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REAL HYPERSURFACES WITH SEMI-PARALLEL NORMAL JACOBI OPERATOR IN THE REAL GRASSMANNIANS OF RANK TWO

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Abstract. In this paper, we introduce the notion of a semi-parallel normal Jacobi operator for a real hypersurface in the real Grassmannian of rank two, denoted by $\mathbb{Q}^m(\varepsilon)$, where $\varepsilon = \pm 1$. Here, $\mathbb{Q}^m(\varepsilon)$ represents the complex quadric $\mathbb{Q}^m(1) = SO_{m+2}/SO_mSO_2$ for $\varepsilon = 1$ and $\mathbb{Q}^m(-1) = SO^0_{m,2}/SO_mSO_2$ for $\varepsilon = -1$, respectively. In general, the notion of semi-parallel is weaker than the notion of parallel normal Jacobi operator. In this paper we prove that the unit normal vector field of a Hopf real hypersurface in $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$, with semi-parallel normal Jacobi operator is singular. Moreover, the singularity of the normal vector field gives a nonexistence result for Hopf real hypersurfaces in $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$, admitting a semi-parallel normal Jacobi operator.

1. Introduction

As one of typical examples of real Grassmannians of rank two, we can consider the complex quadric Q^m, which is a complex hypersurface in the complex projective space $\mathbb{C}P^{m+1}$. The other is the complex hyperbolic quadric Q^{m*}, which can be regarded as the real Grassmann manifold of all oriented spacelike 2-dimensional subspaces in indefinite Euclidean space $\mathbb{R}^{2m+2}_2$ (see Kobayashi-Nomizu [6], Romero [24] and [25], Smyth [26]). The sectional curvature 4ε of the complex quadric Q^m and the complex hyperbolic quadric Q^{m*} differ from

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each other as $\varepsilon = \pm 1$. Therefore, let us denote them by $\mathbb{Q}^m(\varepsilon)$, that is,

$$\mathbb{Q}^m(\varepsilon) = \begin{cases} Q^m = SO_{m+2}/SO_mSO_2 & \text{for } \varepsilon = 1, \\ Q^{m*} = SO^0_{m,2}/SO_mSO_2 & \text{for } \varepsilon = -1. \end{cases}$$

It is well known that the real Grassmannian of rank two $\mathbb{Q}^m(\varepsilon)$ admit two kinds of geometric structures. One is a rank two vector subbundle $\mathfrak{A} = \{A_{\lambda\bar{z}} \mid \lambda \in S^1\}$ which is the set of real structures. The other is a complex structure J on $T_p(\mathbb{Q}^m(\varepsilon))$, $p \in \mathbb{Q}^m(\varepsilon)$, which anti-commutes with real structure A, $AJ = -JA$. Then for $m \geq 2$, the triple $(\mathbb{Q}^m(\varepsilon), J, g)$ is a Hermitian symmetric space of rank 2 with the Riemannian metric g and whose sectional curvatures are equal to ± 4 (see Klein [4], Kobayashi-Nomizu [6], Reckziegel [23], Suh [29] and [30]). In particular, Q^1 is isomorphic to the sphere S^2, and Q^2 is isomorphic to the Riemannian product of two 2-spheres $S^2 \times S^2$ with constant holomorphic sectional curvature. Additionally, the 1-dimensional complex hyperbolic quadric Q^{1^*} is isomorphic to the real hyperbolic space $\mathbb{R}H^2 = SO^0_{1,2}/SO_2$, and the 2-dimensional complex hyperbolic quadric Q^{2^*} is isomorphic to the Hermitian product of complex hyperbolic spaces $\mathbb{C}H^1 \times \mathbb{C}H^1$. For these reasons, we suppose $m \geq 3$ throughout this paper (see Klein-Suh [5], Smyth [26] and [27], Suh [28]).

For any $A \in \mathfrak{A}_p$ and $p \in \mathbb{Q}^m(\varepsilon)$, the real structure A induces a splitting $T_p\mathbb{Q}^m(\varepsilon) = V(A) \oplus JV(A)$ into two orthogonal, maximal totally real subspaces of the tangent space $T_p\mathbb{Q}^m(\varepsilon)$. Here $V(A)$ and $JV(A)$ are the $(+1)$-eigenspace and the (-1)-eigenspace of A, respectively. This implies that for every unit vector $W \in T_p\mathbb{Q}^m(\varepsilon)$, there exist $t \in [0, \frac{\pi}{4}]$, $A \in \mathfrak{A}_p$, and orthonormal vectors $Z_1, Z_2 \in V(A)$ such that

$$W = \cos(t)Z_1 + \sin(t)JZ_2$$

(see Proposition 3 in Reckziegel [23]). Here, t is uniquely determined by the vector W. In particular, the vector W is *singular*, i.e., contained in more than one Cartan subalgebra of $\mathfrak{m} \cong T_p\mathbb{Q}^m(\varepsilon)$, if and only if either $t = 0$ or $t = \frac{\pi}{4}$ holds. The vectors with $t = 0$ are called $\mathfrak{A}$-*principal*, whereas the vectors with $t = \frac{\pi}{4}$ are called $\mathfrak{A}$-*isotropic*. If W is regular, i.e., $0 < t < \frac{\pi}{4}$ holds, then also A and Z_1, Z_2 are uniquely determined by the unit vector W.

As a remarkable classification of real hypersurfaces in $\mathbb{Q}^m(\varepsilon)$, we introduce the notion of *isometric Reeb flow* of a real hypersurface M in $\mathbb{Q}^m(\varepsilon)$, which means that the Reeb flow on M in $(\mathbb{Q}^m(\varepsilon), J, g)$ satisfies the property $\mathcal{L}_\xi g = 0$, where $\mathcal{L}_\xi$ is the Lie derivative with respect to $\xi = -JN$ (N is a (local) unit normal vector field of M in $\mathbb{Q}^m(\varepsilon)$). Then the complete classification of real hypersurfaces with isometric Reeb flow are introduced in [8] and [29] as given in the following theorem.

Theorem A. *Let M be a real hypersurface with isometric Reeb flow in the real Grassmannian of rank two $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$. Then m is even, say*

$m = 2k$, and M is locally congruent to an open part of one of the following hypersurfaces:

- *For $\varepsilon = 1$*
  ($\mathcal{T}_A$) *a tube around the totally geodesic $\mathbb{C}P^k$ in the complex quadric $\mathbb{Q}^{2k}(1) = Q^{2k}$, $k \geq 2$.*
- *For $\varepsilon = -1$*
  ($\mathcal{T}_A^*$) *a tube around the totally geodesic $\mathbb{C}H^k$ in the complex hyperbolic quadric $\mathbb{Q}^{2k}(-1) = Q^{2k^*}$, $k \geq 2$, or*
  ($\mathcal{H}_A^*$) *a horosphere in $\mathbb{Q}^m(-1) = Q^{m*}$ whose center at infinity is the equivalence class of $\mathfrak{A}$-isotropic singular geodesic in Q^{m*}.*

As mentioned above, we say that the unit normal vector field N of a real hypersurface M in $\mathbb{Q}^m(\varepsilon)$ is *singular*, if the unit normal vector field N is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [10, 29]). In fact, ($\mathcal{T}_A$), ($\mathcal{T}_A^*$) and ($\mathcal{H}_A^*$) in Theorem A can be regarded as model spaces with $\mathfrak{A}$-isotropic unit normal vector field N (see Proposition 4.1 in Suh [29]). Apart from this, the following model spaces have an $\mathfrak{A}$-principal unit normal vector field N in $\mathbb{Q}^m(\varepsilon)$:

- for $\varepsilon = 1$
  ($\mathcal{T}_B$): *a tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around the m-dimensional sphere S^m in Q^m.*
- for $\varepsilon = -1$ (see Propositions 3.1, 3.2 and 4.1 in Klein-Suh [5])
  ($\mathcal{T}_{B_1}^*$): *a tube of radius r around the Hermitian symmetric space Q^{m-1^*} which is embedded in Q^{m*} as a totally geodesic complex hypersurface,*
  ($\mathcal{T}_{B_2}^*$): *a tube of radius r around the m-dimensional real hyperbolic space $\mathbb{R}H^m$ which is embedded in Q^{m*} as a real space form of Q^{m*}, and*
  ($\mathcal{H}_B^*$): *a horosphere in Q^{m*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m*}.*

On the other hand, a real hypersurface M of $\mathbb{Q}^m(\varepsilon)$ is said to be *Hopf* if the Reeb vector field ξ of M is principal for the shape operator S, meaning $S\xi = g(S\xi, \xi)\xi = \alpha\xi$. Specifically, if the Reeb function $\alpha = g(S\xi, \xi)$ vanishes identically on M, we say that M has *vanishing geodesic Reeb flow*. Otherwise, it has *non-vanishing geodesic Reeb flow*.

Recently, a common tool in studying submanifold theory is given by Jacobi operators (see, for example, [1, 3, 7, 8, 12, 14, 16, 18, 22]). Jacobi operators of a Riemannian manifold $(\bar{M}, g, \bar{R})$ are defined as follows: If $\bar{R}$ is the curvature tensor of $\bar{M}$, then the Jacobi operator with respect to a unit vector W at $p \in \bar{M}$ is defined by

$$(\bar{R}_W Z)(p) := \bar{R}((Z, W)W)(p)$$

for any $Z \in T_p\bar{M}$, $p \in \bar{M}$. Consequently, $\bar{R}_W \in \mathrm{End}(T_p\bar{M})$ is a self-adjoint endomorphism of the tangent space $T\bar{M}$ of $\bar{M}$. Clearly, each tangent vector field W to $\bar{M}$ provides a Jacobi operator with respect to W.

Regarding such Jacobi operators, several classification results have been provided for real hypersurfaces of Hermitian symmetric spaces. For real hypersurfaces in the complex two-plane Grassmannians $\mathbb{G}_2(\mathbb{C}^{m+2})$, in [2,13], Jeong, Machado, Pérez and Suh gave nonexistence theorems concerning the concept of parallelism for Jacobi operators. Additionally, in [31], Wang proved the nonexistence of Hopf hypersurfaces in $\mathbb{G}_2(\mathbb{C}^{m+2})$ admitting a parallel normal Jacobi operator with respect to the generalized Tanaka-Webster connection. On the other hand, Wang [32, 33] classified real hypersurfaces in the complex projective space $\mathbb{C}P^2$ with either constant Reeb sectional curvature or a GTW-parallel structure Jacobi operator, respectively. Furthermore, Pérez and others gave some classification results for real hypersurfaces in complex space forms in terms of the structure Jacobi operator (see Pérez [15], Pérez-Santos [19], Pérez-Santos-Suh [20], Pérez-Suh [21]).

For a real hypersurface M in $\mathbb{Q}^m(\varepsilon)$, the *normal Jacobi operator* $\bar{R}_N$ can be defined as follows:

$$\bar{R}_N := \bar{R}(\,\cdot\,, N)N \in \mathrm{End}(T_pM), \quad p \in M,$$

where N represents a unit normal vector field of M in $\mathbb{Q}^m(\varepsilon)$. Here, $\bar{R}$ denotes the (Riemannian) curvature tensor of $\mathbb{Q}^m(\varepsilon)$. We recall that the normal Jacobi operator $\bar{R}_N$ of M is called *parallel* if $\nabla_X\bar{R}_N = 0$ and is said to be *semi-parallel* when

$$R(X,Y)\bar{R}_N = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})\bar{R}_N = 0$$

for every tangent vector fields X, Y on M, where the curvature tensor R of M acts as a derivation on $\bar{R}_N$. The notion of semi-parallelism of the normal Jacobi operator is a generalization of parallelism.

Based on such notions, if the normal Jacobi operator $\bar{R}_N$ of M in the real Grassmannian of rank two $\mathbb{Q}^m(\varepsilon)$ is semi-parallel, we can assert that the unit normal vector field N is singular as follows.

THEOREM 1. *Let M be a Hopf real hypersurface in the real Grassmannian of rank two $\mathbb{Q}^m(\varepsilon)$ for $m \geq 3$. If the normal Jacobi operator $\bar{R}_N$ of M in $\mathbb{Q}^m(\varepsilon)$ is semi-parallel, then the unit normal vector field N is singular. That is, N is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

By Theorem 1, we can give a classification for Hopf real hypersurfaces in the real Grassmannian of rank two $\mathbb{Q}^m(\varepsilon)$ satisfying semi-parallelism of the normal Jacobi operator $\bar{R}_N$ as follows.

THEOREM 2. *There does not exist any Hopf real hypersurface in the real Grassmannian of rank two $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$, with semi-parallel normal Jacobi operator.*

This paper is organized as follows. In Section 2, we provide preliminaries on $\mathbb{Q}^m(\varepsilon)$ and real hypersurfaces in it. In this section, we introduce some general equations and results for Hopf real hypersurfaces in $\mathbb{Q}^m(\varepsilon)$. In Sections 3, 4, and 5, we can prove these results, respectively. In Section 3, we derive some general equations related to the normal Jacobi operator $\bar{R}_N$ of a Hopf real hypersurface M in $\mathbb{Q}^m(\varepsilon)$. Using these formulas, we show that the unit normal vector field N of M admitting semi-parallel normal Jacobi operator in $\mathbb{Q}^m(\varepsilon)$ is singular. Based on the singularity of N, in Sections 4 and 5, we will consider the classification problem for Hopf real hypersurfaces in $\mathbb{Q}^m(\varepsilon)$ admitting semi-parallel normal Jacobi operator.

2. Preliminaries

We use some references [5, 7, 8, 11] and [17] to recall the Riemannian geometry of the real Grassmannian of rank two $\mathbb{Q}^m(\varepsilon)$, $\varepsilon = \pm 1$, and some fundamental formulas including the Codazzi and Gauss equations for a real hypersurface in $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$. Through this paper all manifolds, vector fields, etc., are considered of class C^∞.

Let M be a connected real hypersurface in the real Grassmannian of rank two $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$, and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. As mentioned before, the ambient space $\mathbb{Q}^m(\varepsilon)$ is equipped with a Kähler structure (J, g) and a real structure A. With respect to the Kähler structure we write $JX = \phi X + \eta(X)N$ and $JN = -\xi$, where N is a (local) unit normal vector field of M and η the corresponding 1-form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathcal{C}^\perp$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$. Moreover, since $\mathbb{Q}^m(\varepsilon)$ has also a real structure A, we decompose AX into its tangential and normal components for a fixed $A \in \mathfrak{A}$ and $X \in TM$:

$$AX = BX + g(AX, N)N \tag{2.1}$$

where BX denotes the tangential component of AX. Since A is symmetric, that is, $g(AX, Y) = g(X, AY)$, we see that the operator B is also symmetric.

By virtue of Proposition 3 in [23], at each point $p \in M$ we can choose a real structure $A \in \mathfrak{A}_p$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2 \tag{2.2}$$

for some orthonormal vectors $Z_1,\ Z_2 \in V(A) := \{Z \in T_p\mathbb{Q}^m(\varepsilon)\,|\,AZ = Z\}$ and $0 \leq t \leq \frac{\pi}{4}$. This implies

$$\begin{cases} JN = \cos(t)JZ_1 - \sin(t)Z_2 \ \ (\text{i.e. } \xi = \sin(t)Z_2 - \cos(t)JZ_1), \\ AN = \cos(t)Z_1 - \sin(t)JZ_2, \\ A\xi = \cos(t)JZ_1 + \sin(t)Z_2, \end{cases} \tag{2.3}$$

and therefore $g(A\xi, N) = g(AN, \xi) = 0$ and $g(A\xi, \xi) = -g(AN, N) = -\cos(2t)$ on M. From this, we assert that the unit vector $A\xi$ of $\mathbb{Q}^m(\varepsilon)$ is tangent to M. Since the real structure A anti-commutes with the Kähler structure J, that is, $JA = -AJ$, we obtain

$$AN = AJ\xi = -JA\xi = -\phi A\xi - g(A\xi, \xi)N, \tag{2.4}$$

and

$$\phi BX + g(X, \phi A\xi)\xi = JAX = -AJX = -B\phi X + \eta(X)\phi A\xi, \tag{2.5}$$

for any $X \in TM$. In addition, from the property of $A^2 = I$, we get

$$B^2 X = X - g(\phi A\xi, X)\phi A\xi, \quad B\phi A\xi = g(A\xi, \xi)\phi A\xi. \tag{2.6}$$

In [5] and [28], the Riemannian curvature tensor $\bar{R}$ of $\mathbb{Q}^m(\varepsilon)$ was introduced as follows:

$$\begin{aligned} \bar{R}(U, V)W = \varepsilon\big\{ & g(V, W)U - g(U, W)V + g(JV, W)JU \\ & - g(JU, W)JV - 2g(JU, V)JW + g(AV, W)AU \\ & - g(AU, W)AV + g(JAV, W)JAU - g(JAU, W)JAV \big\} \end{aligned} \tag{2.7}$$

for any complex conjugation $A \in \mathfrak{A}$ and any vector fields U, V, and $W \in T\mathbb{Q}^m(\varepsilon)$. By virtue of the Gauss and Weingarten formulas, given respectively as $\bar{\nabla}_X Y = \nabla_X Y + g(SX, Y)N$ and $\bar{\nabla}_X N = -SX$, the left-hand side of (2.7) becomes

$$\begin{aligned} \bar{R}(X, Y)Z &= \bar{\nabla}_U \bar{\nabla}_V W - \bar{\nabla}_V \bar{\nabla}_U W - \bar{\nabla}_{[U,V]} W \\ &= R(X, Y)Z - g(SY, Z)SX + g(SX, Z)SY \\ &\quad + g((\nabla_X S)Y, Z)N - g((\nabla_Y S)X, Z)N, \end{aligned}$$

where S is the shape operator of M and R is the Riemannian curvature tensor of M defined as $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ for any vector fields X, Y and $Z \in TM$. From this formula and the expression of the curvature tensor $\bar{R}$ of $\mathbb{Q}^m(\varepsilon)$ in (2.7), the Gauss and Codazzi equations for a real hypersurface M in $\mathbb{Q}^m(\varepsilon)$ can be derived as follows:

$$\begin{aligned} & R(X, Y)Z - g(SY, Z)SX + g(SX, Z)SY \\ & = \varepsilon\big\{ g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & \quad - 2g(\phi X, Y)\phi Z + g(BY, Z)BX - g(BX, Z)BY \\ & \quad + g(\phi BY, Z)\phi BX + g(\phi BY, Z)g(X, \phi A\xi)\xi \\ & \quad + g(Y, \phi A\xi)\eta(Z)\phi BX - g(\phi BX, Z)\phi BY \\ & \quad - g(\phi BX, Z)g(Y, \phi A\xi)\xi - g(X, \phi A\xi)\eta(Z)\phi BY \big\} \end{aligned} \tag{2.8}$$

and

$$
\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, Z) \\
&= \varepsilon\big\{\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\
&\quad - g(BY, Z)g(\phi A\xi, X) + g(BX, Z)g(\phi A\xi, Y) \\
&\quad + g(A\xi, X)g(\phi BY, Z) + \eta(Z)g(A\xi, X)g(\phi A\xi, Y) \\
&\quad - g(A\xi, Y)g(\phi BX, Z) - \eta(Z)g(A\xi, Y)g(\phi A\xi, X)\big\}
\end{aligned} \tag{2.9}
$$

for any $X, Y, Z \in TM$.

When the Reeb vector field ξ on M is a principal vector field with Reeb curvature function $\alpha = g(S\xi, \xi)$, M is said to be a *Hopf* real hypersurface in $\mathbb{Q}^m(\varepsilon)$. Then, differentiating the equation $S\xi = \alpha\xi$ and using the equation of Codazzi, we obtain the following lemma.

Lemma 2.1 ([8,29]). *Let M be a Hopf real hypersurface in $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$. Then we obtain*

$$Y\alpha = (\xi\alpha)\eta(Y) - 2\varepsilon g(A\xi, \xi)g(\phi A\xi, Y) \tag{2.10}$$

and

$$
\begin{aligned}
&2S\phi SY - \alpha(S\phi + \phi S)Y \\
&= 2\varepsilon\big\{\phi Y - g(A\xi, \xi)g(\phi A\xi, Y)\xi + \eta(Y)g(A\xi, \xi)\phi A\xi \\
&\quad + g(\phi A\xi, Y)A\xi - g(A\xi, Y)\phi A\xi\big\}
\end{aligned} \tag{2.11}
$$

for any tangent vector fields X and Y of M.

According to the fact mentioned above, we already knew that $A\xi$ is a tangent vector field on M, that is, $A\xi \in T_pM$ for any point p of M. By using the Gauss formula and the covariant derivative formula given by $(\bar{\nabla}_U A)V = q(U)JAV$ for any $U, V \in T_p\mathbb{Q}^m(\varepsilon)$, it leads to

$$
\begin{aligned}
\nabla_X(A\xi) &= \bar{\nabla}_X(A\xi) - g(SX, A\xi)N \\
&= q(X)JA\xi + A(\nabla_X\xi) + g(SX, \xi)AN - g(SX, A\xi)N \\
&= q(X)\big\{\phi A\xi + g(A\xi, \xi)N\big\} + B\phi SX + g(A\phi SX, N)N \\
&\quad - g(SX, \xi)\big\{\phi A\xi + g(A\xi, \xi)N\big\} - g(SX, A\xi)N
\end{aligned}
$$

for any $X \in T_pM$. Then, by comparing the tangential and the normal components of the above equation, we get, respectively,

$$\nabla_X(A\xi) = q(X)\phi A\xi + B\phi SX - g(SX, \xi)\phi A\xi, \tag{2.12}$$

$$
\begin{aligned}
q(X)g(A\xi, \xi) &= -g(A\phi SX, N) + g(SX, \xi)g(A\xi, \xi) + g(SX, A\xi) \\
&= g(\phi SX, \phi A\xi) + g(SX, \xi)g(A\xi, \xi) + g(SX, A\xi) \\
&= 2g(SX, A\xi).
\end{aligned} \tag{2.13}
$$

In particular, if M is Hopf, then the equation (2.13) becomes

$$q(\xi)g(A\xi, \xi) = 2\alpha g(A\xi, \xi). \tag{2.14}$$

In [9], the authors prove the following result from (2.10).

Lemma 2.2 ([9]). *Let M be a Hopf real hypersurface in the real Grassmannian of rank two $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$. If the Reeb function $\alpha = g(S\xi, \xi)$ identically vanishes on any subset $\mathcal{V} \subset M$, then the normal vector field N is singular on $\mathcal{V}$.*

3. Singularity of the unit normal vector field on a Hopf real hypersurface admitting semi-parallel normal Jacobi operator

In this section, from the Gauss equation in the ambient space of the real Grassmannian of rank two $\mathbb{Q}^m(\varepsilon)$ we can define the normal Jacobi operator $\bar{R}_N$ of M in $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$. As mentioned in the introduction, we say that the normal Jacobi operator $\bar{R}_N$ of M is *semi-parallel* if the $(1, 1)$-type tensor field $\bar{R}_N$ satisfies

$$R \cdot \bar{R}_N = 0, \tag{3.1}$$

where the curvature tensor R acts on $\bar{R}_N$ as a derivation. More precisely, it means that $(R(X, Y)\bar{R}_N)Z = 0$ for any tangent vector fields X, Y, Z of M. Using $R(X, Y)(\bar{R}_N Z) = (R(X, Y)\bar{R}_N)Z + \bar{R}_N(R(X, Y)Z)$, we see that equation (3.1) is equivalent to

$$R(X, Y)(\bar{R}_N Z) = \bar{R}_N(R(X, Y)Z). \tag{3.2}$$

On the other hand, the normal Jacobi operator $\bar{R}_N$ induced from the curvature tensor $\bar{R}$ of $\mathbb{Q}^m(\varepsilon)$ introduced in Section 2 is given, for any vector field U in $T\mathbb{Q}^m(\varepsilon)$, by

$$\begin{aligned} \bar{R}_N U &= \bar{R}(U, N)N \\ &= \varepsilon\big\{U - g(U, N)N + 3g(U, \xi)\xi + g(AN, N)AU \\ &\quad - g(AN, U)AN - g(A\xi, U)A\xi\big\} \in T\mathbb{Q}^m(\varepsilon), \end{aligned} \tag{3.3}$$

where $JN = -\xi$ and $g(JAN, N) = -g(AJN, N) = g(A\xi, N) = 0$. It means that the tensor field $\bar{R}_N$ of type (1,1) defined by (3.3) is an endomorphism of $T\mathbb{Q}^m(\varepsilon)$. From this and $T\mathbb{Q}^m(\varepsilon) = TM \oplus \operatorname{span}\{N\}$, the vector field $\bar{R}_N Y$ for any tangent vector field Y of M is decomposed as follows:

$$\bar{R}_N Y = (\bar{R}_N Y)^\top + g(\bar{R}_N Y, N)N,$$

where $(\,\cdot\,)^\top$ denotes the tangential part of $(\,\cdot\,)$.

Bearing in mind $g(\bar{R}(Y, N)N, N) = 0$, the normal part of $\bar{R}_N Y$ vanishes on M in $\mathbb{Q}^m(\varepsilon)$. Hence, we see that $\bar{R}_N \in \operatorname{End}(TM)$. In fact, by applying (2.1), (2.4), and $g(AN, N) = -g(A\xi, \xi)$ in (3.3), the normal Jacobi operator $\bar{R}_N$ of M is given by:

$$\bar{R}_N Y = \varepsilon\big\{Y + 3\eta(Y)\xi - g(A\xi, \xi)BY - g(\phi A\xi, Y)\phi A\xi - g(A\xi, Y)A\xi\big\} \tag{3.4}$$

for any $Y \in TM$, which means $\bar{R}_N \in \mathrm{End}(TM)$. Then, we obtain

$$\bar{R}_N\xi = \varepsilon\{4\xi - 2g(A\xi,\xi)A\xi\} = \varepsilon\{4\xi - 2\beta A\xi\}, \tag{3.5}$$

$$\bar{R}_N A\xi = 2\varepsilon g(A\xi,\xi)\xi = 2\varepsilon\beta\xi, \tag{3.6}$$

$$\bar{R}_N\phi A\xi = \varepsilon\{\phi A\xi - g(A\xi,\xi)B\phi A\xi - g(\phi A\xi,\phi A\xi)\phi A\xi\} = 0, \tag{3.7}$$

where we have used $A\xi = B\xi$, $BA\xi = A^2\xi - g(A^2\xi, N)N = \xi$, $g(A\xi, A\xi) = g(\xi,\xi) = 1$, $B\phi A\xi = \beta\phi A\xi$, and $g(\phi A\xi,\phi A\xi) = 1 - \beta^2$. Here, β denotes the smooth function $\beta = g(A\xi,\xi)$ on M.

To give a proof of Theorem 1, let us assume that M is a Hopf real hypersurface in $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$, with semi-parallel normal Jacobi operator. That is, the normal Jacobi operator $\bar{R}_N$ of M satisfies (3.2). Taking $Y = Z = \xi$ in (3.2) gives

$$R(X,\xi)(\bar{R}_N\xi) = \bar{R}_N(R(X,\xi)\xi). \tag{3.8}$$

Now, let us define another Jacobi operator $R_\xi \in \mathrm{End}(TM)$ distinguished from the normal Jacobi operator $\bar{R}_N \in \mathrm{End}(TM)$. We call such an operator the *structure Jacobi operator* and define it as the Jacobi operator with respect to $\xi = -JN$, that is, $R_\xi X = R(X,\xi)\xi$ for any $X \in TM$. Here, R stands for the Riemannian curvature tensor field of M. In fact, by virtue of (2.8), the structure Jacobi operator R_ξ of a Hopf real hypersurface M in $\mathbb{Q}^m(\varepsilon)$ is given by:

$$\begin{aligned} R_\xi X &= R(X,\xi)\xi \\ &= \varepsilon\{X - \eta(X)\xi + \beta BX - g(A\xi,X)A\xi - g(X,\phi A\xi)\phi A\xi\} \\ &\quad + \alpha SX - \alpha^2\eta(X)\xi. \end{aligned} \tag{3.9}$$

Bearing in mind the notation of structure Jacobi operator R_ξ, (3.8) is rearranged using (3.5) as follows:

$$\varepsilon\{4R_\xi X - 2\beta R(X,\xi)A\xi\} = \bar{R}_N R_\xi X. \tag{3.10}$$

Taking the inner product of (3.10) with ξ and using the symmetry of $\bar{R}_N$, we obtain

$$\varepsilon\{4g(R_\xi X,\xi) - 2\beta g(R(X,\xi)A\xi,\xi)\} = g(R_\xi X, \bar{R}_N\xi). \tag{3.11}$$

By the skew-symmetries of R, we have

$$g(R_\xi X,\xi) = g(R(X,\xi)\xi,\xi) = -g(R(X,\xi)\xi,\xi) = -g(R_\xi X,\xi).$$

Thus, it implies $g(R_\xi X,\xi) = 0$. Similarly, we get

$$g(R(X,\xi)A\xi,\xi) = -g(R(X,\xi)\xi, A\xi) = -g(R_\xi X, A\xi).$$

Furthermore, by (3.5), the right-hand side of (3.11) becomes

$$g(R_\xi X, \bar{R}_N\xi) = 4\varepsilon g(R_\xi X,\xi) - 2\varepsilon\beta g(R_\xi X, A\xi).$$

Using the above three equations, (3.11) leads to

$$\varepsilon\beta g(R_\xi X, A\xi) = 0. \tag{3.12}$$

Applying properties of R, we obtain

$$\begin{aligned} g(R_\xi X, A\xi) &= g(R(X,\xi)\xi, A\xi) = g(R(\xi, A\xi)X, \xi) \\ &= g(R(A\xi,\xi)\xi, X) = g(R_\xi A\xi, X) \end{aligned}$$

for any $X \in TM$. From this, together with the fact that $\varepsilon = \pm 1$, (3.12) yields two cases: either $\beta = 0$ or $R_\xi A\xi = 0$.

If $\beta = 0$, it naturally implies that the unit normal vector field N is $\mathfrak{A}$-isotropic. In fact, from (2.3), we have

$$\begin{aligned} \beta = g(A\xi,\xi) &= g(\cos(t)JZ_1 + \sin(t)Z_2, \sin(t)Z_2 - \cos(t)JZ_1) \\ &= -\cos^2(t)g(JZ_1, JZ_1) + \sin^2(t)g(Z_2, Z_2) \\ &= -\cos^2(t) + \sin^2(t) = -\cos(2t), \end{aligned}$$

where $0 \le t \le \frac{\pi}{4}$. Here, Z_1, Z_2 are unit orthonormal vector fields of $T\mathbb{Q}^m(\varepsilon)$ such that $AZ_\nu = Z_\nu$ for $\nu = 1, 2$. Thus, the condition of $\beta = 0$ implies $t = \frac{\pi}{4}$, which means that there exists a real structure $A \in \mathfrak{A}$ such that $N = \frac{1}{\sqrt{2}}\{Z_1 + JZ_2\}$ by (2.2), ensuring that N is $\mathfrak{A}$-isotropic.

Now, let us consider the case where $\beta \ne 0$. From (3.12), we have

$$R_\xi A\xi = 0.$$

Since $BA\xi = \xi$ and $g(\phi X, X) = 0$, (3.9) yields

$$\alpha S A\xi = \alpha^2\beta\xi. \tag{3.13}$$

As a direct consequence of Lemma 2.2, if the smooth function $\alpha = g(S\xi, \xi)$ vanishes identically on any open subset of M, then the unit normal vector field N is singular. Therefore, in the remaining part of this section, we focus on the case where $\alpha \ne 0$. With (3.13), we obtain the following lemma.

Lemma 3.1. *Let M be a Hopf real hypersurface in $\mathbb{Q}^m(\varepsilon)$, $m \ge 3$, with semi-parallel normal Jacobi operator. If the smooth functions $\alpha = g(S\xi,\xi)$ and $\beta = g(A\xi,\xi)$ are non-vanishing on an open subset $\mathcal{U} \subset M$, then it holds that*

$$SA\xi = \alpha\beta\xi \quad \textit{and} \quad S\phi A\xi = \kappa\phi A\xi,$$

where $\kappa = -\frac{2\varepsilon\beta^2}{\alpha}$.

Proof. On $\mathcal{U}$, the two smooth functions $\alpha = g(S\xi,\xi)$ and $\beta = g(A\xi,\xi)$ are non-vanishing. Therefore, from (3.13), we have $SA\xi = \alpha\beta\xi$. This implies $\phi SA\xi = 0$. By substituting $A\xi$ for Y in (2.11) and utilizing these expressions, we obtain

$$-\alpha S\phi A\xi = 2\varepsilon\beta^2\phi A\xi.$$

Consequently, we have

$$S\phi A\xi = -\frac{2\varepsilon\beta^2}{\alpha}\phi A\xi,$$

which completes a proof of Lemma 3.1. ∎

Using Lemma 3.1, we can state the following lemma:

LEMMA 3.2. *Let M be a Hopf real hypersurface in $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$, with semi-parallel normal Jacobi operator. If the smooth functions $\alpha = g(S\xi,\xi)$ and $\beta = g(A\xi,\xi)$ are non-vanishing on an open subset $\mathcal{U} \subset M$, then the unit normal vector field N is singular on $\mathcal{U}$.*

PROOF. Let $\mathcal{U}$ be an open subset of a Hopf real hypersurface M in $\mathbb{Q}^m(\varepsilon)$ such that $\mathcal{U} = \{p \in M \,|\, \alpha(p) \neq 0,\ \beta(p) \neq 0\}$. Assume that M has semi-parallel normal Jacobi operator. Then, by virtue of (3.2) and (3.7), the normal Jacobi operator $\bar{R}_N$ satisfies

$$\bar{R}_N(R(\phi A\xi,\xi)\phi A\xi) = 0 \tag{3.14}$$

for $X = Z = \phi A\xi$ and $Y = \xi$. On the other hand, (2.8) provides us

$$\begin{aligned}
&R(\phi A\xi,\xi)\phi A\xi + \alpha g(S\phi A\xi,\phi A\xi)\xi \\
&= \varepsilon\big\{ - g(\phi A\xi,\phi A\xi)\xi - \beta g(\phi A\xi,\phi A\xi)A\xi \\
&\quad + \beta g(\phi A\xi,\phi A\xi)\phi^2 A\xi + g(\phi A\xi,\phi A\xi)g(\phi A\xi,\phi A\xi)\xi\big\} \\
&= -2\varepsilon\beta(1-\beta^2)A\xi,
\end{aligned}$$

where we have used $B\xi = A\xi$, $B\phi A\xi = \beta\phi A\xi$, $\phi^2 A\xi = -A\xi + \beta\xi$ and $g(\phi A\xi,\phi A\xi) = 1-\beta^2$. Making use of Lemma 3.1, it leads to

$$\begin{aligned}
R(\phi A\xi,\xi)\phi A\xi &= -2\varepsilon\beta(1-\beta^2)A\xi - \alpha\kappa g(\phi A\xi,\phi A\xi)\xi \\
&= -2\varepsilon\beta(1-\beta^2)A\xi + 2\varepsilon\beta^2(1-\beta^2)\xi.
\end{aligned}$$

Moreover, by (3.5) and (3.6), together with $\varepsilon^2 = 1$, we get

$$\begin{aligned}
\bar{R}_N(R(\phi A\xi,\xi)\phi A\xi) &= -2\varepsilon\beta(1-\beta^2)\bar{R}_N A\xi + 2\varepsilon\beta^2(1-\beta^2)\bar{R}_N\xi \\
&= 4\beta^2(1-\beta^2)\big\{\xi - \beta A\xi\big\}.
\end{aligned}$$

From this, (3.14) is rearranged as

$$(1-\beta^2)\big\{\xi - \beta A\xi\big\} = 0 \tag{3.15}$$

on $\mathcal{U} = \{p \in M \,|\, \alpha(p) \neq 0,\ \beta(p) \neq 0\}$.

Taking the inner product of (3.15) with ξ yields $(1-\beta^2)^2 = 0$. This implies $1-\beta^2 = 0$ on $\mathcal{U}$. By virtue of (2.3), the smooth function $\beta = g(A\xi,\xi)$ satisfies $\beta = -\cos 2t$, where $0 \leq t \leq \frac{\pi}{4}$. From this, $\beta^2 = 1$ implies $t = 0$. Then, according to (2.2), the unit normal vector field N is expressed as $N = Z_1 \in V(A)$, meaning that N is $\mathfrak{A}$-principal on $\mathcal{U}$. ∎

From the discussion above and Lemma 2.2, we have provided a complete proof of Theorem 1, and we will prove our Theorem 2 in sections 4 and 5, respectively. This will depend on whether the normal vector field N of M is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.

4. $\mathfrak{A}$-principal unit normal vector field

In this section, we assume that a Hopf real hypersurface M admitting a semi-parallel normal Jacobi operator has an $\mathfrak{A}$-principal unit normal vector field N in $\mathbb{Q}^m(\varepsilon)$. The assumption of N being $\mathfrak{A}$-principal yields $t = 0$ in (2.2). Applying this fact to (2.3), we get $A\xi = -\xi$ and $AN = N$. Moreover, these properties lead to the following result.

Lemma 4.1. *Let M be a real hypersurface in the real Grassmannian of rank two $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$, with $\mathfrak{A}$-principal normal vector field N. Then we obtain:*

(i) $AX = BX$
(ii) $A\phi X = -\phi AX$
(iii) $A\phi SX = -\phi SX$ *and* $q(X) = 2g(SX, \xi)$
(iv) $ASX = SX - 2g(SX, \xi)\xi$

for any $X \in T_pM$, $p \in M$, where $BX = (AX)^T$ denotes the tangential part of the vector field AX on M in $\mathbb{Q}^m(\varepsilon)$.

Proof. As M has an $\mathfrak{A}$-principal unit normal vector field N in $\mathbb{Q}^m(\varepsilon)$, we have $g(AX, N) = 0$ for any vector field $X \in TM$. It means that AX is always orthogonal to N. Consequently, we can assert $AX \in TM$, that is, $AX = BX$ for any $X \in TM$.

As already mentioned in Section 1, we know that the complex structure J anti-commutes with the real structure $A \in \mathfrak{A}$ of $\mathbb{Q}^m(\varepsilon)$, that is, $JA = -AJ$. Making use of (i), $JX = \phi X + \eta(X)N$ yields

$$\begin{aligned}\phi AX - \eta(X)N &= \phi AX + \eta(AX)N = JAX \\ &= -AJX = -A\big(\phi X + \eta(X)N\big) = -A\phi X - \eta(X)N.\end{aligned}$$

It implies $A\phi X = -\phi AX$, which gives a complete proof of (ii).

Now, differentiating the equations $A\xi = -\xi$ and $AN = N$ with respect to the Levi-Civita connection $\bar{\nabla}$ of $\mathbb{Q}^m(\varepsilon)$ respectively yields:

$$\begin{aligned}& - q(X)N + A\phi SX + g(SX, \xi)N \\ &= q(X)JA\xi + A\phi SX + g(SX, \xi)AN \\ &= (\bar{\nabla}_X A)\xi + A(\bar{\nabla}_X \xi) = -\bar{\nabla}_X \xi = -\phi SX - g(SX, \xi)N\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}-q(X)\xi - ASX &= q(X)JAN - ASX \\ &= (\bar{\nabla}_X A)N + A(\bar{\nabla}_X N) = \bar{\nabla}_X N = -SX.\end{aligned} \tag{4.2}$$

Here, we have used $\bar{\nabla}_X Y = \nabla_X Y + g(SX, Y)N$, $\bar{\nabla}_X N = -SX$ (known as the Gauss and Weingarten formulas), and $\nabla_X \xi = \phi SX$. By comparing the tangential and normal parts of (4.1), we obtain both formulas in (iii). Moreover, (iv) follows from $q(X) = 2g(SX, \xi)$ obtained in (iii) and (4.2). □

By using these formulas for real hypersurfaces in $\mathbb{Q}^m(\varepsilon)$ with $\mathfrak{A}$-principal normal vector field, we assert the following lemma.

LEMMA 4.2. *There does not exist any Hopf real hypersurface admitting a semi-parallel normal Jacobi operator in* $\mathbb{Q}^m(\varepsilon)$ *for* $m \geq 3$, *with* $\mathfrak{A}$*-principal normal vector field* N.

PROOF. Let us suppose that M is a Hopf real hypersurface with semi-parallel normal Jacobi operator and $\mathfrak{A}$-principal normal vector field N in $\mathbb{Q}^m(\varepsilon)$. From the assumption that N is $\mathfrak{A}$-principal, we obtain $A\xi = -\xi$, implying $\beta = g(A\xi, \xi) = -1$. Thus, (3.5) yields $\bar{R}_N \xi = 2\varepsilon\xi$. Using this fact and inserting $Y = Z = \xi$ into (3.2), we obtain $2\varepsilon R(X, \xi)\xi = \bar{R}_N(R(X, \xi)\xi)$, which can be expressed as follows:

$$2\varepsilon R_\xi X = \bar{R}_N R_\xi X. \tag{4.3}$$

On the other hand, when the unit normal vector field N is $\mathfrak{A}$-principal, the structure Jacobi operator R_ξ given in (3.9) becomes

$$R_\xi X = \varepsilon\{X - 2\eta(X)\xi - AX\} + \alpha SX - \alpha^2 \eta(X)\xi. \tag{4.4}$$

Due to (4.4) and $\varepsilon^2 = 1$, (4.3) can be rearranged as

$$\begin{aligned} &2\{X - 2\eta(X)\xi - AX\} + 2\varepsilon\alpha SX - 2\varepsilon\alpha^2\eta(X)\xi \\ &= \varepsilon\{\bar{R}_N X - 2\eta(X)\bar{R}_N\xi - \bar{R}_N AX\} + \alpha\bar{R}_N SX - \alpha^2\eta(X)\bar{R}_N\xi. \end{aligned} \tag{4.5}$$

Now, according to (3.4), the normal Jacobi operator $\bar{R}_N$ of M with $\mathfrak{A}$-principal normal vector field satisfies

$$\bar{R}_N X = \varepsilon\{X + 2\eta(X)\xi + AX\}$$

for any tangent vector field X on M. We obtain

$$\begin{aligned} \bar{R}_N\xi &= 2\varepsilon\xi, \\ \bar{R}_N AX &= \varepsilon\{AX - 2\eta(X)\xi + X\}, \\ \bar{R}_N SX &= \varepsilon\{SX + 2\alpha\eta(X)\xi + ASX\}. \end{aligned}$$

Substituting the above four formulas into (4.5) yields

$$2\{X - 2\eta(X)\xi - AX\} = \varepsilon\alpha\{ASX - SX + 2\alpha\eta(X)\xi\}. \tag{4.6}$$

By (iv) in Lemma 4.1, (4.6) becomes

$$X - 2\eta(X)\xi - AX = 0,$$

which implies

$$AX = X - 2\eta(X)\xi \tag{4.7}$$

for any $X \in TM$.

Take a local orthonormal frame field $\mathfrak{B}$ of $\mathbb{Q}^m(\varepsilon)$ as follows

$$\mathfrak{B} = \big\{e_1, e_2 = \phi e_1, e_3, e_4 = \phi e_3, e_5, e_6 = \phi e_5, \dots \\ \dots, e_{2m-3}, e_{2m-2} = \phi e_{2m-3}, e_{2m-1} = \xi, e_{2m} = N\big\}.$$

It follows from (4.7) that the trace of the real structure A, defined as $\mathrm{Tr}A = \sum_{k=1}^{2m} g(Ae_k, e_k)$, is given by

$$\begin{aligned} \mathrm{Tr}A &= \sum_{k=1}^{2m-2} g(Ae_k, e_k) + g(A\xi, \xi) + g(AN, N) \\ &= 2m - 2, \end{aligned}$$

where we have used $A\xi = -\xi$ and $AN = N$. It is well known that the trace of the real structure A vanishes on $\mathbb{Q}^m(\varepsilon)$, i.e., $\mathrm{Tr}A = 0$. Therefore, we obtain $m = 1$, which leads to a contradiction for $m \geq 3$. This completes the proof of Lemma 4.2. □

5. $\mathfrak{A}$-isotropic unit normal vector field

Expanding upon Theorem 1 and Lemma 4.2 discussed previously, we assert that if M is a Hopf real hypersurface in $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$, with semi-parallel normal Jacobi operator, then the unit normal vector field N of M is $\mathfrak{A}$-isotropic.

By the definition of $\mathfrak{A}$-isotropic tangent vector field on $\mathbb{Q}^m(\varepsilon)$, (2.2) implies $N = \frac{1}{\sqrt{2}}\big(Z_1 + JZ_2\big)$, where $t = \frac{\pi}{4}$. From (2.3), we also obtain $\beta = g(A\xi, \xi) = -g(AN, N) = 0$. This implies that the vector field $AN = -\phi A\xi$ is tangent to M. By differentiating the tangent vector field AN and using the Gauss and Weingarten formulas, we obtain the following:

$$\begin{aligned} \nabla_X(AN) &= \bar{\nabla}_X(AN) - \sigma(X, AN) \\ &= \big(\bar{\nabla}_X A\big)N + A\big(\bar{\nabla}_X N\big) - g(SX, AN)N \\ &= q(X)JAN - ASX - g(SX, AN)N. \end{aligned} \tag{5.1}$$

Here, we have used $\big(\bar{\nabla}_U A\big)W = q(U)JAW$ for U, $W \in T\mathbb{Q}^m(\varepsilon)$ (see [26]). Moreover, taking the inner product of (5.1) with N, we get $SAN = 0$.

Similarly, by differentiating the tangent vector field $A\xi$, we get the following:

$$\begin{aligned} \nabla_X(A\xi) &= \bar{\nabla}_X(A\xi) - \sigma(X, A\xi) \\ &= \big(\bar{\nabla}_X A\big)\xi + A\big(\bar{\nabla}_X \xi\big) - g(SX, A\xi)N \\ &= q(X)JA\xi + A(\nabla_X \xi) + g(SX, \xi)AN - g(SX, A\xi)N. \end{aligned} \tag{5.2}$$

Then, by taking the inner product of (5.2) with the unit normal vector field N, we get $SA\xi = 0$.

Summing up these discussions, in general it holds:

Lemma 5.1. *Let M be a real hypersurface in the real Grassmannian of rank two $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$, with $\mathfrak{A}$-isotropic normal vector field N. Then the tangent vector fields $A\xi$ and AN are principal, satisfying $SA\xi = SAN = S\phi A\xi = 0$ for the shape operator S of M in $\mathbb{Q}^m(\varepsilon)$.*

And, in the case where a Hopf real hypersurface M in $\mathbb{Q}^m(\varepsilon)$ possesses an $\mathfrak{A}$-isotropic normal vector field N, it can be deduced from (3.4) and (3.9) that the normal and structure Jacobi operators of M are given by the following expressions:

$$\begin{aligned} \bar{R}_N X &= \varepsilon\big\{X + 3\eta(X)\xi - g(A\xi, X)A\xi - g(\phi A\xi, X)\phi A\xi\big\}, \\ R_\xi X &= \varepsilon\big\{X - \eta(X)\xi - g(A\xi, X)A\xi - g(X, \phi A\xi)\phi A\xi\big\} \\ &\quad + \alpha SX - \alpha^2\eta(X)\xi. \end{aligned} \tag{5.3}$$

Hereafter, M denotes a Hopf real hypersurface in $\mathbb{Q}^m(\varepsilon)$ with $m \geq 3$, admitting a semi-parallel normal Jacobi operator. We take $\mathfrak{B}$ as the orthonormal frame field on M given by

$$\begin{aligned} \mathfrak{B} = \big\{&e_1, e_2 = \phi e_1, e_3, e_4 = \phi e_3, \dots \\ &\dots, e_{2m-5}, e_{2m-4} = \phi e_{2m-5}, e_{2m-3} = A\xi, e_{2m-2} = \phi A\xi, e_{2m-1} = \xi\big\}. \end{aligned}$$

Then, it follows from (5.3) that the normal Jacobi operator of M is expressed as follows:

$$\bar{R}_N X = \begin{cases} 4\varepsilon\xi & \text{if } X = \xi, \\ 0 & \text{if } X = A\xi, \\ 0 & \text{if } X = \phi A\xi, \\ \varepsilon X & \text{if } X \perp \xi, A\xi, \phi A\xi. \end{cases} \tag{5.4}$$

Take $Y = \phi A\xi$ and $Z = A\xi$ in (3.2). Then, according to (5.4), the assumption of $\bar{R}_N$ being semi-parallel implies

$$\bar{R}_N(R(X, \phi A\xi)A\xi) = 0. \tag{5.5}$$

On the other hand, using the equation of Gauss in (2.8) and Lemma 5.1, we obtain

$$R(X, \phi A\xi)A\xi = \varepsilon\big\{-3g(X, A\xi)\phi A\xi - \phi X + g(\phi X, A\xi)A\xi\big\}, \tag{5.6}$$

along with (2.6) and $\phi^2 A\xi = -A\xi$. Substituting (5.6) into (5.5) and using (5.4) again yields

$$\begin{aligned} 0 &= \varepsilon\big\{-3g(X, A\xi)\bar{R}_N\phi A\xi - \bar{R}_N\phi X + g(\phi X, A\xi)\bar{R}_N A\xi\big\} \\ &= -\varepsilon\bar{R}_N\phi X \end{aligned} \tag{5.7}$$

for any tangent vector field X on M. Due to (5.3) and $\varepsilon^2 = 1$, we obtain

$$-\varepsilon\bar{R}_N\phi X = -\big\{\phi X - g(A\xi, \phi X)A\xi - g(\phi A\xi, \phi X)\phi A\xi\big\},$$

which implies that (5.7) can be rewritten as:

$$\phi X - g(A\xi, \phi X)A\xi - g(\phi A\xi, \phi X)\phi A\xi = 0. \tag{5.8}$$

Applying the structure tensor field ϕ from (5.8) yields, for any $X \in TM$:

$$X = \eta(X)\xi + g(\phi A\xi, X)\phi A\xi + g(A\xi, X)A\xi,$$

where we have used $\phi^2 X = -X + \eta(X)\xi$ and $\beta = g(A\xi, \xi) = 0$. According to the construction of $\mathfrak{B}$ for M, it implies that the dimension of M is exactly 3, i.e., $\dim M = 2m - 1 = 3$. Consequently, we obtain $m = 2$. This leads to a contradiction for $m \geq 3$, providing a comprehensive proof of our Theorem 2 in the introduction.

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REALNE HIPERPLOHE S POLUPARALELNIM NORMALNIM JACOBIJEVIM OPERATOROM U REALNIM GRASSMANNOVIM MNOGOSTRUKOSTIMA RANGA DVA

Sažetak. U ovom radu uvodimo pojam poluparalelnog normalnog Jacobijevog operatora za realne hiperplohe u realnim Grassmannovim mnogostrukostima ranga dva, kojeg označavamo s $\mathbb{Q}^m(\varepsilon)$, gdje je $\varepsilon = \pm 1$. Ovdje $\mathbb{Q}^m(\varepsilon)$ predstavlja kompleksnu kvadriku $\mathbb{Q}^m(1) = SO_{m+2}/SO_mSO_2$ za $\varepsilon = 1$ i $\mathbb{Q}^m(-1) = SO^0_{m,2}/SO_mSO_2$ za $\varepsilon = -1$, redom. Općenito, pojam poluparalelnog je slabiji od pojma paralelnog normalnog Jacobijevog operatora. U ovom radu dokazujemo da je jedinično normalno vektorsko polje Hopfove realne hiperplohe u $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$, s poluparalelnim normalnim Jacobijevim operatorom singularno. Štoviše, singularnost normalnog vektorskog polja daje rezultat nepostojanja za Hopfove realne hiperplohe u $\mathbb{Q}^m(\varepsilon)$, $m \geq 3$, koje dopuštaju poluparalelni normalni Jacobijev operator.