac{1}{2}\big(&(m_1 + 1 - 6a_1)^2 + d(k_1 - 6b_1)^2 - 4m, \\ &2(m_1 + 1 - 6a_1)(k_1 - 6b_1) - 4k\big) \times (6a_1 + 1, -6b_1). \end{aligned}$$

We are looking for b satisfying $b \in \mathbb{Z}[\sqrt{d}]$, so that m_1 should be odd and k_1 should be even. These choices of a, b and r provide infinitely many $D(n)$-quadruples of the form $\{a, b, a + b + 2r, a + 4b + 4r\}$ in $\mathbb{Z}[\sqrt{d}]$.

On the other hand for even m_1, we choose $a = 4(6a_1 + 1, 6b_1)$ with $a_1, b_1 \in \mathbb{Z}$ and $\mathrm{Nm}(a) = 16$. Then as before we get

$$r = (m_1 - 12a_1, k_1 - 12b_1),$$

which provides

$$b = \frac{1}{4}\left((m_1 - 12a_1)^2 + d(k_1 - 12b_1)^2 - 4m, 2(m_1 - 12a_1)(k_1 - 12b_1) - 4k\right)$$

$\times (6a_1 + 1, -6b_1)$.

Clearly $b \in \mathbb{Z}[\sqrt{d}]$ when k_1 is even. These give the required elements a, b and r. Utilizing Lemma 2.2, this implies that the set $\mathcal{A} = \{a, b, a + b + 2r, a + 4b + 4r\}$ forms a Diophantine quadruple in $\mathbb{Z}[\sqrt{d}]$ with the property $D(n)$, under the condition that all the elements of $\mathcal{A}$ must be non-zero and distinct from each other. These conditions can be verified by using Lemma 2.4, except $a + 4b + 4r \neq 0$ and $a + 2r \neq 0$. We handle these exceptions separately since they do not fit into Lemma 2.4. We first consider $a + 2r = 0$. This gives $m_1 = -2$ and $k_1 = 0$. This gives $n = -12$. Now if $a + 4b + 4r = 0$, then $(m_1, k_1) = (0, 0)$ or $(m_1, k_1) = (4, 0)$. This gives $n = 1, 36$, which are already known.

The case $n = -12$ gives $3n = -18 \times 2$. We now choose $\alpha_1 = -18$ and $\alpha_2 = 2$. As before, we choose $a = 4(6a_1 + 1, 6b_1)$ with $a_1, b_1 \in \mathbb{Z}$ and $\mathrm{Nm}(a) = 16$, and thus $r = (-2 - 12a_1, -12b_1)$. This gives

$$b = ((1 + 6a_1, 6b_1)^2 + 3)(6a_1 + 1, -6b_1).$$

Owing to the guaranteed existence of infinitely many a's, there exist infinitely many $D(n)$-quadruples.

The possibility of m_1 even and k_1 odd needs to be examined. In this case $n = (16m + 4, 16k + 8) = 2^2(4m + 1, 4k + 2)$, and thus the existence of infinitely many $D(n)$-quadruples in $\mathbb{Z}[\sqrt{d}]$ is guaranteed by [7, Theorem 1.1] and Lemma 2.1.

Case III: m is even and k is odd. In this case, we consider $a = (6a_1 + 3, 6b_1 + 1)$ with $a_1, b_1 \in \mathbb{Z}$ and $\mathrm{Nm}(a) = -1$. This provides us

$$\begin{aligned} b = \big(&(m/2 - 3a_1)^2 + d((k-1)/2 - 3b_1)^2 - 4m, \\ &2(m/2 - 3a_1)((k-1)/2 - 3b_1) - 4k\big) \times (-6a_1 - 3, 6b_1 + 1), \end{aligned}$$

(for the value of r we use (3.1)). As dealt with in the previous cases, these values of a, b, r will guarantee infinitely many $D(n)$- quadruples in $\mathbb{Z}[\sqrt{d}]$.

Case IV: Both m and k are odd. This case is bit more involved. Clearly n can be expressed as $n = (8m_1 + 4, 8k_1 + 4)$ for some $m_1, k_1 \in \mathbb{Z}$. Then

$$3n = 6(4m_1 + 2, 4k_1 + 2).$$

Let $\alpha_1 = 6$ and $\alpha_2 = (4m_1 + 2, 4k_1 + 2)$. That would imply (by Lemma 2.2)

$$a + 2r = (2m_1 + 4, 2k_1 + 1). \tag{3.2}$$

In what follows we will apply Lemma 2.3 (iv), (v), with $M, N \in \mathbb{Z}$. First, set $a = (12M + 4, 6N + 1)$, with $\mathrm{Nm}(a) = 6$. Thus (3.2) implies that

$$r = (m_1 - 6M, k_1 - 3N).$$

Employing $ab+n=r^2$ and $d \equiv 10 \pmod{48}$ (see, (iii) of Lemma 2.3), we get

$$b = \frac{1}{6}\big((m_1-6M)^2 + d(k_1-3N)^2 - 8m_1 - 4,\\ 2(m_1-6M)(k_1-3N) - 8k_1 - 4\big) \times (12M+4, -6N-1).$$

To ensure the existence of b in $\mathbb{Z}[\sqrt{d}]$, we must have,

$$(m_1, k_1) \equiv (0,0), (0,1), (2,0), (2,2), (4,1), (4,2) \pmod{(6,3)}.$$

As before, we assume $a = (12M+4, 6N-1)$, with $\mathrm{Nm}(a) = 6$. Then we arrive at

$$b = \frac{1}{6}\big((m_1-6M)^2 + d(k_1-3N+1)^2 - 8m_1 - 4,\\ 2(m_1-6M)(k_1-3N+1) - 8k_1 - 4\big) \times (12M+4, -6N+1).$$

As $b \in \mathbb{Z}[\sqrt{d}]$, so that we have additional cases of (m_1, k_1), where

$$(m_1, k_1) \equiv (0,2), (4,0) \pmod{(6,3)}.$$

Similarly, we set $a = (12M+2, 6N+1)$ with $\mathrm{Nm}(a) = -6$ to get

$$b = -\frac{1}{6}\big((m_1+1-6M)^2 + d(k_1-3N)^2 - 8m_1 - 4,\\ 2(m_1+1-6M)(k_1-3N) - 8k_1 - 4\big) \times (12M+2, -6N-1).$$

For b to be in $\mathbb{Z}[\sqrt{d}]$,

$$(m_1, k_1) \equiv (1,0), (1,1), (3,2), (5,0), (5,2) \pmod{(6,3)}.$$

Again we choose $a = (12M+2, 6N-1)$, with $\mathrm{Nm}(a) = -6$, which gives

$$b = -\frac{1}{6}\big((m_1-6M+1)^2 + d(k_1-3N+1)^2 - 8m_1 - 4,\\ 2(m_1-6M+1)(k_1-3N+1) - 8k_1 - 4\big) \times (12M+2, -6N+1).$$

Thus for $b \in \mathbb{Z}[\sqrt{d}]$,

$$(m_1, k_1) \equiv (1,2), (3,0), (3,1) \pmod{(6,3)}.$$

Finally for $a = (12M-2, 6N-1)$ one gets the same values for (m_1, k_1) as in the case $a = (12M+2, 6N+1)$. This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

The proof of Theorem 1.3 goes along the lines of that of Theorem 1.2, except the factorization of $3n$. However, we provide the outlines of the proof for convenience to the readers. The notations α_1 and α_2 are as in §3. Assume that $n = (4m+2, 4k)$, where $m, k \in \mathbb{Z}$.

Case I: Both m and k are even. Let $M, N \in \mathbb{Z}$, and let

$$\begin{aligned} 3n &= 6(2m+1, 2k) \\ &= (12M+4, -6N-1)(12M+4, 6N+1)(2m+1, 2k) \\ &\text{(Using Lemma 2.3(iv))} \\ &= \alpha_1\alpha_2, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \alpha_1 &= (12M+4, -6N-1), \\ \alpha_2 &= \big(24Mm + 12M + 8m + 4 + d(12Nk + 2k), \\ &\quad 24Mk + 8k + 12Nm + 2m + 6N + 1\big). \end{aligned}$$

Now, $a = 4(6a_1 + 1, 6b_1)$ with $a_1, b_1 \in \mathbb{Z}$ and $\mathrm{Nm}(a) = 16$, which gives

$$r = \big(6Mm+6M+2m+(d/2)(6Nk+k)-12a_1, 6Mk+2k+3Nm+m/2-12b_1\big)$$

and

$$\begin{aligned} b = \frac{1}{4}\Big(&(6Mm + 6M + 2m + (d/2)(6Nk + k) - 12a_1)^2 \\ &+ d(6Mk + 2k + 3Nm + m/2 - 12b_1)^2 - 4m - 2, \\ &2(6Mm + 6M + 2m + (d/2)(6Nk + k) - 12a_1) \\ &\times (6Mk + 2k + 3Nm + m/2 - 12b_1) - 4k\Big) \\ &\times (6a_1 + 1, -6b_1). \end{aligned}$$

Now for $r, b \in \mathbb{Z}[\sqrt{d}]$, since $d \equiv 2 \pmod 4$, we must have $m \equiv 2 \pmod 4$. Assume that

$$(\alpha, \beta) = \big(6Mm + 6M + 2m + (d/2)(6Nk + k), 6Mk + 2k + 3Nm + m/2\big).$$

Then $r = (\alpha - 12a_1, \beta - 12b_1)$.

Now if $a + 4b + 4r = 0$, then

$$\begin{aligned} 4 + \alpha^2 + d\beta^2 - 4m - 2 + 4\alpha &= 0, \\ 2\alpha\beta - 4k + 4\beta &= 0. \end{aligned}$$

By Lemma 2.4, we conclude that there exist only finitely many α and β which satisfy the above system of equations. We now rewrite α and β as follows,

$$\begin{aligned} \alpha &= 6M(m+1) + N(3dk) + 2m + (d/2)k \\ \beta &= 6Mk + 3Nm + (m/2) + 2k. \end{aligned}$$

These can be written as

$$\begin{pmatrix} \alpha - 2m - (d/2)k \\ \beta - (m/2) - 2k \end{pmatrix} = \begin{pmatrix} 6(m+1) & 3dk \\ 6k & 3m \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix}.$$

Since $m \equiv 2 \pmod 4$, k is even, and $d \equiv 2 \pmod 4$, so that the determinant of

$$\begin{pmatrix} 6(m+1) & 3dk \\ 6k & 3m \end{pmatrix}$$

is non-zero. As we have infinitely many choices for M and N, so that there exist infinitely many α and β for which $a+4b+4r \neq 0$. Hence we can take such M and N for which $a+4b+4r \neq 0$. Using these values of a, b and r, we can get infinitely many quadruples with the property $D(n)$ from Lemma 2.2, since we have infinitely many choices of a, by using Lemma 2.3 (i) and for checking the condition of non-zero and distinct elements of the set $\{a, b, a+b+2r, a+4b+4r\}$ (given in Lemma 2.2), we use Lemma 2.4.

In the case $m \equiv 0 \pmod 4$, we replace n by $n = (16m_1+2, 8k_1)$ and then consider (4.1) with

$$\begin{aligned} \alpha_1 &= (-12M-2, 6N+1), \\ \alpha_2 &= \big(96Mm_1 + 12M + 16m_1 + 2 + d(24Nk_1 + 4k_1), \\ &\qquad 48Mk_1 + 8k_1 + 48Nm_1 + 8m_1 + 6N + 1\big), \end{aligned}$$

where $m_1, k_1 \in \mathbb{Z}$. This gives by utilizing $a = (12a_1+4, 6b_1+1)$ with $a_1, b_1 \in \mathbb{Z}$ and $\mathrm{Nm}(a) = 6$,

$$\begin{aligned} r = \big(&24Mm_1 + 4m_1 + d(6Nk_1 + k_1) - 6a_1 - 2, \\ &12Mk_1 + 2k_1 + 12Nm_1 + 2m_1 + 3N - 3b_1\big) \end{aligned}$$

and

$$\begin{aligned} b = \frac{1}{6}\Big(&(24Mm_1 + 4m_1 + d(6Nk_1 + k_1) - 6a_1 - 2)^2 \\ &+ d(12Mk_1 + 2k_1 + 12Nm_1 + 2m_1 + 3N - 3b_1)^2 - 16m_1 - 2, \\ &2(24Mm_1 + 4m_1 + d(6Nk_1 + k_1) - 6a_1 - 2) \\ &\times (12Mk_1 + 2k_1 + 12Nm_1 + 2m_1 + 3N - 3b_1) - 8k_1\Big) \\ &\times (12a_1 + 4, -6b_1 - 1). \end{aligned}$$

Using $d \equiv 10 \pmod{48}$ (from Lemma 2.3(iii)), these further imply that

$$(m_1, k_1) \equiv (0,1), (0,2), (1,0), (1,1), (2,0), (2,2) \pmod{(3,3)}.$$

Similarly, for $a = (12a_1 - 4, 6b_1 + 1)$ with $a_1, b_1 \in \mathbb{Z}$ and $\mathrm{Nm}(a) = 6$, we have

$$\begin{aligned} r = \big(&24Mm_1 + 4m_1 + d(6Nk_1 + k_1) - 6a_1 + 2, \\ &12Mk_1 + 2k_1 + 12Nm_1 + 2m_1 + 3N - 3b_1\big) \end{aligned}$$

and

$$b = \frac{1}{6}\Big((24Mm_1 + 4m_1 + d(6Nk_1 + k_1) - 6a_1 + 2)^2$$

$$+\, d(12Mk_1 + 2k_1 + 12Nm_1 + 2m_1 + 3N - 3b_1)^2 - 16m_1 - 2,$$
$$2(24Mm_1 + 4m_1 + d(6Nk_1 + k_1) - 6a_1 + 2)$$
$$\times (12Mk_1 + 2k_1 + 12Nm_1 + 2m_1 + 3N - 3b_1) - 8k_1\Big)$$
$$\times (12a_1 - 4, -6b_1 - 1).$$

For b to be in $\mathbb{Z}[\sqrt{d}]$,

$$(m_1, k_1) \equiv (1, 2) \pmod{(3, 3)}.$$

The factorization (4.1) with

$$\alpha_1 = (12M + 2, 6N + 1),$$
$$\alpha_2 = \big(-96Mm_1 - 12M - 16m_1 - 2 + d(24Nk_1 + 4k_1),$$
$$-48Mk_1 - 8k_1 + 48m_1N + 8m_1 + 6N + 1\big),$$

as well as $a = (12a_1 + 4, 6b_1 - 1)$ with $a_1, b_1 \in \mathbb{Z}$ and $\mathrm{Nm}(a) = 6$ provides

$$r = \big(-24Mm_1 - 4m_1 + d(6Nk_1 + k_1) - 6a_1 - 2,$$
$$-12Mk_1 - 2k_1 + 12m_1N + 2m_1 + 3N + 1 - 3b_1\big)$$

and

$$b = \frac{1}{6}\Big((-24Mm_1 - 4m_1 + d(6Nk_1 + k_1) - 6a_1 - 2)^2$$
$$+\, d(-12Mk_1 - 2k_1 + 12m_1N + 2m_1 + 3N + 1 - 3b_1)^2 - 16m_1 - 2,$$
$$2(-24Mm_1 - 4m_1 + d(6Nk_1 + k_1) - 6a_1 - 2)$$
$$\times (-12Mk_1 - 2k_1 + 12m_1N + 2m_1 + 3N + 1 - 3b_1) - 8k_1\Big)$$
$$\times (12a_1 + 4, -6b_1 + 1).$$

For $b \in \mathbb{Z}[\sqrt{d}]$,

$$(m_1, k_1) \equiv (2, 1) \pmod{(3, 3)}.$$

Finally, owing to Lemma 2.3, there are infinitely many choices of M and N, and hence there are infinitely many choices for such a, b and r.

To conclude this case, we have covered all possibilities for (m_1, k_1), except $(m_1, k_1) \not\equiv (0, 0) \pmod{(3, 3)}$. Hence, there exist infinitely many Diophantine quadruples in $\mathbb{Z}[\sqrt{d}]$ with the property $D(16m_1 + 2, 8k_1)$, where $(m_1, k_1) \not\equiv (0, 0) \pmod{(3, 3)}$.

Case II: m is even and k is odd. In this case too we work with the factorization (4.1). We use

$$\alpha_1 = (12M + 4, -6N - 1),$$
$$\alpha_2 = \big(24Mm + 12M + 8m + 4 + d(12Nk + 2k),$$
$$24Mk + 8k + 12Nm + 2m + 6N + 1\big)$$

and $a = 2(6a_1 + 1, 6b_1)$ with $a_1, b_1 \in \mathbb{Z}$ and $\mathrm{Nm}(a) = 4$. These provide us

$$r = \big(6Mm + 6M + 2m + 2 + (d/2)(6Nk + k) - 6a_1 - 1,\\ 6Mk + 2k + 3Nm + (m/2) - 6b_1\big)$$

and

$$b = \frac{1}{2}\Big((6Mm + 6M + 2m + 2 + (d/2)(6Nk + k) - 6a_1 - 1)^2\\ + d(6Mk + 2k + 3Nm + m/2 - 6b_1)^2 - 4m - 2,\\ 2(6Mm + 6M + 2m + 2 + (d/2)(6Nk + k) - 6a_1 - 1)\\ \times (6Mk + 2k + 3Nm + m/2 - 6b_1) - 4k\Big)\\ \times (6a_1 + 1, -6b_1).$$

Case III: m is odd and k is even. Here, we use (4.1) with

$$\alpha_1 = (-12M - 2, 6N + 1),\\ \alpha_2 = \big(24Mm + 12M + 4m + 2 + d(12Nk + 2k),\\ 24Mk + 4k + 12Nm + 2m + 6N + 1\big).$$

Then, $a = 2(6a_1 + 1, 6b_1)$ with $a_1, b_1 \in \mathbb{Z}$ and $\mathrm{Nm}(a) = 4$ gives

$$r = \big(12Mm + 2m + d(6Nk + k), 12Mk + 2k + 6Nm + m + 6N + 1\big)$$

and

$$b = \frac{1}{2}\Big((12Mm + 2m + d(6Nk + k))^2\\ + d(12Mk + 2k + 6Nm + m + 6N + 1)^2 - 4m - 2,\\ 2(12Mm + 2m + d(6Nk + k))\\ \times (12Mk + 2k + 6Nm + m + 6N + 1) - 4k\Big)\\ \times (6a_1 + 1, -6b_1).$$

Case IV: Both m and k are odd. The choices of α_1 and α_2 as in Case III work in this case too. We set $a = 4(6a_1 + 1, 6b_1)$ with $a_1, b_1 \in \mathbb{Z}$ and $\mathrm{Nm}(a) = 16$ to get

$$r = \big(6Mm + m + (d/2)(6Nk + k) - 12a_1 - 2,\\ 6Mk + k + 3Nm + (m + 1)/2 + 3N - 12b_1\big)$$

and

$$b = \frac{1}{4}\Big((6Mm + m + (d/2)(6Nk + k) - 12a_1 - 2)^2\\ + d(6Mk + k + 3Nm + (m + 1)/2 + 3N - 12b_1)^2 - 4m - 2,\\ 2(6Mm + m + (d/2)(6Nk + k) - 12a_1 - 2)$$

$$\times\,(6Mk+k+3Nm+(m+1)/2+3N-12b_1)-4k\Big)$$
$$\times\,(6a_1+1,-6b_1).$$

These would imply $m \equiv 3 \pmod 4$ whenever $r, b \in \mathbb{Z}[\sqrt{d}]$. The existence of infinitely many quadruples can be seen by similar argument of $n = (4m+2, 4k)$ in Case I with $m \equiv 2 \pmod 4$ and even k.

The next case is $m \equiv 1 \pmod 4$ and here n can be replaced by $n = (16m_1+6, 8k_1+4)$ with $m_1, k_1 \in \mathbb{Z}$. The factorization uses in this case is:

$$3n = \alpha_1\alpha_2, \tag{4.2}$$

where,

$$\alpha_1 = (12M+4, -6N-1),$$
$$\alpha_2 = \big(96Mm_1+36M+32m_1+12+d(24Nk_1+12N+4k_1+2),$$
$$48Mk_1+24M+16k_1+11+48Nm_1+18N+8m_1\big).$$

We set $a = (12a_1+2, 6b_1+1)$ with $a_1, b_1 \in \mathbb{Z}$ and $\mathrm{Nm}(a) = -6$, which gives

$$r = \big(24Mm_1+12M+8m_1+4+(d/2)(12Nk_1+6N+2k_1+1)-6a_1-1,$$
$$12Mk_1+6M+4k_1-3a_1+2+12Nm_1+3N+2m_1\big)$$

and

$$b = -\frac{1}{6}\Big((24Mm_1+12M+8m_1+4$$
$$+(d/2)(12Nk_1+6N+2k_1+1)-6a_1-1)^2$$
$$+d(12Mk_1+6M+4k_1-3a_1+2+12Nm_1+3N+2m_1)^2-16m_1-6,$$
$$2(24Mm_1+12M+8m_1+4+(d/2)(12Nk_1+6N+2k_1+1)-6a_1-1)$$
$$\times\,(12Mk_1+6M+4k_1-3a_1+2+12Nm_1+3N+2m_1)-8k_1-4\Big)$$
$$\times\,(12a_1+2,-6b_1-1).$$

Now, using $d \equiv 10 \pmod{48}$ (from Lemma 2.3(iii)),

$$(m_1, k_1) \equiv (0,0), (0,1), (1,1), (2,0), (2,2) \pmod{(3,3)},$$

for $b \in \mathbb{Z}[\sqrt{d}]$.

Similarly, $a = (12a_1-2, 6b_1+1)$ with $a_1, b_1 \in \mathbb{Z}$ and $\mathrm{Nm}(a) = -6$ provides

$$r = \big(24Mm_1+12M+8m_1+5+(d/2)(12Nk_1+6N+2k_1+1)-6a_1,$$
$$12Mk_1+6M+4k_1+2+12Nm_1+3N+2m_1-3b_1\big)$$

and

$$b = -\frac{1}{6}\Big((24Mm_1+12M+8m_1+5+(d/2)(12Nk_1+6N+2k_1+1)-6a_1)^2$$
$$+d(12Mk_1+6M+4k_1+2+12Nm_1+3N+2m_1-3b_1)^2-16m_1-6,$$

$$\begin{aligned}&2(24Mm_1+12M+8m_1+5+(d/2)(12Nk_1+6N+2k_1+1)-6a_1)\\&\times(12Mk_1+6M+4k_1+2+12Nm_1+3N+2m_1-3b_1)-8k_1-4\Big)\\&\times(12a_1-2,-6b_1-1).\end{aligned}$$

For $b\in\mathbb{Z}[\sqrt{d}]$,

$$(m_1,k_1)\equiv(0,2)\pmod{(3,3)}.$$

Again, we use (4.2) by taking

$$\begin{aligned}\alpha_1&=(12M+4,6N+1),\\\alpha_2&=\big(96Mm_1+36M+32m_1+12+d(-24Nk_1-12N-4k_1-2),\\&\qquad 48Mk_1+24M+16k_1+8-48Nm_1-18N-8m_1-3).\end{aligned}$$

Then we choose $a=(12a_1+2,6b_1-1)$ with $a_1,b_1\in\mathbb{Z}$ and $\mathrm{Nm}(a)=-6$, which gives

$$\begin{aligned}r=\big(&24Mm_1+12M+8m_1+3+(d/2)(-12Nk_1-6N-2k_1-1)-6a_1,\\&12Mk_1+6M+4k_1+2-12Nm_1-3N-2m_1-3b_1\big)\end{aligned}$$

and

$$\begin{aligned}b=-\frac{1}{6}\Big(&(24Mm_1+12M+8m_1+3\\&+(d/2)(-12Nk_1-6N-2k_1-1)-6a_1)^2\\&+d(12Mk_1+6M+4k_1+2-12Nm_1-3N-2m_1-3b_1)^2-16m_1-6,\\&2(24Mm_1+12M+8m_1+3+(d/2)(-12Nk_1-6N-2k_1-1)-6a_1)\\&\times(12Mk_1+6M+4k_1+2-12Nm_1-3N-2m_1-3b_1)-8k_1-4\Big)\\&\times(12a_1+2,-6b_1+1).\end{aligned}$$

For $b\in\mathbb{Z}[\sqrt{d}]$,

$$(m_1,k_1)\equiv(1,0)\pmod{(3,3)}.$$

The existence of infinitely many $D(n)$-quadruples in $\mathbb{Z}[\sqrt{d}]$ is guaranteed by the above choices of a,b and r in each case.

To conclude this case, we have covered all possibilities for (m_1,k_1) except $(m_1,k_1)\not\equiv(2,1)\pmod{(3,3)}$. Therefore, there exist infinitely many Diophantine quadruples in $\mathbb{Z}[\sqrt{d}]$ with the property $D(16m_1+6,8k_1+4)$, where $(m_1,k_1)\not\equiv(2,1)\pmod{(3,3)}$.

5. Concluding Remarks

Given a square-free integer $d\equiv 2\pmod 4$, the existence of $D(n)$-quadruples in the ring $\mathbb{Z}[\sqrt{d}]$ for some $n\in\mathbb{Z}[\sqrt{d}]$ has been investigated in [7,17]. We investigate this problem for the remaining values of n. However, our method does not work for a few values of n, i.e., $n \in \{4(12r+5, 6s+3), 4(12r+11, 6s+3), (48r+38, 24s+12), (48r+2, 24s)\}$ with $r, s \in \mathbb{Z}$.

We discuss some examples for the existence of $D(n)$-quadruples in $\mathbb{Z}[\sqrt{d}]$ for these exceptions. We first shorten these exceptions with the help of [7, Theorem 1.1], and then we provide some examples for the remaining cases.

Let $d = 2N$ such that (1.1) and (1.2) are solvable in integers, where $N \in \mathbb{N}$. Assume that $n = 4(12m+5, 6k+3)$ with $m = \alpha N + \beta$ and $k = \alpha_1 N + \beta_1$, where $\alpha, \beta, \alpha_1, \beta_1 \in \mathbb{Z}$. Then $n = 4(12\alpha N + 12\beta + 5, 6\alpha_1 N + 6\beta_1 + 3)$. Utilizing (iii) of Lemma 2.3, we get $2, 3 \nmid N$ and thus we can choose β, β_1 such that $12\beta + 5$ and $6\beta_1 + 3$ are of the form $N\gamma$ and $N\gamma_1$, respectively with odd integers γ and γ_1. Thus $n = 2N(24\alpha + 2\gamma, 12\alpha_1 + 2\gamma_1)$, since $2\gamma, 2\gamma_1 \equiv 2 \pmod 4$, so that $24\alpha + 2\gamma$ and $12\alpha_1 + 2\gamma_1$ are of the form $4t_1 + 2$ for some integer $t_1 \geq 1$.

Again $2N$ is square in $\mathbb{Z}[\sqrt{d}]$, and thus [7, Theorem 1.1] and Lemma 2.1 together show that there exist infinitely many $D(n)$-quadruples in $\mathbb{Z}[\sqrt{d}]$. Analogously, we can draw a similar conclusion for $n = 4(12m + 11, 6k + 3)$. We now consider $n = (4(12m + 9) + 2, 4(6k + 3))$. As in the above, $n = 2(24N\alpha + 24\beta + 19, 12N\alpha_1 + 12\beta_1 + 6)$. Since $2, 3 \nmid N$, so that we can choose β, β_1 such that $24\beta + 19$ and $12\beta_1 + 6$ are of the form $N\gamma$ and $N\gamma_1$, respectively. Using (iii) of Lemma 2.3, we get $N \equiv 1 \pmod 4$, and thus $\gamma \equiv 3 \pmod 4$ and $\gamma_1 \equiv 2 \pmod 4$. Finally we use [7, Theorem 1.1] and Lemma 2.1 to conclude that there exist infinitely many $D(n)$-quadruples in $\mathbb{Z}[\sqrt{d}]$. Analogously we can establish the same for $n = (4(12m) + 2, 4(6k))$.

We now provide some examples supporting the existence of $D(n)$-quadruple in $\mathbb{Z}[\sqrt{10}]$ for the exceptional values of n.

Example 1. We consider $d = 10$ and $n = 4(12m+5, 6k+3)$ with $m, k \in \mathbb{Z}$. Let $m = 5M$ and $k = 5K+2$, where $M, K \in \mathbb{Z}$. Then $n = 4(5(12M+1), 30K+15)$, which can be written as $n = 10(24M+2, 12K+6)$. Thus n is of the form $10(4m' + 2, 4k' + 2)$ with $m', k' \in \mathbb{Z}$. Therefore using [7, Theorem 1.1] and Lemma 2.1, we conclude that there exist infinitely many $D(n)$-quadruples in $\mathbb{Z}[\sqrt{10}]$. Analogously, we can show the same for $n = 4(12m + 11, 6k + 3)$ by putting $m = 5M + 2$ and $k = 5K + 2$.

Example 2. Suppose $d = 10$ and $n = (4(12m + 9) + 2, 4(6k + 3)) = 2(24m+19, 12k+6)$. Let $m = 5M + 4$ and $k = 5K + 2$. Then $n = 10(24M + 23, 12k + 6)$. Since $24M + 23 \equiv 3 \pmod 4$ and $12K + 6 \equiv 2 \pmod 4$, so that by [7, Theorem 1.1] and Lemma 2.1, we can conclude that there exist infinitely many $D(n)$-quadruples in $\mathbb{Z}[\sqrt{10}]$. Similar conclusion can be drawn for $n = (4(12m) + 2, 4(6k))$ by taking $m \equiv 1 \pmod 5$ and $k \equiv 0 \pmod 5$.

Example 3. Assume that $d = 10$ and $n = 4(12m+5, 6k+3)$ with $m \equiv 2, 3 \pmod 5$. We factorize $3n$ as follows:

$$\begin{aligned} 3n &= 12(12m+5, 6k+3) \\ &= (-18, 6)(3, 1)(24m+10, 12k+6) \\ &= (-18, 6)(120k+72m+90,\ 36k+24m+28). \end{aligned}$$

We take α_1 and α_2 to be the first and the second factor of the above equation, respectively. Further utilizing Lemma 2.2 we get

$$a + 2r = (60k + 36m + 36, 18k + 12m + 17).$$

We choose $a = (19, 6)^t(0, 1)$ with $\mathrm{Nm}(a) = -10$, where $t \in \mathbb{N}$. This implies that there exist $\alpha, \beta \in \mathbb{Z}$ such that $a = (20\alpha, 10\beta - 1)$, and thus $r = (30k + 18m - 10\alpha + 18,\ 9k + 6m + 9 - 5\beta)$. Further $ab + n = r^2$ implies

$$b = \frac{(r^2 - n)(20\alpha, -10\beta + 1)}{-10}.$$

Since $m \equiv 2,$ or $3 \pmod 5$, $b \in \mathbb{Z}[\sqrt{10}]$ and we have infinitely many a's, therefore by using Lemmas 2.2 and 2.4, we get infinitely many $D(n)$-quadruples in $\mathbb{Z}[\sqrt{10}]$. Analogously, we can show the existence of infinitely many $D(n)$-quadruples in $\mathbb{Z}[\sqrt{10}]$ for $n = 4(12m+11, 6k+3)$ when $m \equiv 0,$ or $4 \pmod 5$.

Example 4. Suppose that $d = 10$ and $n = (4(12m + 9) + 2,\ 4(6k + 3))$ with $m \equiv 1,$ or $2 \pmod 5$. We factorize

$$\begin{aligned} 3n &= (4, 1)(4, -1)(2(12m + 9) + 1,\ 2(6k + 3)) \\ &= (4, 1)(-120k + 96m + 16,\ 48k - 24m + 5). \end{aligned}$$

We choose α_1 and α_2 to be the first and the second factor of the last equation, respectively. We use Lemma 2.2 to get $a + 2r = (-60k + 48m + 10,\ 24k - 12m + 3)$. Let $a = (19, 6)^t(10, 3)$ with $\mathrm{Nm}(a) = 10$, where $t \in \mathbb{N}$. Thus there exist $\alpha,\ \beta \in \mathbb{Z}$ such that $a = (20\alpha + 10, 10\beta + 3)$ and thus $r = (24m - 30k - 10\alpha, -6m + 12k - 5\beta)$. Therefore Lemma 2.2 gives

$$b = \frac{r^2 - n}{a}.$$

Since $m \equiv 1,$ or $2 \pmod 5$, so that $b \in \mathbb{Z}[\sqrt{10}]$. Hence there exist infinitely many $D(n)$-quadruples in $\mathbb{Z}[\sqrt{10}]$. Analogously, we can construct $D(n)$-quadruples for $n = (4(12m) + 2, 4(6k))$ when $m \equiv 3,$ or $4 \pmod 5$.

The problem of existence of infinitely many $D(n)$-quadruples in $\mathbb{Z}[\sqrt{10}]$ for $n \in \mathbb{Z}[\sqrt{10}]$ is solved, except for $n \in \mathcal{S}_0 := S_1 \cup S_2 \cup S_3 \cup S_4$, where

$$\begin{aligned} S_1 =& \{4(12m + 5, 6k + 3) : (m, k) \equiv (0, 0), (0, 1), (0, 3), (0, 4) \pmod{(5, 5)} \\ & \text{or } m \equiv 1, 4 \pmod 5\}, \\ S_2 =& \{4(12m + 11, 6k + 3) : (m, k) \equiv (2, 0), (2, 1), (2, 3), (2, 4) \pmod{(5, 5)} \end{aligned}$$

$$\begin{aligned}
&\text{or } m \equiv 1,3 \pmod 5\},\\
S_3 =&\{(4(12m+9)+2, 4(6k+3)) : (m,k) \equiv (4,0),(4,1),(4,3),(4,4)\\
&\quad \pmod{(5,5)} \text{ or } m \equiv 0,3 \pmod 5\}, \text{ and}\\
S_4 =&\{48m+2, 36k) : (m,k) \equiv (1,1),(1,2),(1,3),(1,4) \pmod{(5,5)}\\
&\text{or } m \equiv 0,2 \pmod 5\}.
\end{aligned}$$

Finally, we pose the following question for $n \in \mathcal{S}_0$.

QUESTION 5.1. *Do there exist infinitely many $D(n)$-quadruples in $\mathbb{Z}[\sqrt{10}]$ when $n \in S_0$?*

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DIOFANTOVE $D(n)$-ČETVORKE U $\mathbb{Z}[\sqrt{4k+2}]$

Sažetak. Neka je d kvadratno-slobodan cijeli broj i $\mathbb{Z}[\sqrt{d}]$ prsten cijelih kvadratnog polja. Za dani $n \in \mathbb{Z}[\sqrt{d}]$, skup od m različitih ne-nul elemenata iz $\mathbb{Z}[\sqrt{d}]$ zove se Diofantova $D(n)$-m-torka (ili jednostavno $D(n)$-m-torka) u $\mathbb{Z}[\sqrt{d}]$ ako je produkt bilo koja dva od njih uvećan za n kvadrat u $\mathbb{Z}[\sqrt{d}]$. Pretpostavimo da je $d \equiv 2 \pmod 4$ pozitivni cijeli broj takav da su $x^2 - dy^2 = -1$ i $x^2 - dy^2 = 6$ rješive u cijelim brojevima. U ovom članku dokazujemo postojanje beskonačno mnogo $D(n)$-četvorki u $\mathbb{Z}[\sqrt{d}]$ za $n = 4m + 4k\sqrt{d}$, pri čemu $m, k \in \mathbb{Z}$ zadovoljavaju $m \not\equiv 5 \pmod 6$ i $k \not\equiv 3 \pmod 6$. Štoviše, isto dokazujemo i za $n = (4m+2) + 4k\sqrt{d}$ kada je ili $m \not\equiv 9 \pmod{12}$ i $k \not\equiv 3 \pmod 6$, ili $m \not\equiv 0 \pmod{12}$ i $k \not\equiv 0 \pmod 6$. Na kraju dajemo neke primjere koji potvrđuju postojanje četvorki u $\mathbb{Z}[\sqrt{d}]$ sa svojstvom $D(n)$ za gore istaknute n i za $d = 10$.