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POLYNOMIALS VANISHING ON A BASIS OF $S_m(\Gamma_0(N))$

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Abstract. In this paper we compute the bases of homogeneous polynomials of degree d such that they vanish on cuspidal modular forms of even weight $m \geq 4$ that form a basis for $S_m(\Gamma_0(N))$. Among them we find the irreducible ones.

1. Introduction

Let $N > 1$, $m \geq 4$ be an even number and $f_0, \dots, f_{t-1}$ be elements of the basis of the space of cuspidal modular forms $S_m(\Gamma_0(N))$ of weight m with $\dim S_m(\Gamma_0(N)) = t$. Let $X_0(N)$ be the modular curve for $\Gamma_0(N)$. We look at the holomorphic map $X_0(N) \to \mathbb{P}^{t-1}$ defined by

$$\mathfrak{a}_z \mapsto (f_0(z) : \cdots : f_{t-1}(z)) \tag{1.1}$$

and we denote the image curve of this map by

$$\mathcal{C}(N, m) \subseteq \mathbb{P}^{t-1}.$$

Let us set $g = f_{t-1}$. Then the map (1.1) can be written as

$$\mathfrak{a}_z \mapsto (f_0(z)/g(z) : \cdots : f_{t-1}(z)/g(z)) \tag{1.2}$$

and it is a rational map of algebraic curves. Here, we are continuing the work in [18], where it is shown that the complete linear system attached to this map, consisting of integral divisors of degree $t + g - 1$ attached to modular forms f_i, obtained from usual $\mathrm{div}(f_i)$ by subtracting contributions at elliptic points and cusps, satisfy the conditions for the map to be an embedding. Namely the linear system is base point free when $t \geq g + 1$ and very ample

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when $t \geq g+2$. We repeat the following facts about the image curve $\mathcal{C}(N,m)$ from [18].

LEMMA 1.1. *Assume that $m \geq 4$ is even. Let $t = \dim S_m(\Gamma_0(N))$, $f_0, \ldots, f_{t-1}$ be a basis of $S_m(\Gamma_0(N))$, g be the genus of $\Gamma_0(N)$ and we denote by*

$$\mathcal{C}(N,m) = C(f_0, \cdots, f_{t-1})$$

the image of the map (1.1). Then

i) $\mathcal{C}(N,m)$ is an irreducible smooth projective curve in $\mathbb{P}^{t-1}$.
ii) If $m \geq 4$, then $t \geq g+2$ and the degree of the curve is $t+g-1$.

PROOF. The i) part of Lemma 1.1 follows from Chow's theorem, while ii) and its proof can be found in [18, Corollary 3-4a]. □

In our previous work we have used map (1.1) for $t = 3$ to map the modular curve $X_0(N)$ to a projective plane, find its irreducible equation and check conditions for birationality, ([12, 18, 19]).

Maps to higher-dimensional projective spaces generate a projective curve in $\mathbb{P}^{t-1}$, $t \geq 3$. Here curves are no longer defined by just one equation. Our goal is to adapt the algorithms used in [12] and compute all linearly independent homogeneous polynomials of a certain given degree that vanish on the curve $\mathcal{C}(N,m)$. Geometrically, these polynomials define hypersurfaces in $\mathbb{P}^{t-1}$ lying over the curve.

In weight 2, the space of cusp forms $S_2(\Gamma_0(N))$ is isomorphic to holomorphic 1-forms, we have $\dim S_2(\Gamma_0(N)) = g$, divisors of cusp forms defining the map (1.1) make the canonical linear system of the map and (1.1) is a canonical embedding ([11, Chapter IV.5]). In [9] the bases of $S_2(\Gamma_0(N))$ are used to obtain canonical models for modular curves.

Canonical curves and their ideals are well studied ([4, Chapter III],[8, 23]), their ideal is generated by quadrics except when the curve is trigonal or isomorphic to a smooth plane quintic and then at least one cubic generator appears in the minimal generating system of the ideal. In [10] one can find the complete list of trigonal modular curves $X_0(N)$.

For $m > 2$ the space of cusp forms of weight m is bigger than the set of differentials of degree $m/2$, ([15]). But the complete linear system of integral divisors attached to cusp forms consists of special divisors ([18]), so by results from [3] the ideal of our image curve is generated by quadrics.

In Section 2 we present the algorithm to compute homogeneous polynomials vanishing on $\mathcal{C}(N,m)$ and in Section 3 we present the results of computations and some examples.

2. Computing homogeneous polynomials vanishing on cusp forms

Let $\mathcal{P} = \mathbb{Q}[X_0, \ldots, X_{t-1}]$ be the ring of polynomials in t variables and $\mathcal{P}_d = \mathbb{Q}[X_0, \ldots, X_{t-1}]_d$ the subring of homogeneous polynomials of degree d. We regard $\mathcal{P}$ as the graded ring $\mathcal{P} = \bigoplus_{d\geq 0} \mathcal{P}_d$.

Let $I(\mathcal{C}(N,m)) \subseteq \mathcal{P}$ be the homogenous ideal of the curve $\mathcal{C}(N,m)$ consisting of all homogenous polynomials that vanish on $\mathcal{C}(N,m)$. Then $f \in I(\mathcal{C}(N,m))$ defines a hypersurface $\mathcal{C}(N,m) \subset V(f)$ in $\mathbb{P}^{t-1}$. There is a graded structure on the ideal $I(\mathcal{C}(N,m))$

$$I(\mathcal{C}(N,m)) = \bigoplus_{d\geq 0} I(\mathcal{C}(N,m))_d.$$

If we set

$$I(\mathcal{C}(N,m))_d = \mathcal{P}_d \cap I(\mathcal{C}(N,m)) \tag{2.1}$$

we get the vector space $I(\mathcal{C}(N,m))_d$ of all homogenous polynomials of degree d which vanish on $\mathcal{C}(N,m)$. Product of two homogeneous polynomials of degrees d_1 and d_2 is again a homogeneous polynomial of degree $d_1 + d_2$. We can view this graded structure as vector spaces or modules,

$$\mathcal{P}_j I(\mathcal{C}(N,m))_d \subseteq I(\mathcal{C}(N,m))_{jd}.$$

Let $f_0, \ldots, f_{t-1} \in S_m(\Gamma_0(N))$ be a basis of the space of cuspidal modular forms for the congruence subgroup $\Gamma_0(N)$ of weight $m \geq 4$.

Let $P \in \mathbb{Q}[x_0, \ldots, x_{t-1}]$ be a homogeneous polynomial of degree d

$$P(x_0, \ldots, x_{t-1}) = \sum_{\substack{0\leq i_0,\ldots,i_{t-1}\leq d\\ i_0+\cdots+i_{t-1}=d}} a_{i_0,\ldots,i_{t-1}} x_0^{i_0} \cdots x_{t-1}^{i_{t-1}}.$$

For a given degree $d \geq 0$, we are interested in those polynomials which vanish on the elements of the basis $f_0, \ldots, f_{t-1}$,

$$P(f_0(z), \cdots, f_{t-1}(z)) = \sum_{\substack{0\leq i_0,\ldots,i_{t-1}\leq d\\ i_0+\cdots+i_{t-1}=d}} a_{i_0,\ldots,i_{t-1}} f_0^{i_0} \cdots f_{t-1}^{i_{t-1}} = 0 \tag{2.2}$$

for all $\mathfrak{a}_z \in X_0(N)$.

Vector space $\mathcal{P}_d$ of all homogeneous polynomials of degree d is generated with monomials and its dimension can be viewed as the number of coefficients $a_{i_0,\ldots,i_{t-1}}$ with respect to the indexing set of the set of monomials of degree d,

$$I = \{(i_0, \ldots, i_{t-1}) : 0 \leq i_0, \ldots, i_{t-1} \leq d, i_0 + \cdots + i_{t-1} = d\}.$$

Determination of the cardinality of $|I|$ is known as the weak composition problem in combinatorics and the solution is

$$d' = \dim \mathcal{P}_d = |I| = \binom{d+t-1}{d}. \tag{2.3}$$

We will order I using the lexicographical ordering ([7]), so that we consider a polynomial P as a finite linear array of its coefficients

$$P \longrightarrow (a_0, \dots, a_{d'-1}) \tag{2.4}$$

satisfying the order of corresponding monomials, as basis representation of P.

We are interested in subspaces $I(\mathcal{C}(N,m))_d \subseteq \mathcal{P}_d$ containing polynomials that vanish on the basis $f_0, \dots, f_{t-1}$ of $S_m(\Gamma_0(N))$ for certain choices of d, N, m and their dimensions,

$$I(\mathcal{C}(N,m))_d = \{P \in \mathcal{P}_d : \, P(f_0, \cdots, f_{t-1}) = 0\}\,. \tag{2.5}$$

Each modular form is in practical computations given by finitely many coefficients of its integral Fourier expansion in the cusp ∞.

The polynomial combination $P(f_0, \dots, f_{t-1})$ is again a modular form of weight md, where d is the degree of the polynomial P, since cuspidal forms on a given group also make a graded ring $S(\Gamma_0(N)) = \oplus_m S_m(\Gamma_0(N))$.

The condition of vanishing of the modular form $P(f_0, \dots, f_{t-1})$ is known as the Sturm bound saying that we only consider a finite number B of coefficients of the q-expansion of the form to distinguish forms,

$$B_m = \left\lfloor \frac{m\,[SL_2(\mathbb{Z}) : \Gamma_0(N)]}{12} \right\rfloor . \tag{2.6}$$

Similar to [9, 12, 15, 19], the algorithm for computing polynomials vanishing on a basis of $S_m(\Gamma_0(N))$ is based on the following linear algebra considerations: for fixed values of d, N, m we are solving a homogeneous system of equations, where the unknowns are coefficients $a_0, \dots, a_{d'-1}$ of a polynomial P, as in (2.4) and the coefficients of the system are values of q-expansions of evaluated monomials $f_0^{i_0} \dots f_t^{i_{t-1}}$ over the indexing set I,

$$\begin{aligned} P(f_0, \dots, f_{t-1}) &= \sum_{\substack{0 \le i_0, \dots, i_{t-1} \le d \\ i_0 + \cdots + i_{t-1} = d}} a_{i_0, \dots, i_{t-1}} f_0^{i_0} \cdots f_{t-1}^{i_{t-1}} \\ &= \sum_{\substack{0 \le i_0, \dots, i_{t-1} \le d \\ i_0 + \cdots + i_{t-1} = d}} a_{i_0, \dots, i_{t-1}} \left(a_0^{(i_0, \dots, i_{t-1})} + a_1^{(i_0, \dots, i_{t-1})} q + \dots \right) \\ &= p_0 + p_1 q + p_2 q^2 + \cdots . \end{aligned}$$

The homogeneous system is $p_0 = p_1 = \cdots = p_{B_{md}} = 0$ and its solutions are obtained as the basis of the right kernel of the transpose of $d' \times B_{md}$ matrix whose rows are made of coefficients of $f_0^{i_0} \dots f_{t-1}^{i_{t-1}}$, after ordering the index set I.

Here is the algorithm, for a given N and weight m, with the use of lexicographic ordering on the set of monomials of degree d:

Input: q-expansions of $f_0, \dots, f_{t-1}$ basis of $S_m(\Gamma_0(N))$.

- For a degree $d \ge 0$:

- for each monomial index $(i_0, \dots, i_{t-1}) \in I$ in the ordered set of monomials of degree d:
  compute $f_0^{i_0} \dots f_{t-1}^{i_{t-1}}$,
- create a $d' \times B_{md}$ matrix A, whose rows are first B_{md} coefficients of q-expansion of $f_0^{i_0} \dots f_{t-1}^{i_{t-1}}$,
- return elements of the right kernel of A.

Output: linearly independent homogeneous polynomials of degree $d \geq 0$ vanishing on all forms, i.e. such that $P(f_0, \dots, f_{t-1}) = 0$.

In our computations we are using the SAGE software system [22] and the cusp form basis we are using is generated by command

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CuspForms(Gamma_0(N),m).q_integral_basis(prec).
```

3. Results

Let $t = \dim S_m(\Gamma_0(N))$, g be the genus of $\Gamma_0(N)$. The formula for t is derived from Riemann-Roch theorem, ([14, Proposition 6.1])

$$t = \dim S_m(\Gamma_0(N)) = (m-1)(g-1) + \left(\frac{m}{2} - 1\right) c_0 + \mu_{0,2} \left\lfloor \frac{m}{4} \right\rfloor \mu_{0,3} \left\lfloor \frac{m}{3} \right\rfloor \tag{3.1}$$

for even $m \geq 4$ where c_0 is the number of inequivalent cusps and $\mu_{0,i}$ is the number of inequivalent elliptic points of order i of $\Gamma_0(N)$.

t	g	(N,m)
2	0	(2,12), (2,14), (3,10), (4,8)
	1	(11,4)
3	0	(2,16), (2,18), (3,12), (3,14), (4,10), (5,8), (5,10), (6,6), (7,6), (7,8), (8,6), (9,6), (10,4), (12,4), (13,4), (16,4)
4	0	(2,20), (2,22), (3,16), (4,12)
	1	(14,4), (15,4), $(17,4)^*$, $(19,4)^*$, (11,6)
5	0	(2,24), (2,26), (3,18), (3,20), (4,14), (5,12), $(5,14)^*$, (6,8), (7,10), (8,8), (9,8), $(10,6)^*$, (13,6), (18,4), (25,4)
	2	(23,4)
6	0	(2,28),(2,30), (3,22), (4,16)
	1	(11,8), (17,6), $(20,4)^{**}$, $(21,4)^{***}$, $(27,4)^*$
7	0	(2,32), (2,34), (3,24), (3,26), (4,18), (5,16), (5,18) (6,10), (7,12), $(7,14)^{**}$, (8,10), (9,10), (12,6), (13,8), $(16,6)^*$
	2	(22,4), $(29,4)^*$, (31,4)
8	0	(2,36), (2,38), (3,28), (4,20)
	1	(11,10), $(14,6)^*$, (15,6), (19,6), (24,4), $(32,4)^\dagger$

Table 1: (N, m) for $2 \leq \dim S_m(\Gamma_0(N)) \leq 8$

REMARK 3.1. For the ordered pairs denoted with asterisk $(N,m)^*$, $(N,m)^{**}$, $(N,m)^{***}$ the number of irreducible polynomials differs from other in the group (Table 8), and for $(N,m)^\dagger$ no computation could be made.

Using the algorithm in Section 2 we were able to compute homogeneous polynomials that vanish on all elements of basis of $S_m(\Gamma_0(N))$, and the irreducible ones among them, for small degrees d up to 10 or at times lower due to the limitations of calculations on huge numbers. For $g \geq 0$ of the modular curve $X_0(N)$ we will denote possible cases for maps (1.1) defined by basis of $S_m(\Gamma_0(N))$ by listing ordered pairs (N,m) in Table 1.

t	g	degree d of P: 2	3	4	5	6	7	8	9	10
2	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0
3	0	1	3	6	10	15	21	28	36	45
4	0	3	10	22	40	65	98	140	192	255
	1	2	8	19	36	60	92	133	184	246
5	0	6	22	53	105	185	301	462	678	960
	2	4	18	47	97	175	289	448	662	942
6	0	10	40	105	226	431	756	1246	1956	2952
	1	9	38	102	222	426	750	1239	1948	2943
7	0	15	65	185	431	887	1673	2954	4950	7947
	2	13	61	179	423	877	1661	2940	4934	7929
8	0	21	98	301	756	1673	3382	6378	11376	19377
	1	20	96	298	752	1668	3376	6371	11368	19368

Table 2: Number of polynomials for $2 \leq t \leq 8$ and $2 \leq d \leq 10$

REMARK 3.2. The blue colored numbers present the assumed numbers but they are not calculated due to the computational limitation.

REMARK 3.3. There are no homogeneous polynomial of degree $d \in \{0,1\}$ that vanish on all elements of the basis of $S_m(\Gamma_0(N))$, therefore they are omitted in the table.

PROPOSITION 3.4. *In Table 2 for* $2 \leq t \leq 8$ *are given the numbers of all linearly independent homogeneous polynomials of degree* $2 \leq d \leq 10$ *that vanish on the basis of* $S_m(\Gamma_0(N))$, $P(f_0, f_1, \cdots, f_{t-1}) = 0$.

3.1. *Case* $t = 3$. Since $X_0(N)$ is mapped by (1.1) to $\mathbb{P}^2$ its image is a planar curve, given by one irreducible equation. The degree of this equation is the degree of the curve and for all higher degrees we can find more than

$(16,4)$	$\mathbf{d{=}2}:\ p_2 = xz - y^2,$
	$\mathbf{d{=}3}:\ xp_2, yp_2, zp_2,$
	$\mathbf{d{=}4}:\ x^2p_2, z^2p_2, xyp_2, xzp_2, yzp_2, (xz+y^2)p_2$
$(13,4)$	$\mathbf{d{=}2}:\ q_2 = xz - y^2 + yz - 3z^2,$
	$\mathbf{d{=}3}:\ xq_2, yq_2, (x+y+3z)q_2$
	$\mathbf{d{=}4}:\ x^2q_2, y^2q_2, xyq_2,$ $x(x+y+3z)q_2,\ y(x+y+3z)q_2,$ $(x^2+(2y+3z)x-2y^2+3yz+9z^2)q_2$

Table 3: Basis of $I(\mathcal{C}(N,m))_d$ for $X_0(N)$ with $g=0$ and $t=3$

one polynomial vanishing on the curve. These higher degree polynomials are reducible, because they have the defining polynomial as a factor.

The numbers appearing in Proposition 3.4 for $t=3$ are the initial part of the integer sequence called triangular numbers A000217, [20]. They also appear in the usual genus-degree formula for curves ([2, Theorem 2.1]). This happens because to raise the degree we multiply a polynomial with a monomial.

The formula relating the degree d of the image curve $\mathcal{C}(N,m)$ and the degree $d(f_0,f_1,f_2)$ of the map (1.1) is ([12,19])

$$d \cdot d(f_0,f_1,f_2) = \dim S_m(\Gamma_0(N)) + g(\Gamma_0(N)) - 1 - \epsilon_m, \tag{3.2}$$

where $\epsilon_2 = 1$ and $\epsilon_m = 0$ for $m \geq 4$ is the number of possible common zeroes of the basis cusp forms. Given $t=3$, the right-hand side of (3.2) can attain values $3+0-1-0=2$ for $g=0$ and even $m \geq 4$. Since we have computed irreducible equation for $\mathcal{C}(N,m)$ of that exact degree we can conclude that the map is birational.

Corollary 3.5. *Assume that* $\dim S_m(\Gamma_0(N)) = 3$ *and let* $\{f_0,f_1,f_2\}$ *be the basis of* $S_m(\Gamma_0(N))$. *Then the map* $X_0(N) \to \mathbb{P}^2$ *given by*

$$\mathfrak{a}_z \mapsto (f_0(z) : f_1(z) : f_2(z))$$

is birational equivalence of $X_0(N)$ *and the image curve* $C(f_0,f_1,f_2)$ *is a conic if* $g(X_0(N)) = 0$.

3.2. *Number of computed polynomials.* The numbers for $g=0$, $3 \leq t \leq 8$ in Table 2 appear to be diagonals of the number sequence A124326 from OEIS database [20] written as a triangle of numbers. This sequence of numbers can be obtained as a difference of Pascal triangle A00731 and rascal triangle

A077028 omitting zeros and satisfies the formula

$$T(m,n) = \binom{m}{n} - (1 + n(m-n)). \tag{3.3}$$

The table 2 is filled with the assumed blue numbers $T(m,n)$ which could not be computed by the algorithm.

We can deduce the following result.

LEMMA 3.6. *The numbers in Table 2 of all linearly independent homogeneous polynomials that vanish on the basis of $S_m(\Gamma_0(N))$, for $3 \leq t \leq 8$ and $3 \leq d \leq 10$ can be obtained as:*

i) first six diagonals of the number sequence A124326 written as a triangle, for $g = 0$,

ii) number of polynomials of same degree as genus 0 subtracted by $g(d-1)$, for $g = 1, 2$.

This is in accordance with what is known for the dimensions of ideals of projective curves. For $d \geq 0$ the Hilbert function ([13, Chapter 5]) of the curve $\mathcal{C}(N,m)$ is the Hilbert function of its coordinate ring:

$$HF_{\mathcal{C}(N,m)}(d) = HF_{\mathcal{P}/I(\mathcal{C}(N,m))}(d) = \dim \mathcal{P}_d - \dim I_d. \tag{3.4}$$

For the polynomial ring $\mathcal{P}$ we have

$$HF_{\mathcal{P}}(d) = \dim \mathcal{P}_d = \binom{t+d-1}{d}. \tag{3.5}$$

By Hilbert-Serre theorem ([11, Theorem 7.5]) for a projective curve there is a unique linear polynomial such that for $d \gg 0$

$$HP_{\mathcal{C}(N,m)}(d) = HF_{\mathcal{C}(N,m)}(d).$$

But the condition $\gg$ here is excessive. Bounds for the regularity index of the Hilbert function, minimal index from which it coincides with this linear polynomial are known ([6, Proposition 4.2.12], [5, 21]) and they show that the two functions coincide for d close to zero. We have used CoCoA System ([1]) to compute the Hilbert polynomial of ideals generated by polynomials of degree 2 and 3 we have computed and the numbers coincide for $d \geq 2$. This linear polynomial has known form

$$HP_{\mathcal{C}(N,m)}(d) = \deg \mathcal{C}(N,m) \cdot d + 1 - g.$$

THEOREM 3.7.

$$\dim(I(\mathcal{C}(N,m))_d) = \binom{t+d-1}{d} - (t+g-1)d - 1 + g. \tag{3.6}$$

PROOF. From (3.3) for $n = d$ and $m - n = t - 1$ and (3.5) we obtain the formula (3.6) for the case $g = 0$ in which the linear polynomial $HP_{\mathcal{C}(N,m)}(d)$ appears. For $g = 1, 2$ we use Lemma 3.6[ii] to get (3.6). ∎

We give examples of computed polynomials in Tables 4, 5, 6 and 7.

d=2 : $p_2 = y^2 + 40wy - xz - 20z^2,\ q_2 = xw - zy + 20zw,$

$r_2 = xz - y^2 - 20wy + 800w^2$

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d=3 : $xp_2,\ xq_2,\ xr_2,\ yp_2,\ yq_2,\ yr_2,$

$p_3 = (x - 60z)y^2 + 80xwy - x^2z + 400z^3,$

$q_3 = y^3 + 40wy^2 - 2xzy + x^2w - 400z^2w$

$r_3 = (x - 40z)y^2 + 60xwy - x^2z + 16000zw^2,$

$s_3 = y^3 + 60wy^2 - 3xzy + 2x^2w - 32000w^3$

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d=4, $x^2p_2,\ x^2q_2,\ x^2r_2,\ xyp_2,\ xyq_2,\ xyr_2,\ y^2p_2,\ y^2q_2,\ y^2r_2,$

$xp_3,\ xq_3,\ xr_3,\ xs_3,\ yp_3,\ yq_3,\ yr_3,\ ys_3,$

$60y^4 + 2400wy^3 + (x^2 - 160xz)y^2 + 120x^2wy - x^3z - 8000z^4,$

$(2x - 60z)y^3 + 120xwy^2 - 3x^2zy + x^3w + 8000z^3w,$

$40y^4 + 1600wy^3 - (120xz - x^2)y^2 + 100x^2wy - x^3z - 320000z^2w^2,$

$(3x - 80z)y^3 + 180xwy^2 - 5x^2zy + 2x^3w + 640000zw^3,$

$30y^4 + 1000wy^3 - (90xz - x^2)y^2 + 80x^2wy - x^3z - 12800000w^4$

Table 4: Basis of $I(\mathcal{C}(4,12))_d$

d=2 : $p_2 = 2y^2 - (z + 3w)y - 2xz + 4z^2 + xw - 4zw,$

$q_2 = y^2 - (z + w)y - xz + 2z^2 + xw - w^2$

---

d=3 : $xp_2,\ xq_2,\ yp_2,\ yq_2,$

$p_3 = 12y^3 + (6x - 4z - 20w)y^2 + (12z^2 - 13xz + xw)y - 6x^2z$

$+4xz^2 + 16z^3 + x^2w,$

$q_3 = 18y^3 + (4x - 15z - 29w)y^2 + (28z^2 - 18xz + 7xw)y - 4x^2z$

$+8xz^2 + 16z^2w,$

$r_3 = 16y^3 + (2x - 14z - 26w)y^2 + (28z^2 - 15xz + 9xw)y - 2x^2z$

$+4xz^2 - x^2w + 8zw^2,$

$s_3 = 30y^3 - (25z + 51w)y^2 + (52z^2 - 24xz + 19xw)y - 6x^2w + 8w^3$

---

d=4 : $x^2p_2,\ x^2q_2,\ y^2p_2,\ y^2q_2,\ xyp_2,\ xyq_2,\ xp_3,\ xq_3,\ xr_3,\ xs_3,$

$yp_3,\ yq_3,\ yr_3,\ ys_3,$

$38y^4 + (14x - 40z - 60w)y^3 + (2x^2 - 53xz + 66z^2 + 3xw)y^2+$

$(36xz^2 - 14x^2z + 3x^2w)y - 2x^3z + 12x^2z^2 - 32z^4,$

$142y^4 + (52x - 129z - 227w)y^3 + (12x^2 - 166xz + 260z^2 + 17xw)y^2$

$-(54x^2z - 96xz^2 - 2x^2w)y - 12x^3z + 24x^2z^2 + 2x^3w - 64z^3w,$

$68y^4 + (18x - 59z - 109w)y^3 + (4x^2 - 73xz + 122z^2 + 20xw)y^2$

$-(19x^2z - 32xz^2 + 2x^2w)y - 4x^3z + 8x^2z^2 + x^3w - 16z^2w^2,$

$258y^4 + (40x - 223z - 413w)y^3 + (12x^2 - 252xz + 460z^2 + 121xw)y^2$

$-(46x^2z - 72xz^2 + 24x^2w)y - 12x^3z + 24x^2z^2 + 6x^3w - 32zw^3,$

$122y^4 + (6x - 106z - 194w)y^3 + (6x^2 - 109xz + 218z^2 + 81xw)y^2$

$-(12x^2z - 12xz^2 + 19x^2w)y - 6x^3z + 12x^2z^2 + 6x^3w - 8w^4$

Table 5: Basis of $I(\mathcal{C}(15,4))_d$

d=2 : $p_2 = y^2 - wy - xz + z^2,\ q_2 = (2u - z)y + xw - zw,$
$r_2 = wy - xu + zu,\ s_2 = y^2 - 2wy - xz - w^2 + 2xu,$
$t_2 = (u - z)y + xw + wu,\ u_2 = y^2 - wy - xz + xu + 2u^2$
d=3 : $xp_2,\ xq_2,\ xr_2,\ xs_2,\ xt_2,\ xu_2,\ yp_2,\ yq_2,\ yr_2,\ ys_2,\ yt_2,\ yu_2,$
$(2u - x - 2z)y^2 + 2xwy + x^2z - z^3,\ y^3 - 3wy^2 + (4xu - 2xz)y + x^2w - z^2w,$
$(2u - z)y^2 + 2xwy - x^2u + z^2u,\ (x + 3z - 4u)y^2 - 5xwy - x^2z - zw^2 + 2x^2u,$
$y^3 - 2wy^2 + (3xu - 2xz)y + x^2w + zwu,$
$(x + 2z - 2u)y^2 - 3xwy - x^2z + x^2u + 2zu^2,$
$2y^3 - 5wy^2 + (8xu - 5xz)y + 3x^2w + w^3,$
$(x + 2z - 3u)y^2 - 4xwy - x^2z + 2x^2u + w^2u,$
$3wy^2 - y^3 + (3xz - 5xu)y - 2x^2w + 2wu^2,$
$(x + 2z - 4u)y^2 - 5xwy - x^2z + 3x^2u - 4u^3$

Table 6: Basis of $I(\mathcal{C}(25,4))_d$

d=2 : $p_2 = y^2 - 2uy - xz - z^2 + 3zw + xu + 2zu,$
$q_2 = y^2 + (4z - 3w)y - xz + 4z^2 - 4xw - 9zw + 6w^2 + 3xu,$
$r_2 = y^2 + (4z - 2w - 2u)y - xz + 3z^2 - 4xw - 5zw + 3xu + 4wu,$
$s_2 = 5y^2 + (8z - 14u)y - 5xz + 5z^2 - 8xw - zw + 7xu + 8u^2$
d=3 : $xp_2,\ xq_2,\ xr_2,\ xs_2,\ yp_2,\ yq_2,\ yr_2,\ ys_2,$
$(23x - 20z + 72u)y^2 - 36y^3 + (32xz + 4z^2 - 30xw - 162zw - 78xu)y$
$-23x^2z - 11xz^2 - 38z^3 + 4x^2w + 185xzw + 33x^2u,$
$(11x - 36z + 84u)y^2 - 42y^3 + (50xz + 30z^2 - 16xw - 170zw - 72xu)y$
$-11x^2z + 3xz^2 - 8x^2w + 105xzw - 76z^2w + 29x^2u,$
$22y^3 + (8z - 51x - 44u)y^2 + (10xz + 6z^2 + 12xw + 42zw + 92xu)y + 51x^2z$
$-7xz^2 - 32x^2w - 169xzw - 17x^2u + 152z^2u,$
$(29x - 240z + 180u)y^2 - 90y^3 + (194xz - 66z^2 + 96xw - 6zw - 252xu)y$
$-29x^2z + 77xz^2 - 104x^2w + 111xzw - 456zw^2 + 111x^2u,$
$70y^3 + (136z - 31x - 140u)y^2 + (102z^2 - 134xz - 100xw - 46zw + 196xu)y$
$+31x^2z - 43xz^2 + 64x^2w - 61xzw - 61x^2u + 304zwu,$
$214y^3 + (216z - 47x - 428u)y^2 + (86z^2 - 262xz - 284xw + 146zw + 356xu)y$
$+47x^2z - 227xz^2 + 48x^2w + 339xzw + 35x^2u + 608zu^2,$
$(35x - 2208z + 228w - 396u)y^2 + (926xz - 2226z^2 + 2388xw + 4254zw - 1908xu)y$
$-30y^3 - 35x^2z + 431xz^2 - 896x^2w - 39xzw - 2736w^3 + 417x^2u,$
$(15x + 2072z - 456w + 156u)y^2 + (2162z^2 - 602xz - 2516xw - 3714zw + 1788xu)y$
$74y^3 - 15x^2z - 141xz^2 + 528x^2w - 147xzw - 147x^2u + 1824w^2u,$
$(37x + 1848z - 304w - 284u)y^2 + (-350xz + 1918z^2 - 2396xw - 3142zw + 1492xu)y$
$142y^3 - 37x^2z + 17xz^2 + 208x^2w - 241xzw + 63x^2u + 1216wu^2,$
$(517x + 4312z - 3196u)y^2 + (3918z^2 - 1374xz + 3918z^2 - 5692xw - 4646zw$
$+2772xu)y + 1294y^3 - 517x^2z - 239xz^2 + 80x^2w + 1135xzw + 831x^2u + 2432u^3$

Table 7: Basis of $I(\mathcal{C}(23,4))_d$

3.3. *Irreducibility.* For the computed polynomials we check irreducibility by standard argument.

Lemma 3.8. *If $P(X_0, \dots, X_n) \in \mathbb{C}[X_0, \cdots, X_n]$ is irreducible as an univariate polynomial $P(X_i) \in \mathbb{C}[X_0, \cdots X_{i-1}, X_{i+1}, \cdots, X_n][X_i]$ then P is irreducible.*

t	g	degree d of P								
		2	3	4	5	6	7	8	9	10
3	0	1	0	0	0	0	0	0	0	0
4	0	3	4	5	6	7	8	9	10	11
	1	2	4	5	6	7	8	9	10	11
	1*	2	3	5	6	7	8	9	10	11
5	0	6	10	15	21	28	36	45	55	66
	0*	6	12	23	39	61	90	127	173	229
	2	4	10	15	21	28	36	45	55	66
6	0	10	20	35	56	84	120	165	220	286
	1	9	20	35	56	84	120	165	220	286
	1*	9	17	35	56	84	120	165	220	286
	1**	9	20	44	82	139	214	324	454	
	1***	9	25	55	107	187	303	464	680	
7	0	15	35	70	126	210	330	495	715	1001
	0*	15	38	82	182	322	552	877		
	0**	15	39	89	180	334				
	2	13	35	70	126	210	330	495	715	1001
	2*	13	39	96	205	394	699			
8	0	21	56	126	252	462	792	1287	2002	3003
	1	20	56	126	252	462	792	1287	2002	3003
	1*	20	56	131						

Table 8: Number of irreducible polynomials for $3 \leq t \leq 8$

Proposition 3.9. *In Table 8 for $3 \leq t \leq 8$ we give the numbers of computed irreducible polynomials of degree $2 \leq d \leq 10$ among all linearly independent homogeneous polynomials that vanish on the basis of $S_m(\Gamma_0(N))$, $P(f_0, f_1, \cdots, f_{t-1}) = 0$.*

Proposition 3.10. *Let $2 \leq t \leq 8$.*

i) There are $\dfrac{t(t-3)}{2} - g + 1$ homogeneous polynomials of degree 2 vanishing on the basis of $S_m(\Gamma_0(N))$ and all are irreducible.

ii) For $d \geq 3$ the number of linearly independent homogeneous polynomials of degree d vanishing on the basis of $S_m(\Gamma_0(N))$ is greater than the number of irreducible polynomials of degree d.

If the ordered pairs denoted with asterisk (see Table 1) are omitted from the Table 8 then we can deduce the following conjecture

CONJECTURE 3.11. *For $t \geq 4$ and $d \geq 3$ the number of irreducible polynomials of degree d is $\binom{d+1}{t-3}$.*

Specially for $t = 5$ we have triangular numbers A00217, for $t = 6$ tetrahedral numbers A000292, for $t = 7$ binomial coefficient $C(n, 4)$ A000332, $t = 8$ binomial coefficient $C(n, 5)$ A000389, [20].

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POLINOMI KOJI IŠČEZAVAJU NA BAZI ZA $S_m(\Gamma_0(N))$

Sažetak. U ovom radu računamo baze homogenih polinoma stupnja d koji iščezavaju na kuspidalnim modularnim formama parne težine $m \geq 4$ koje čine bazu za $S_m(\Gamma_0(N))$. Među njima nalazimo i ireducibilne.