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# MINIMAL DYNAMICAL SYSTEMS WITH CLOSED RELATIONS

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Abstract. We introduce dynamical systems $(X,G)$ with closed relations $G$ on compact metric spaces $X$ and discuss different types of minimality of such dynamical systems, all of them generalizing minimal dynamical systems $(X,f)$ with continuous function $f$ on a compact metric space $X$.

## 1. Introduction

In dynamical systems theory, the study of chaotic behaviour of a dynamical system is often based on some topological properties or properties of continuous functions. One of the commonly used properties is the minimality of a dynamical system $(X,f)$ or the minimality of the function $f$. According to [KS], minimal dynamical systems were defined by Birkhoff in 1912 [B] as the systems which have no nontrivial closed subsystems: they are considered to be the most fundamental dynamical systems; see [KS] where more references can be found. Minimal dynamical systems $(X,f)$ (i.e., with a minimal map $f$) have the property that each point moves under iteration of $f$ from one non-empty open set to another. This property has been studied intensively by mathematicians since it is an important property in dynamical system theory.

In this paper, we generalize the notion of topological dynamical systems to topological dynamical systems with closed relations and introduce the notion of minimality of such dynamical systems. A similar generalization of a

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dynamical object was presented in 2004 by Ingram and Mahavier [IM, M] introducing inverse limits of inverse sequences of compact metric spaces $X$ with upper semi-continuous set-valued bonding functions $f$ (their graphs $\Gamma(f)$ are examples of closed relations on $X$ with certain additional properties). These inverse limits provide a valuable extension to the role of inverse limits in the study of dynamical systems and continuum theory. For example, Kennedy and Nall have developed a simple method for constructing families of $\lambda$-dendroids [KN]. Their method involves inverse limits of inverse sequences with upper semi-continuous set-valued functions on closed intervals with simple bonding functions. Such generalizations have proven to be useful (also in applied areas); frequently, when constructing a model for empirical data, continuous (single-valued) functions fall short, and the data are better modelled by upper semi-continuous set-valued functions, or sometimes, even closed relations that are not set-valued functions are required. The Christiano-Harrison model from macroeconomics is one such example [CH]. The study of inverse limits of inverse sequences with upper semi-continuous set-valued functions is rapidly gaining momentum - the recent books by Ingram [I], and by Ingram and Mahavier [IM], give a comprehensive exposition of this research prior to 2012.

Also, several papers on the topic of dynamical systems with (upper semi-continuous) set-valued functions have appeared recently, see [BEGK, BEK, CP, LP, LYY, LWZ, KN, KW, MRT, R, SWS], where more references may be found. However, there is not much known of such dynamical systems and therefore, there are many properties of such set-valued dynamical systems that are yet to be studied. In this paper, we study the minimality of such dynamical systems. We also extend the notion of dynamical systems with (upper semi-continuous) set-valued functions to dynamical systems with closed relations.

We proceed as follows. In the sections that follow Section 2, where basic definitions are given, we discuss the following topics:

1. Minimal dynamical systems with closed relations and invariant sets (Section 3).
2. Minimal dynamical systems with closed relations and forward orbits (Section 4).
3. Minimal dynamical systems with closed relations and omega limit sets (Section 5).
4. Backward minimal dynamical systems with closed relations (Section 6).
5. Minimal dynamical systems with closed relations and backward orbits (Section 7).
6. Minimal dynamical systems with closed relations and alpha limit sets (Section 8).
7. Preserving minimality by topological conjugation (Section 9).

In Sections 3, 4, 5, 7, 8, and 9, we first revisit minimal dynamical systems $(X, f)$ and then, we generalize the asserted property from dynamical systems $(X, f)$ to dynamical systems with closed relations $(X, G)$ by making the identification $(X, f) = (X, \Gamma(f))$. Results about dynamical systems $(X, f)$, presented in the first part of each of the above mentioned sections, are well-known. Their proofs are short, rather straight forward and elementary. Therefore, we omit the proofs. The reader can track these proofs by a little help from S. Kolyada's and L. Snoha's paper "Minimal dynamical systems" [KS], where a wonderful overview of minimal dynamical systems is given, or by E. Akin's book "General Topology of Dynamical Systems" [A], where dynamical systems using closed relations are presented.

## 2. Definitions and notation

In this section, basic definitions and well-known results that are needed later in the paper are presented.

Definition 2.1. *Let $X$ and $Y$ be metric spaces, and let $f : X \to Y$ be a function. We use*

$$\Gamma(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$$

*to denote* the graph of the function $f$.

Definition 2.2. *If $X$ is a compact metric space, then $2^X$ denotes the set of all non-empty closed subsets of $X$.*

Definition 2.3. *Let $X$ be a compact metric space and let $G \subseteq X \times X$ be a relation on $X$. If $G \in 2^{X \times X}$, then we say that $G$ is* a closed relation on $X$.

Definition 2.4. *Let $X$ be a set and let $G$ be a relation on $X$. Then we define*

$$G^{-1} = \{(y, x) \in X \times X \mid (x, y) \in G\}$$

*to be* the inverse relation of the relation $G$ *on $X$.*

Definition 2.5. *Let $X$ be a compact metric space and let $G$ be a closed relation on $X$. Then we call*

$$\star_{i=1}^{m} G = \Big\{(x_1, x_2, \ldots, x_{m+1}) \in \prod_{i=1}^{m+1} X \mid \text{ for each } i \in \{1, 2, \ldots, m\}, \\ (x_i, x_{i+1}) \in G\Big\}$$

*for each positive integer $m$,* the $m$-th Mahavier product of $G$*, and*

$$\star_{i=1}^{\infty} G = \Big\{(x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty} X \mid \text{ for each positive integer } i, (x_i, x_{i+1}) \in G\Big\}$$

the infinite Mahavier product of $G$.

Observation 2.6. *Let $X$ be a compact metric space, let $f : X \to X$ be a continuous function. Then*

$$\star_{n=1}^{\infty}\Gamma(f)^{-1} = \varprojlim(X, f).$$

In this paper, we use various projections that are defined in the following two definitions.

Definition 2.7. *Let $X$ be a metric space. We use $p_1, p_2 : X \times X \to X$ to denote* the standard projections *defined by*

$$p_1(x, y) = x \text{ and } p_2(x, y) = y$$

*for all $(x, y) \in X \times X$.*

Definition 2.8. *Let $X$ be a compact metric space. For each positive integer $k$, we use $\pi_k : \prod_{i=1}^{\infty} X \to X$ to denote* the $k$-th standard projection *from $\prod_{i=1}^{\infty} X$ to $X$, defined by*

$$\pi_k(x_1, x_2, x_3, \ldots, \ldots) = x_k$$

*for each $(x_1, x_2, x_3, \ldots, \ldots) \in \prod_{i=1}^{\infty} X$.*

## 3. Minimal dynamical systems with closed relations

First, we revisit minimal dynamical systems and then, we introduce dynamical systems with closed relations and generalize the notion of minimality of a dynamical system to minimality of dynamical systems with closed relations.

Definition 3.1. *Let $X$ be a compact metric space and let $f : X \to X$ be a continuous function. We say that $(X, f)$ is* a dynamical system.

Definition 3.2. *Let $(X, f)$ be a dynamical system and let $A \subseteq X$. We say that*

1. *$A$ is* $f$-invariant, *if $f(A) \subseteq A$.*
2. *$A$ is* strongly $f$-invariant, *if $f(A) = A$.*

Definition 3.3. *Let $(X, f)$ be a dynamical system. We say that $(X, f)$ is* a minimal dynamical system, *if for each closed subset $A$ of $X$,*

$$A \text{ is } f\text{-invariant} \Longrightarrow A \in \{\emptyset, X\}.$$

First, we state the following well-known result. One can easily prove it by using Zorn's lemma. We leave the proof to the reader.

Theorem 3.4. *Let $(X, f)$ be a dynamical system. The following statements are equivalent.*

1. *$(X, f)$ is a minimal dynamical system.*
2. *For each closed subset $A$ of $X$,*

$$f(A) = A \Longrightarrow A \in \{\emptyset, X\}.$$

Next, we introduce dynamical systems with closed relations. Before we do that, we give Observation 3.5, which will serve as a motivation for the rest of this section.

OBSERVATION 3.5. *Let* $(X, f)$ *be a dynamical system and let* $A \subseteq X$*. The following statements are equivalent:*

1. *$A$ is $f$-invariant.*
2. *For each* $(x, y) \in \Gamma(f)$*,*

$$x \in A \Longrightarrow y \in A.$$

3. *For each* $x \in A$*,*

$$x \in p_1(\Gamma(f)) \Longrightarrow \text{ there is } y \in A \text{ such that } (x, y) \in \Gamma(f).$$

Motivated by Observation 3.5, we introduce two different types of invariant sets with respect to a closed relation on a compact metric space.

DEFINITION 3.6. *Let* $X$ *be a compact metric space and let* $G$ *be a nonempty closed relation on* $X$*. We say that* $(X, G)$ *is* a dynamical system with a closed relation *or, briefly,* a CR-dynamical system.

DEFINITION 3.7. *Let* $(X, G)$ *be a CR-dynamical system and let* $A \subseteq X$*. We say that the set* $A$ *is*

1. 1-invariant in $(X, G)$*, if for each* $x \in A$*,*

$$x \in p_1(G) \Longrightarrow \text{ there is } y \in A \text{ such that } (x, y) \in G.$$

2. $\infty$-invariant in $(X, G)$*, if for each* $(x, y) \in G$*,*

$$x \in A \Longrightarrow y \in A.$$

OBSERVATION 3.8. *Let* $(X, G)$ *be a CR-dynamical system, let* $A$ *be an* $\infty$*-invariant set in* $(X, G)$*, and let* $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \star_{i=1}^{\infty} G$*. If* $x_1 \in A$*, then* $x_k \in A$ *for any positive integer* $k$*.*

OBSERVATION 3.9. *Let* $(X, f)$ *be a dynamical system and let* $A \subseteq X$*. Then* $(X, \Gamma(f))$ *is a CR-dynamical system and by Observation 3.5, the following statements are equivalent.*

1. *The set* $A$ *is* $f$*-invariant.*
2. *The set* $A$ *is* 1*-invariant in* $(X, \Gamma(f))$*.*
3. *The set* $A$ *is* $\infty$*-invariant in* $(X, \Gamma(f))$*.*

Next, we show that every $\infty$-invariant set in $(X, G)$ is also 1-invariant in $(X, G)$.

PROPOSITION 3.10. *Let* $(X, G)$ *be a CR-dynamical system and let* $A \subseteq X$*. If* $A$ *is* $\infty$*-invariant in* $(X, G)$*, then* $A$ *is* 1*-invariant in* $(X, G)$*.*

Proof. Suppose that $A$ is $\infty$-invariant in $(X,G)$. If $A\cap p_1(G)=\emptyset$, then there is nothing to show, so, let $x\in A\cap p_1(G)$ and let $y\in X$ be any point such that $(x,y)\in G$. Such a point exists since $x\in p_1(G)$. Since $x\in A$ and since $A$ is $\infty$-invariant in $(X,G)$, $y\in A$. So, there is a point $y\in A$ such that $(x,y)\in G$. Therefore, $A$ is 1-invariant in $(X,G)$. □

The following example shows that there are CR-dynamical systems $(X,G)$ and subsets $A$ of $X$ such that $A$ is 1-invariant in $(X,G)$ but it is not $\infty$-invariant in $(X,G)$.

Example 3.11. Let $X=[0,1]$ and let $G=([0,1]\times\{\frac{1}{2}\})\cup(\{\frac{1}{2}\}\times[0,1])$, see Figure 1.

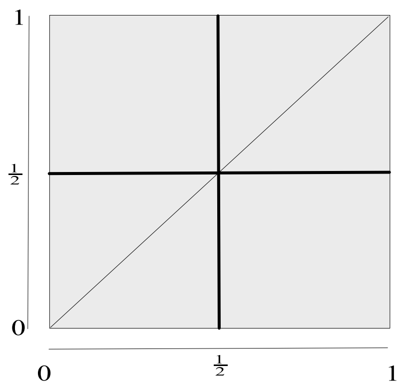


Figure 1. The relation $G$ from Example 3.11

Then $(X,G)$ is a CR-dynamical system. Let $A=\{\frac{1}{2}\}$. Then $A$ is 1-invariant in $(X,G)$ but it is not $\infty$-invariant in $(X,G)$.

Definition 3.12. *Let $(X,G)$ be a CR-dynamical system. We say that*

1. $(X,G)$ *is* 1-minimal *if for each closed subset $A$ of $X$,*
$$A \text{ is 1-invariant in } (X,G) \Longrightarrow A\in\{\emptyset,X\}.$$
2. $(X,G)$ *is* $\infty$-minimal *if for each closed subset $A$ of $X$,*
$$A \text{ is } \infty\text{-invariant in } (X,G) \Longrightarrow A\in\{\emptyset,X\}.$$

Theorem 3.13. *Let $(X,G)$ be a CR-dynamical system. If $(X,G)$ is 1-minimal, then $(X,G)$ is $\infty$-minimal.*

Proof. Let $(X,G)$ be 1-minimal and let $A$ be a non-empty closed subset of $X$ such that $A$ is $\infty$-invariant in $(X,G)$. Then $A$ is 1-invariant in $(X,G)$ by Proposition 3.10. Therefore, $A=X$ and it follows that $(X,G)$ is $\infty$-minimal. □

In the following example, we show that there is a $\infty$-minimal CR-dynamical system which is not 1-minimal.

EXAMPLE 3.14. Let $X = [0,1]$ and let $G = ([0,1] \times \{\frac{1}{2}\}) \cup (\{\frac{1}{2}\} \times [0,1])$, see Figure 1. Then $(X, G)$ is $\infty$-minimal but it is not 1-minimal. Let $A = \{\frac{1}{2}\}$. Then $A$ is 1-invariant in $(X, G)$ but $A \notin \{\emptyset, X\}$. Therefore, $(X, G)$ is not 1-minimal. To see that $(X, G)$ is $\infty$-minimal, let $A$ be a non-empty closed subset of $X$ such that $A$ is $\infty$-invariant in $(X, G)$. Let $x \in A$. Then $(x, \frac{1}{2}) \in G$ and $\frac{1}{2} \in A$ follows. Since $(\frac{1}{2}, t) \in G$, it follows that $t \in A$ for any $t \in X$. Therefore, $A = X$ and it follows that $(X, G)$ is $\infty$-minimal.

## 4. Minimality and forward orbits

First, we revisit forward orbits of dynamical systems $(X, f)$ and then we generalize these to forward orbits of CR-dynamical systems $(X, G)$.

DEFINITION 4.1. *Let $(X, f)$ be a dynamical system and let $x_0 \in X$. The sequence*

$$\mathbf{x} = (x_0, f(x_0), f^2(x_0), f^3(x_0), \dots) \in \star_{i=1}^{\infty}\Gamma(f)$$

*is called* the trajectory of $x_0$. *The set*

$$\mathcal{O}_f^{\oplus}(\mathbf{x}) = \{x_0, f(x_0), f^2(x_0), f^3(x_0), \dots\}$$

*is called* the forward orbit of $x_0$.

First, we state the following well-known result.

THEOREM 4.2. *Let $(X, f)$ be a dynamical system. The following statements are equivalent.*

1. *$(X, f)$ is a minimal dynamical system.*
2. *For each $x \in X$,*

$$\mathrm{Cl}(\mathcal{O}_f^{\oplus}(x)) = X.$$

DEFINITION 4.3. *Let $(X, G)$ be a CR-dynamical system and let $x_0 \in X$. We use $T_G^{+}(x_0)$ to denote the set*

$$T_G^{+}(x_0) = \{\mathbf{x} \in \star_{i=1}^{\infty}G \mid \pi_1(\mathbf{x}) = x_0\} \subseteq \star_{i=1}^{\infty}G.$$

DEFINITION 4.4. *Let $(X, G)$ be a CR-dynamical system, let $\mathbf{x} \in \star_{i=1}^{\infty}G$, and let $x_0 \in X$.*

1. *We say that $\mathbf{x}$ is* a trajectory of $x_0$ in $(X, G)$, *if $\pi_1(\mathbf{x}) = x_0$.*
2. *We use $\mathcal{O}_G^{\oplus}(\mathbf{x})$ to denote the set*

$$\mathcal{O}_G^{\oplus}(\mathbf{x}) = \{\pi_k(\mathbf{x}) \mid k \text{ is a positive integer}\} \subseteq X.$$

*If $\pi_1(\mathbf{x}) = x_0$, then we call this set* a forward orbit of $x_0$.

3. *We use $\mathcal{U}_G^{\oplus}(x_0)$ to denote the set*

$$\mathcal{U}_G^{\oplus}(x_0) = \bigcup_{\mathbf{x} \in T_G^{+}(x_0)} \mathcal{O}_G^{\oplus}(\mathbf{x}) \subseteq X.$$

EXAMPLE 4.5. Let $X = [0,1]$ and consider the graph of the singleton point $G = \{(1,0)\}$. Then $\star_{i=1}^{1} G \neq \emptyset$ and for each $m \neq 1$, $\star_{i=1}^{m} G = \emptyset$. Therefore, in this CR-dynamical system, there are no trajectories in $(X,G)$.

DEFINITION 4.6. *Let $(X,G)$ be a CR-dynamical system. We say that*

1. *$(X,G)$ is* $1^{\oplus}$-minimal *if for each $x \in X$, $T_G^+(x) \neq \emptyset$, and for each $\mathbf{x} \in \star_{i=1}^{\infty} G$,*
$$\mathrm{Cl}\left(\mathcal{O}_G^{\oplus}(\mathbf{x})\right) = X.$$
2. *$(X,G)$ is* $2^{\oplus}$-minimal *if for each $x \in X$ there is $\mathbf{x} \in T_G^+(x)$ such that*
$$\mathrm{Cl}\left(\mathcal{O}_G^{\oplus}(\mathbf{x})\right) = X.$$

THEOREM 4.7. *Let $(X,G)$ be a CR-dynamical system. Then $(X,G)$ is* $2^{\oplus}$-minimal *if and only if for each $x_0 \in X$,*
$$\mathrm{Cl}\left(\mathcal{U}_G^{\oplus}(x_0)\right) = X.$$

PROOF. Suppose that $(X,G)$ is $2^{\oplus}$-minimal. Let $x \in X$ be any point. Since $(X,G)$ is a $2^{\oplus}$-minimal dynamical system, there is a point $\mathbf{x}_0 \in \star_{i=1}^{\infty} G$ such that $\pi_1(\mathbf{x}_0) = x$ and $\mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x}_0)) = X$. It follows from $\mathcal{O}_G^{\oplus}(\mathbf{x}_0) \subseteq \mathcal{U}_G^{\oplus}(x)$ that $\mathrm{Cl}(\mathcal{U}_G^{\oplus}(x)) = X$.

Now, we show the other implication. Let $x_0$ be any point in $X$. We construct a point $\mathbf{x_0} \in T_G^+(x_0)$ with a dense forward orbit set. Since $X$ is a compact metric space, it is also second-countable. Let $\{U_1, U_2, U_3, \ldots\}$ be a countable basis for $X$.

STEP 1: Since $\mathrm{Cl}\left(\mathcal{U}_G^{\oplus}(x_0)\right) = X$, it follows that
$$U_1 \cap \Big(\bigcup_{\mathbf{x} \in T_G^+(x_0)} O_G^{\oplus}(\mathbf{x})\Big) = \bigcup_{\mathbf{x} \in T_G^+(x_0)} (O_G^{\oplus}(\mathbf{x}) \cap U_1) \neq \emptyset.$$
Let $\mathbf{x_1} \in T_G^+(x_0)$ and let $i_1$ be a positive integer such that
$$\pi_{i_1}(\mathbf{x_1}) \in U_1.$$

STEP 2: Since $\mathrm{Cl}\left(\mathcal{U}_G^{\oplus}(x_0)\right) = X$, it follows that
$$U_2 \cap \Big(\bigcup_{\mathbf{x} \in T_G^+(\pi_{i_1}(\mathbf{x_1}))} O_G^{\oplus}(\mathbf{x})\Big) = \bigcup_{\mathbf{x} \in T_G^+(\pi_{i_1}(\mathbf{x_1}))} (O_G^{\oplus}(\mathbf{x}) \cap U_2) \neq \emptyset.$$
Let $\mathbf{x_2} \in T_G^+(x_1)$ and let $i_2$ be a positive integer such that
$$\pi_{i_2}(\mathbf{x_2}) \in U_2.$$

STEP K: Since $\mathrm{Cl}\left(\mathcal{U}_G^{\oplus}(x_0)\right) = X$, it follows that
$$U_k \cap \Big(\bigcup_{\mathbf{x} \in T_G^+(\pi_{i_{k-1}}(\mathbf{x_{k-1}}))} O_G^{\oplus}(\mathbf{x})\Big) = \bigcup_{\mathbf{x} \in T_G^+(\pi_{i_{k-1}}(\mathbf{x_{k-1}}))} (O_G^{\oplus}(\mathbf{x}) \cap U_k) \neq \emptyset.$$

Let $\mathbf{x_k} \in T_G^+(x_{k-1})$ and let $i_k$ be a positive integer such that

$$\pi_{i_k}(\mathbf{x_k}) \in U_k.$$

We repeat these steps inductively to obtain the sequence of points $(\mathbf{x}_n)$ and the sequence $(i_n)$ of positive integers such that for each positive integer $n$,

1. $\mathbf{x_n} \in T_G^+(\pi_{i_{n-1}}(\mathbf{x_{n-1}}))$ and
2. $\pi_{i_n}(\mathbf{x_n}) \in U_n$.

Let

$$\mathbf{x_0} = (x_1, x_2, x_3, \dots)$$

be such a point that

$$(x_1, x_2, x_3, \dots, x_{i_1}) = (\pi_1(\mathbf{x_1}), \pi_2(\mathbf{x_1}), \pi_3(\mathbf{x_1}), \dots, \pi_{i_1}(\mathbf{x_1}))$$

and for each positive integer $k$,

$$\begin{aligned}&(x_{i_k}, x_{i_k+1}, x_{i_k+3}, \dots, x_{i_{k+1}})\\ &\qquad = (\pi_1(\mathbf{x}_{k+1}), \pi_2(\mathbf{x}_{k+1}), \pi_3(\mathbf{x}_{k+1}), \dots, \pi_{i_{k+1}}(\mathbf{x}_{k+1})).\end{aligned}$$

It is clear that $\mathbf{x_0} \in T_G^+(x)$ and $\mathbf{x_0}$ has a dense forward orbit set in $X$, which concludes the proof. □

Theorem 4.8. *Let $(X, G)$ be a CR-dynamical system. Then the following holds.*

1. *$(X, G)$ is 1-minimal if and only if $(X, G)$ is $1^{\oplus}$-minimal.*
2. *If $(X, G)$ is $1^{\oplus}$-minimal, then $(X, G)$ is $2^{\oplus}$-minimal.*
3. *If $(X, G)$ is $2^{\oplus}$-minimal, then $(X, G)$ is $\infty$-minimal.*

Proof. Let $(X, G)$ be a 1-minimal CR-dynamical system. To prove that $(X, G)$ is $1^{\oplus}$-minimal, let $x \in X$. To prove that $T_G^+(x) \neq \emptyset$, we show first that $p_2(G) \subseteq p_1(G)$. Suppose that $p_2(G) \not\subseteq p_1(G)$ and let $x_0 \in p_2(G) \setminus p_1(G)$. Then $A = \{x_0\}$ is trivially 1-invariant in $(X, G)$—a contradiction since $A \neq X$. Therefore, $p_2(G) \subseteq p_1(G)$. Next, we prove that $p_2(G) = X$. Let $A = p_2(G)$ and let $x \in A$ be any point. Since $A \subseteq p_1(G)$, it follows that $x \in p_1(G)$. Then there is $y \in p_2(G)$ such that $(x, y) \in G$. This proves that $A$ is 1-invariant in $(X, G)$. Since $A$ is closed in $X$ and $A \neq \emptyset$, it follows that $A = X$ since $(X, G)$ is 1-minimal. Therefore, $p_2(G) = X$. Also, $p_1(G) = X$ follows since $p_2(G) \subseteq p_1(G)$. Since $p_1(G) = X$, there is a point $\mathbf{x} \in \star_{i=1}^{\infty} G$ such that $\pi_1(\mathbf{x}) = x$ and $T_G^+(x) \neq \emptyset$. This completes the proof that $T_G^+(x) \neq \emptyset$.

Next, let $\mathbf{x} \in \star_{i=1}^{\infty} G$. We show that $\mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x})) = X$. Let $A = \mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x}))$. Then $A$ is a non-empty closed subset of $X$. Let $x \in A$ such that $x \in p_1(G)$. We consider the following possible cases.

(i) $x \in \mathcal{O}_G^{\oplus}(\mathbf{x})$. Let $m$ be a positive integer such that $\pi_m(\mathbf{x}) = x$ and let $y = \pi_{m+1}(\mathbf{x})$. Then $y \in A$ and $(x, y) \in G$.

(ii) $x \notin \mathcal{O}^{\oplus}_G(\mathbf{x})$. Let $(z_n)$ be a sequence of points in $\mathcal{O}^{\oplus}_G(\mathbf{x})$ such that $\lim_{n\to\infty} z_n = x$. For each positive integer $n$, let $i_n$ be a positive integer such that $\pi_{i_n}(\mathbf{x}) = z_n$. For each positive integer $n$, let $y_n = \pi_{i_n+1}(\mathbf{x})$, and let $(y_{j_n})$ be a convergent subsequence of the sequence $(y_n)$ and let $\lim_{n\to\infty} y_{j_n} = y$. Note that for each positive integer $n$, $(z_{j_n}, y_{j_n}) \in G$ . Since $G$ is closed in $X \times X$ and since $\lim_{n\to\infty}(z_{j_n}, y_{j_n}) = (x, y)$, it follows that $(x, y) \in G$. Since $A$ is closed in $X$ and since $y_{i_n} \in \mathcal{O}^{\oplus}_G(\mathbf{x})$ for each positive integer $n$, it follows from $\mathcal{O}^{\oplus}_G(\mathbf{x}) \subseteq A$ that $y \in A$.

We proved that there is $y \in A$ such that $(x, y) \in G$. It follows that $A$ is 1-invariant in $(X, G)$ and, therefore, $A = X$. This proves that $\mathrm{Cl}(\mathcal{O}^{\oplus}_G(\mathbf{x})) = X$ and it follows that $(X, G)$ is $1^{\oplus}$-minimal. This proves the first implication of 1. To prove the other implication, suppose that $(X, G)$ is $1^{\oplus}$-minimal and let $A$ be a non-empty closed subset of $X$ which is 1-invariant in $(X, G)$. Let $a_1 \in A$ be any point. Since $T^+_G(a_1) \neq \emptyset$, there is $\mathbf{x}_1 \in T^+_G(a_1)$. Choose such an element $\mathbf{x}_1 \in T^+_G(a_1)$ and set $x = \pi_2(\mathbf{x}_1)$. Then $(a_1, x) \in G$ and $a_1 \in p_1(G)$ follows. Since $A$ is 1-invariant in $(X, G)$, there is a point $a_2 \in A$ such that $(a_1, a_2) \in G = \star^1_{i=1} G$. Fix such a point $a_2$. Let $n > 1$ be a positive integer and suppose that we have already constructed the points $a_1, a_2, a_3, \ldots, a_n \in A$ such that $(a_1, a_2, a_3, \ldots, a_n) \in \star^{n-1}_{i=1} G$. Since $T^+_G(a_n) \neq \emptyset$, there is $\mathbf{x}_n \in T^+_G(a_n)$. Choose such an element $\mathbf{x}_n \in T^+_G(a_n)$ and set $x = \pi_2(\mathbf{x}_n)$. Then $(a_n, x) \in G$ and $a_n \in p_1(G)$ follows. Since $A$ is 1-invariant in $(X, G)$, there is a point $a_{n+1} \in A$ such that $(a_n, a_{n+1}) \in G$. Fix such a point $a_{n+1}$. Let $\mathbf{x} = (a_1, a_2, a_3, \ldots)$. Then $\mathbf{x} \in \star^{\infty}_{i=1} G$ and $\mathrm{Cl}(\mathcal{O}^{\oplus}_G(\mathbf{x})) = X$ and, since $\mathrm{Cl}(\mathcal{O}^{\oplus}_G(\mathbf{x})) \subseteq A$, it follows that $A = X$. This proves that $(X, G)$ is 1-minimal and we have just proved 1.

To prove 2. suppose that $(X, G)$ is $1^{\oplus}$-minimal. Let $x \in X$ be any point. Since $(X, G)$ is $1^{\oplus}$-minimal, there is a point $\mathbf{x} \in \star^{\infty}_{i=1} G$ such that $\pi_1(\mathbf{x}) = x$. Since $(X, G)$ is $1^{\oplus}$-minimal, it follows that $\mathrm{Cl}(\mathcal{O}^{\oplus}_G(\mathbf{x})) = X$. Therefore, $(X, G)$ is $2^{\oplus}$-minimal.

Finally, to prove 3., suppose first, that $(X, G)$ is $2^{\oplus}$-minimal. Let $A$ be a non-empty closed subset of $X$ such that $A$ is $\infty$-invariant in $(X, G)$. Let $x \in A$. Since $(X, G)$ is $2^{\oplus}$-minimal, it follows from Theorem 4.7 that $\mathrm{Cl}(\mathcal{U}^{\oplus}_G(x)) = X$. We show that $A = X$ by showing that $\mathrm{Cl}(\mathcal{U}^{\oplus}_G(x)) \subseteq A$. First, we show that $\mathcal{U}^{\oplus}_G(x) \subseteq A$. Let $y \in \mathcal{U}^{\oplus}_G(x)$ and let $\mathbf{x}_0 \in T^+_G(x)$ such that $y \in \mathcal{O}^{\oplus}_G(\mathbf{x}_0)$. Since $x \in A$ and since $A$ is $\infty$-invariant in $(X, G)$, it follows that $y \in A$ by Observation 3.8. Therefore, $\mathcal{U}^{\oplus}_G(x) \subseteq A$ and, since $A$ is closed in $X$, it follows that $\mathrm{Cl}(\mathcal{U}^{\oplus}_G(x)) \subseteq A$. □

Among other things, the following theorem says that 1-, $1^{\oplus}$-, $2^{\oplus}$-, and $\infty$-minimality of CR-dynamical systems is a generalization of the notion of the minimality of dynamical systems.

THEOREM 4.9. *Let $(X, f)$ be a dynamical system. The following statements are equivalent.*

1. *$(X, f)$ is minimal.*
2. *$(X, \Gamma(f))$ is 1-minimal.*
3. *$(X, \Gamma(f))$ is $1^{\oplus}$-minimal.*
4. *$(X, \Gamma(f))$ is $2^{\oplus}$-minimal.*
5. *$(X, \Gamma(f))$ is $\infty$-minimal.*

PROOF. Suppose that $(X, f)$ is minimal. To prove that $(X, \Gamma(f))$ is 1-minimal, let $A$ be a closed subset of $X$ such that $A$ is 1-invariant in $(X, \Gamma(f))$. By Observation 3.9, $A$ is $f$-invariant in $(X, \Gamma(f))$. Therefore, $A \in \{\emptyset, X\}$ since $(X, f)$ is minimal. This proves the implication from 1. to 2.

The implications from 2. to 3., from 3. to 4. and from 4. to 5. follow from Theorem 4.8.

Suppose that $(X, \Gamma(f))$ is $\infty$-minimal. To prove that $(X, f)$ is minimal, let $A$ be a closed subset of $X$ such that $A$ is $f$-invariant. By Observation 3.9, $A$ is $\infty$-invariant in $(X, \Gamma(f))$. Therefore, $A \in \{\emptyset, X\}$ since $(X, \Gamma(f))$ is $\infty$-minimal. This proves the implication from 5. to 1. □

THEOREM 4.10. *Let $(X, G)$ be a CR-dynamical system and let $k$ be any element from $\{1, 1^{\oplus}, 2^{\oplus}, \infty\}$. If $(X, G)$ is $k$-minimal, then*

$$p_1(G) = p_2(G) = X.$$

PROOF. Suppose that $(X, G)$ is $\infty$-minimal. First, we show that $p_2(G) \subseteq p_1(G)$. Suppose that $p_2(G) \not\subseteq p_1(G)$ and let $x_0 \in p_2(G) \setminus p_1(G)$. Then $A = \{x_0\}$ is trivially $\infty$-invariant in $(X, G)$—a contradiction. Therefore, $p_2(G) \subseteq p_1(G)$.

Next, we prove that $p_2(G) = X$. Let $A = p_2(G)$ and let $(x, y) \in G$ be any point such that $x \in A$. Since $A \subseteq p_1(G)$, it follows that $y \in p_2(G)$, meaning that $y \in A$. This proves that $A$ is $\infty$-invariant in $(X, G)$. Since $A$ is closed in $X$ and $A \neq \emptyset$, it follows that $A = X$ since $(X, G)$ is $\infty$-minimal. Therefore, $p_2(G) = X$. Also, $p_1(G) = X$ follows since $p_2(G) \subseteq p_1(G)$.

Next, let $k \in \{1, 1^{\oplus}, 2^{\oplus}\}$ and suppose that $(X, G)$ is $k$-minimal. It follows from Theorem 4.8 that $(X, G)$ is also $\infty$-minimal. Therefore, $p_1(G) = p_2(G) = X$. □

In the following example, we show that there is a $2^{\oplus}$-minimal CR-dynamical system which is not $1^{\oplus}$-minimal.

EXAMPLE 4.11. Let $X = [0, 1]$ and let $G = ([0, 1] \times \{\frac{1}{2}\}) \cup (\{\frac{1}{2}\} \times [0, 1])$, see Figure 1. To show that $(X, G)$ is not $1^{\oplus}$-minimal, let $\mathbf{x} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots) \in \star_{i=1}^{\infty} G$. Then $\mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x})) \neq X$. Therefore, $(X, G)$ is not $1^{\oplus}$-minimal.

To show that $(X, G)$ is $2^{\oplus}$-minimal, let $x \in X$ be any point. We show that there is $\mathbf{x} \in T_G^{+}(x)$ such that $\mathrm{Cl}\left(\mathcal{O}_G^{\oplus}(\mathbf{x})\right) = X$. Let $[0, 1] \cap \mathbb{Q} = \{q_1, q_2, q_3, \ldots\}$

be the set of rationals in $[0,1]$, let $x_1 = x$, for each positive integer $n$, let $x_{2n} = \frac{1}{2}$ and $x_{2n+1} = q_n$, and let $\mathbf{x} = (x_1, x_2, x_3, \ldots)$. Then $\mathbf{x} \in T_G^+(x)$ such that $\mathrm{Cl}\left(\mathcal{O}_G^{\oplus}(\mathbf{x})\right) = X$.

In the following example, we show that there is a $\infty$-minimal CR-dynamical system which is not $2^{\oplus}$-minimal.

EXAMPLE 4.12. Let $X = [0,1]$ and let $G$ be the union of the following line segments:

1. the line segment with endpoints $(0, \frac{1}{2})$ and $(1,1)$,
2. the line segment with endpoints $(1,0)$ and $(1,1)$,

see Figure 2.

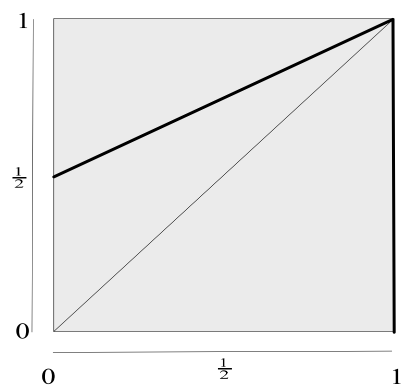


FIGURE 2. The relation $G$ from Example 4.12

To show that $(X,G)$ is not $2^{\oplus}$-minimal, let $x_0 = 0$. Then

$$\mathrm{Cl}(\mathcal{U}_G^{\oplus}(0)) = \mathrm{Cl}\left(\left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\right\}\right) = \left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\right\} \cup \{1\} \neq X.$$

Therefore, $(X,G)$ is not $2^{\oplus}$-minimal by Theorem 4.7.

To show that $(X,G)$ is $\infty$-minimal, let $A$ be a non-empty closed subset of $X$ such that $A$ is $\infty$-invariant in $(X,G)$. First, we show that $1 \in A$. Since $A \neq \emptyset$, it follows that there is $x \in A$. Choose any element $x$ in $A$. If $x = 1$, we are done. Suppose that $x < 1$ and let $f : [0,1] \to [0,1]$ be defined by $f(t) = \frac{1}{2}t + \frac{1}{2}$ for each $t \in [0,1]$. Note that the graph of $f$ is the line segment from $(0, \frac{1}{2})$ to $(1,1)$. Since $A$ is $\infty$-invariant in $(X,G)$, it follows from Observation 3.8 that

$$\mathcal{O}_f^{\oplus}(x) = \{x, f(x), f^2(x), f^3(x), \ldots\} \subseteq A.$$

Since $A$ is closed in $X$, it follows that

$$\mathrm{Cl}(\mathcal{O}_f^{\oplus}(x)) = \{x, f(x), f^2(x), f^3(x), \ldots\} \cup \{1\} \subseteq A.$$

Therefore, $1 \in A$. Next, let $y \in X$ be any point. Then $(1, y) \in G$ and since $1 \in A$, it follows from the fact that $A$ is $\infty$-invariant in $(X, G)$, that $y \in A$. Therefore, $A = X$.

## 5. Minimality and omega limit sets

Theorem 5.5, where results about relations of omega limits sets in CR-dynamical systems $(X, G)$ and minimality are presented, is the main result of this section. First, we revisit omega limit sets in dynamical systems $(X, f)$.

Definition 5.1. *Let $(X, f)$ be a dynamical system and let $x_0 \in X$ and let $\mathbf{x} \in T^+_{\Gamma(f)}(x_0)$ be the trajectory of $x_0$. The set*

$$\omega_f(x_0) = \{x \in X \mid \text{ there is a subsequence of the sequence } \mathbf{x} \text{ with limit } x\}$$

*is called* the omega limit set of $x_0$.

The following is a well-known result.

Theorem 5.2. *Let $(X, f)$ be a dynamical system. The following statements are equivalent.*

1. *$(X, f)$ is minimal.*
2. *For each $x \in X$, $\omega_f(x) = X$.*

Next, we generalize the notion of omega limit sets from dynamical systems to CR-dynamical systems.

Definition 5.3. *Let $(X, G)$ be a CR-dynamical system and let $\mathbf{x} \in \star_{i=1}^{\infty} G$. The set*

$$\omega_G(\mathbf{x}) = \{x \in X \mid \text{ there is a subsequence of the sequence } \mathbf{x} \text{ with limit } x\}$$

*is called* the omega limit set of $\mathbf{x}$. *For each $x \in X$, we use $\psi_G(x)$ to denote the set*

$$\psi_G(x) = \bigcup_{\mathbf{x} \in T_G^+(x)} \omega_G(\mathbf{x}).$$

Observation 5.4. *Note that for each $\mathbf{x} \in \star_{i=1}^{\infty} G$, $\omega_G(\mathbf{x}) \subseteq \mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x}))$.*

Theorem 5.5. *Let $(X, G)$ be a CR-dynamical system. Then the following hold.*

1. *$(X, G)$ is $1^{\oplus}$-minimal if and only if for each $x \in X$, $T_G^+(x) \neq \emptyset$, and for each $\mathbf{x} \in \star_{i=1}^{\infty} G$,*
$$\omega_G(\mathbf{x}) = X.$$
2. *$(X, G)$ is $2^{\oplus}$-minimal if and only if for each $x \in X$ there is $\mathbf{x} \in T_G^+(x)$ such that*
$$\omega_G(\mathbf{x}) = X.$$

*3. $(X,G)$ is $2^{\oplus}$-minimal if and only if for each $x\in X$,*

$$\psi_G(x)=X.$$

PROOF. To prove 1., first suppose that $(X,G)$ is $1^{\oplus}$-minimal. Clearly, for each $x\in X$, $T_G^+(x)\neq\emptyset$. Let $\mathbf{x}\in\star_{i=1}^{\infty}G$. Obviously, $\omega_G(\mathbf{x})\subseteq X$. To prove that $X\subseteq\omega_G(\mathbf{x})$, let $x\in X$. To show that $x\in\omega_G(\mathbf{x})$, we treat the following possible cases.

(i) $x\notin\mathcal{O}_G^{\oplus}(\mathbf{x})$. Since $\mathcal{O}_G^{\oplus}(\mathbf{x})$ is dense in $X$, it follows that for any open set $U$ in $X$,

$$U\neq\emptyset\Longrightarrow U\cap\mathcal{O}_G^{\oplus}(\mathbf{x})\neq\emptyset.$$

Therefore, for any open set $U$ in $X$,

$$x\in U\Longrightarrow U\cap\mathcal{O}_G^{\oplus}(\mathbf{x})\neq\emptyset$$

and since $x\notin\mathcal{O}_G^{\oplus}(\mathbf{x})$, it follows that for any open set $U$ in $X$,

$$x\in U\Longrightarrow (U\setminus\{x\})\cap\mathcal{O}_G^{\oplus}(\mathbf{x})\neq\emptyset.$$

Therefore, $x$ is a limit point of the sequence $\mathbf{x}$ and $x\in\omega_G(\mathbf{x})$ follows.

(ii) $x\in\mathcal{O}_G^{\oplus}(\mathbf{x})$. Suppose that $x\notin\omega_G(\mathbf{x})$. Then there is an open set $U$ in $X$ such that $U\cap\mathcal{O}_G^{\oplus}(\mathbf{x})=\{x\}$ and $\pi_k(\mathbf{x})=x$ only for finitely many positive integers $k$. Let

$$n_0=\max\{n\mid n\text{ is a positive integer such that }\pi_n(\mathbf{x})=x\}$$

and let

$$\mathbf{y}=(\pi_{n_0+1}(\mathbf{x}),\pi_{n_0+2}(\mathbf{x}),\pi_{n_0+3}(\mathbf{x}),\ldots).$$

Then $U\cap\mathcal{O}_G^{\oplus}(\mathbf{y})=\emptyset$—a contradiction since $(X,G)$ is $1^{\oplus}$-minimal and, therefore, $\mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{y}))=X$. It follows that $x\in\omega_G(\mathbf{x})$.

Next, suppose that for each $x\in X$, $T_G^+(x)\neq\emptyset$, and for each $\mathbf{x}\in\star_{i=1}^{\infty}G$,

$$\omega_G(\mathbf{x})=X.$$

Therefore, for each $x\in X$, $T_G^+(x)\neq\emptyset$, and by Observation 5.4, for each $\mathbf{x}\in\star_{i=1}^{\infty}G$,

$$X=\omega_G(\mathbf{x})\subseteq\mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x}))\subseteq X$$

and $\mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x}))=X$ follows. This completes the proof of 1.

Next, we prove 2. Suppose that for each $x\in X$ there is $\mathbf{x}\in T_G^+(x)$ such that

$$\omega_G(\mathbf{x})=X.$$

To show that $(X,G)$ is $2^{\oplus}$-minimal, let $x_0\in X$ be any point and let $\mathbf{x}_0\in T_G^+(x_0)$ be such that $\omega_G(\mathbf{x}_0)=X$. By Observation 5.4,

$$X=\omega_G(\mathbf{x}_0)\subseteq\mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x}_0))\subseteq X.$$

Therefore, $\mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x}_0))=X$. This completes the proof of one implication of 2.

Next, suppose that $(X,G)$ is $2^{\oplus}$-minimal and let $x \in X$ be any point. We will construct $\mathbf{x} \in T_G^+(x)$ such that $\omega_G(\mathbf{x}) = X$. Let

$$\mathbf{x}_1 = (x_1^1, x_2^1, x_3^1, \ldots) \in T_G^+(x)$$

such that $\mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x}_1)) = X$. For each positive integer $n$, let $\ell_n$ be a positive integer and let $y_1^n, y_2^n, y_3^n, \ldots, y_{\ell_n}^n \in X$, such that

$$\mathcal{U}_n = \Big\{B\Big(y_1^n, \frac{1}{n}\Big), B\Big(y_2^n, \frac{1}{n}\Big), B\Big(y_2^n, \frac{1}{n}\Big), \ldots, B\Big(y_{\ell_n}^n, \frac{1}{n}\Big)\Big\}$$

is a finite open cover for $X$. We follow the following steps.

STEP 1. Let $m_1$ be a positive integer such that for each $i \in \{1,2,3,\ldots,\ell_1\}$,

$$\{x_1^1, x_2^1, x_3^1, \ldots, x_{m_1}^1\} \cap B(y_i^1, 1) \neq \emptyset.$$

STEP 2. Let

$$\mathbf{x}_2 = (x_1^2, x_2^2, x_3^2, \ldots) \in T_G^+(x_{m_1}^1)$$

and let $m_2$ be a positive integer such that for each $i \in \{1,2,3,\ldots,\ell_2\}$,

$$\{x_1^2, x_2^2, x_3^2, \ldots, x_{m_2}^2\} \cap B\Big(y_i^2, \frac{1}{2}\Big) \neq \emptyset.$$

STEP 3. Let

$$\mathbf{x}_3 = (x_1^3, x_2^3, x_3^3, \ldots) \in T_G^+(x_{m_2}^2)$$

and let $m_3$ be a positive integer such that for each $i \in \{1,2,3,\ldots,\ell_3\}$,

$$\{x_1^3, x_2^3, x_3^3, \ldots, x_{m_3}^3\} \cap B\Big(y_i^3, \frac{1}{3}\Big) \neq \emptyset.$$

We continue inductively. For each positive integer $j$, the step $j$ is as follows.

STEP $j$. Let

$$\mathbf{x}_j = (x_1^j, x_2^j, x_3^j, \ldots) \in T_G^+(x_{m_{j-1}}^{j-1})$$

and let $m_j$ be a positive integer such that for each $i \in \{1,2,3,\ldots,\ell_j\}$,

$$\{x_1^j, x_2^j, x_3^j, \ldots, x_{m_j}^j\} \cap B\Big(y_i^j, \frac{1}{j}\Big) \neq \emptyset.$$

Finally, let

$$\mathbf{x} = (x_1^1, x_2^1, x_3^1, \ldots, x_{m_1}^1 = x_1^2, x_2^2, x_3^2, \ldots, x_{m_2}^2 = x_1^3, x_2^3, x_3^3, \ldots, x_{m_3}^3, \ldots).$$

Then $\mathbf{x} \in T_G^+(x)$ such that $\omega_G(\mathbf{x}) = X$.

Finally, we prove 3. Suppose that for each $x \in X$, $\psi_G(x) = X$. To prove that $(X,G)$ is $2^{\oplus}$-minimal, let $x_0 \in X$ be any point and we show that $\mathrm{Cl}(\mathcal{U}_G^{\oplus}(x_0)) = X$. Obviously, $\mathrm{Cl}(\mathcal{U}_G^{\oplus}(x_0)) \subseteq X$. To show that $X \subseteq \mathrm{Cl}(\mathcal{U}_G^{\oplus}(x_0))$, let $x \in X$. Then $x \in \psi_G(x_0)$. Let $\mathbf{x}_0 \in T_G^+(x_0)$ such that $x \in \omega_G(\mathbf{x}_0)$. Since $\omega_G(\mathbf{x}_0) \subseteq \mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x}_0))$, it follows that $x \in \mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x}_0))$. Since $\mathcal{O}_G^{\oplus}(\mathbf{x}_0) \subseteq \mathcal{U}_G^{\oplus}(x_0)$, it follows that $\mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x}_0)) \subseteq \mathrm{Cl}(\mathcal{U}_G^{\oplus}(x_0))$ and, therefore,

$$x \in \mathrm{Cl}(\mathcal{U}_G^{\oplus}(x_0)).$$

Now suppose $(X, G)$ is $2^{\oplus}$-minimal and let $x \in X$ be any point. By 2. it follows that there is $\mathbf{x} \in T_G^+(x)$ such that $\omega_G(\mathbf{x}) = X$. Now it follows that $\psi_G(x) = X$. □

Observation 5.6. *Let $(X, f)$ be a dynamical system, let $x \in X$ and let $\mathbf{x} \in T^+_{\Gamma(f)}(x)$ be the trajectory of $x$. Then*

$$\omega_f(x) = \omega_{\Gamma(f)}(\mathbf{x}) = \psi_{\Gamma(f)}(x)$$

*so $(X, f)$ is minimal if and only if $\omega_{\Gamma(f)}(\mathbf{x}) = \psi_{\Gamma(f)}(x) = X$.*

## 6. Backward minimal dynamical systems with closed relations

In this section, we define backward dynamical systems with closed relations.

Definition 6.1. *Let $(X, G)$ be a CR-dynamical system and let $A \subseteq X$. We say that the set $A$ is*

1. 1-backward invariant in $(X, G)$, *if for each $y \in A$,*

$$y \in p_2(G) \Longrightarrow \text{ there is } x \in A \text{ such that } (x, y) \in G.$$

2. $\infty$-backward invariant in $(X, G)$, *if for each $(x, y) \in G$,*

$$y \in A \Longrightarrow x \in A.$$

Observation 6.2. *Let $(X, G)$ be a CR-dynamical system and let $A \subseteq X$. Note that*

1. *$A$ is 1-backward invariant in $(X, G)$ if and only if $A$ is 1-invariant in $(X, G^{-1})$.*
2. *$A$ is $\infty$-backward invariant in $(X, G)$ if and only if $A$ is $\infty$-invariant in $(X, G^{-1})$.*

Proposition 6.3. *Let $(X, G)$ be a CR-dynamical system and let $A \subseteq X$. If $A$ is $\infty$-backward invariant in $(X, G)$, then $A$ is 1-backward invariant in $(X, G)$.*

Proof. The proposition follows from Proposition 3.10 and Observation 6.2. □

Example 6.4. Let $X = [0, 1]$ and let $G = ([0, 1] \times \{\frac{1}{2}\}) \cup (\{\frac{1}{2}\} \times [0, 1])$, see Figure 1. The set $A = \{\frac{1}{2}\}$ is 1-backward invariant in $(X, G)$ but it is not $\infty$-backward invariant in $(X, G)$.

Definition 6.5. *Let $(X, G)$ be a CR-dynamical system. We say that*

1. *$(X, G)$ is* 1-backward minimal *if for each closed subset $A$ of $X$,*

$$A \text{ is 1-backward invariant in } (X, G) \Longrightarrow A \in \{\emptyset, X\}.$$

2. *$(X, G)$ is* $\infty$-backward minimal *if for each closed subset $A$ of $X$,*

$$A \text{ is } \infty\text{-backward invariant in } (X, G) \Longrightarrow A \in \{\emptyset, X\}.$$

OBSERVATION 6.6. *Let $(X, G)$ be a CR-dynamical system and let $k \in \{1, \infty\}$. Then the following holds.*

$$(X, G) \text{ is } k\text{-backward minimal } \iff (X, G^{-1}) \text{ is } k\text{-minimal.}$$

THEOREM 6.7. *Let $(X, G)$ be a CR-dynamical system. If $(X, G)$ is 1-backward minimal, then $(X, G)$ is $\infty$-backward minimal.*

PROOF. Let $(X, G)$ be 1-backward minimal and let $A$ be a non-empty closed subset of $X$ such that $A$ is $\infty$-backward invariant in $(X, G)$. Then $A$ is 1-backward invariant in $(X, G)$ by Proposition 6.3. Therefore, $A = X$ and it follows that $(X, G)$ is $\infty$-backward minimal. □

Note that Example 3.14 is an example of a $\infty$-backward minimal CR-dynamical system which is not 1-backward minimal. In Theorem 7.11, we show (using backward orbits that are defined in Section 7) that for a CR-dynamical system $(X, G)$, the following holds:

$$(X, G) \text{ is 1-backward minimal} \iff (X, G) \text{ is 1-minimal.}$$

The following example gives a CR-dynamical system which is $\infty$-minimal but is not $\infty$-backward minimal.

EXAMPLE 6.8. Let $X = [0, 1]$ and let $G$ be the union of the following line segments:

1. the line segment with endpoints $(0, \frac{1}{2})$ and $(1, 1)$,
2. the line segment with endpoints $(1, 0)$ and $(1, 1)$,

see Figure 2. We proved that $(X, G)$ is $\infty$-minimal in Example 4.12.

To show that $(X, G)$ is not $\infty$-backward minimal, let

$$A = \left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\right\} \cup \{1\}.$$

Then $A$ is a non-empty closed subset of $X$. Let $(x, y) \in G$ such that $y \in A$. If $y = 0$, then $x = 1$ and, therefore $x \in A$. If $y = \frac{1}{2}$, then $x = 0$ and, therefore $x \in A$. If $y = 1$, then $x = 1$ and, therefore $x \in A$. If $y = \frac{2^{n+1}-1}{2^{n+1}}$ for some positive integer $n$, then $x = \frac{2^n-1}{2^n}$ or $x = 1$, therefore, $x \in A$. This proves that $A$ is $\infty$-backward invariant. Since $A \neq X$, it follows that $(X, G)$ is not $\infty$-backward minimal.

The following example gives a CR-dynamical system which is $\infty$-backward minimal but is not $\infty$-minimal.

EXAMPLE 6.9. Let $X = [0, 1]$ and let $H$ be the union of the following line segments:

1. the line segment with endpoints $(0, \frac{1}{2})$ and $(1, 1)$,
2. the line segment with endpoints $(1, 0)$ and $(1, 1)$,

and let $G = H^{-1}$. Then, using a similar approach as in Example 6.8, one can easily prove that $(X, G)$ is $\infty$-backward minimal but is not $\infty$-minimal.

## 7. Minimality and backward orbits

First, we visit the dynamical systems and revisit a well-known result saying that a dynamical system $(X,f)$ is minimal if and only if $f$ is surjective and every backward orbit in $(X,f)$ is dense in $X$.

Definition 7.1. *Let $(X,f)$ be a dynamical system and let $x_0 \in X$. We use $T_f^-(x_0)$ to denote the set*

$$T_f^-(x_0) = \{\mathbf{x} \in \star_{i=1}^{\infty}\Gamma(f)^{-1} \mid \pi_1(\mathbf{x}) = x_0\}.$$

Definition 7.2. *Let $(X,f)$ be a dynamical system, let $x_0 \in X$ be any point and let*

$$\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \star_{i=1}^{\infty}\Gamma(f)^{-1}.$$

*The sequence $\mathbf{x}$ is called* a backward trajectory of $x_0$, *if $\pi_1(\mathbf{x}) = x_0$. We use $\mathcal{O}_f^{\ominus}(\mathbf{x})$ to denote the set*

$$\mathcal{O}_f^{\ominus}(\mathbf{x}) = \{x_1, x_2, x_3, \ldots\}.$$

*If $\mathbf{x}$ is a backward trajectory of $x_0$, then $\mathcal{O}_f^{\ominus}(\mathbf{x})$ is called* a backward orbit of $x_0$.

The following is a well-known result, see [KST, Ma] for more details.

Theorem 7.3. *Let $(X,f)$ be a dynamical system. The following statements are equivalent.*

1. *$(X,f)$ is minimal.*
2. *For each $x \in X$, $T_f^-(x) \neq \emptyset$ and for each $\mathbf{x} \in \star_{i=1}^{\infty}\Gamma(f)^{-1}$,*

$$\mathrm{Cl}(\mathcal{O}_f^{\ominus}(\mathbf{x})) = X.$$

Next, we generalize the notion of backward orbits in $(X,f)$ to the notion of backward orbits in $(X,G)$.

Definition 7.4. *Let $(X,G)$ be a CR-dynamical system and let $x_0 \in X$. We use $T_G^-(x_0)$ to denote the set*

$$T_G^-(x_0) = \{\mathbf{x} \in \star_{i=1}^{\infty}G^{-1} \mid \pi_1(\mathbf{x}) = x_0\}.$$

Definition 7.5. *Let $(X,G)$ be a CR-dynamical system, let $\mathbf{x} \in \star_{i=1}^{\infty}G^{-1}$, and let $x_0 \in X$.*

1. *We say that $\mathbf{x}$ is* a backward trajectory of $x_0$ in $(X,G)$, *if $\pi_1(\mathbf{x}) = x_0$.*
2. *We use $\mathcal{O}_G^{\ominus}(\mathbf{x})$ to denote the set*

$$\mathcal{O}_G^{\ominus}(\mathbf{x}) = \{\pi_k(\mathbf{x}) \mid k \text{ is a positive integer}\}.$$

*If $\mathbf{x}$ is a backward trajectory of $x_0$ in $(X,G)$, then $\mathcal{O}_G^{\ominus}(\mathbf{x})$ is called* a backward orbit of $x_0$ in $(X,G)$.

*3. We use* $\mathcal{U}_G^{\ominus}(x_0)$ *to denote the set*

$$\mathcal{U}_G^{\ominus}(x_0)=\bigcup_{\mathbf{x}\in T_G^-(x_0)}\mathcal{O}_G^{\ominus}(\mathbf{x}).$$

Definition 7.6. *Let* $(X,G)$ *be a CR-dynamical system. We say that*

1. $(X,G)$ *is* $1^{\ominus}$-minimal *if for each* $x\in X$, $T_G^-(x)\neq\emptyset$, *and for each* $\mathbf{x}\in\star_{i=1}^{\infty}G^{-1}$,
$$\mathrm{Cl}\left(\mathcal{O}_G^{\ominus}(\mathbf{x})\right)=X.$$
2. $(X,G)$ *is* $2^{\ominus}$-minimal *if for each* $x\in X$ *there is* $\mathbf{x}\in T_G^-(x)$ *such that*
$$\mathrm{Cl}\left(\mathcal{O}_G^{\ominus}(\mathbf{x})\right)=X.$$

Observation 7.7. *Let* $(X,G)$ *be a CR-dynamical system and let* $k\in\{1,2\}$. *Then the following holds.*

$$(X,G)\text{ is }k^{\ominus}\text{-minimal }\iff(X,G^{-1})\text{ is }k^{\oplus}\text{-minimal}.$$

Theorem 7.8. *Let* $(X,G)$ *be a CR-dynamical system. Then* $(X,G)$ *is* $2^{\ominus}$*-minimal if and only if for each* $x\in X$,

$$\mathrm{Cl}\left(\mathcal{U}_G^{\ominus}(x)\right)=X.$$

Proof. The proof is analogous to the proof of Theorem 4.7. We leave the details to the reader. ∎

Theorem 7.9. *Let* $(X,G)$ *be a CR-dynamical system. Then the following hold.*

1. $(X,G)$ *is* 1*-backward minimal if and only if* $(X,G)$ *is* $1^{\ominus}$*-minimal.*
2. *If* $(X,G)$ *is* $1^{\ominus}$*-minimal, then* $(X,G)$ *is* $2^{\ominus}$*-minimal.*
3. *If* $(X,G)$ *is* $2^{\ominus}$*-minimal, then* $(X,G)$ *is* $\infty$*-backward minimal.*

Proof. The proof is analogous to the proof of Theorem 4.8. We leave the details to the reader. ∎

Using Observations 7.7 and 6.6, one can easily conclude that Example 4.11 is also an example of a $2^{\ominus}$-minimal CR-dynamical system, which is not $1^{\ominus}$-minimal and that Example 4.12 is also an example of a $\infty$-backward minimal CR-dynamical system, which is not $2^{\ominus}$-minimal.

Theorem 7.10. *Let* $(X,G)$ *be a CR-dynamical system. If* $(X,G)$ *is* 1*-backward minimal,* $\infty$*-backward minimal or* $k^{\ominus}$*-minimal for some* $k\in\{1,2,3\}$, *then*

$$p_1(G)=p_2(G)=X.$$

Proof. The theorem follows from Theorem 4.10 and Observations 6.6 and 7.7. ∎

Theorem 7.11. *Let $(X, G)$ be a CR-dynamical system. The following statements are equivalent.*

1. *$(X, G)$ is $1^{\ominus}$-minimal if and only if $(X, G)$ is $1^{\oplus}$-minimal.*
2. *$(X, G)$ is 1-backward minimal if and only if $(X, G)$ is 1-minimal.*

Proof. First, we prove 1. Suppose that $(X, G)$ is $1^{\oplus}$-minimal. Then $p_2(G) = X$ and, therefore, for each $x \in X$, $T_G^-(x) \neq \emptyset$. Let $\mathbf{x} \in \star_{i=1}^{\infty} G^{-1}$. Then we show that

$$\mathrm{Cl}(\mathcal{O}_G^{\ominus}(\mathbf{x})) = X.$$

Let $A$ be the set of all limit points of the sequence $\mathbf{x}$. Then $A \neq \emptyset$, $A$ is closed in $X$, and

$$A \subseteq \mathrm{Cl}(\mathcal{O}_G^{\ominus}(\mathbf{x})).$$

We show that $A$ is 1-invariant in $(X, G)$. Let $x \in A \cap p_1(G) = A$ and let $(x_{i_n})$ be a subsequence of the sequence $\mathbf{x}$ such that $\lim\limits_{n\to\infty} x_{i_n} = x$. Let $(s, t)$ be any limit point of the sequence $(x_{i_n}, x_{i_n - 1})$. Then $s = x$ and, let $y = t$. Since $G$ is closed in $X \times X$, $(x, y) \in G$ and, since $A$ is closed, it follows that $y \in A$. We have just proved that $A$ is 1-invariant in $(X, G)$. Since $(X, G)$ is $1^{\oplus}$-minimal, it is also 1-minimal by Theorem 4.8, and it follows that $A = X$. Therefore, $\mathrm{Cl}(\mathcal{O}_G^{\ominus}(\mathbf{x})) = X$.

Next, suppose that $(X, G)$ is $1^{\ominus}$-minimal. To show that $(X, G)$ is $1^{\oplus}$-minimal, let $x \in X$ and let $\mathbf{x} \in \star_{i=1}^{\infty} G$. We show that $T_G^+(x) \neq \emptyset$ and that $\mathrm{Cl}\left(\mathcal{O}_G^{\oplus}(\mathbf{x})\right) = X$. By Theorem 7.10, $p_1(G) = X$ and $T_G^+(x) \neq \emptyset$ follows. To show that $\mathrm{Cl}\left(\mathcal{O}_G^{\oplus}(\mathbf{x})\right) = X$, let $A$ be the set of all limit points of the sequence $\mathbf{x}$. Then $A \neq \emptyset$, $A$ is closed in $X$, and

$$A \subseteq \mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x})).$$

We show that $A$ is 1-backward invariant in $(X, G)$. Let $y \in A \cap p_2(G) = A$ and let $(x_{i_n})$ be a subsequence of the sequence $\mathbf{x}$ such that $\lim\limits_{n\to\infty} x_{i_n} = y$. Let $(s, t)$ be any limit point of the sequence $(x_{i_n - 1}, x_{i_n})$. Then $y = t$ and, let $x = s$. Since $G$ is closed in $X \times X$, $(x, y) \in G$ and, since $x$ is a limit point of $\mathbf{x}$, it follows that $x \in A$. We have just proved that $A$ is 1-backward invariant in $(X, G)$. Since $(X, G)$ is $1^{\ominus}$-minimal, it is also 1-backward minimal by Theorem 7.9, and it follows that $A = X$. Therefore, $\mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x})) = X$. This completes the proof of 1. Note that this also proves 2. since $(X, G)$ is 1-minimal if and only if $(X, G)$ is $1^{\oplus}$-minimal by Theorem 4.8, and since $(X, G)$ is 1-backward minimal if and only if $(X, G)$ is $1^{\ominus}$-minimal by Theorem 7.9. ☐

Observation 7.12. *Note that in Theorem 4.8, we have proved that $(X, G)$ is $1^{\oplus}$-minimal if and only if $(X, G)$ is 1-minimal. It follows from Theorem 7.11 that the following statements are equivalent.*

1. *$(X, G)$ is $1^{\oplus}$-minimal.*
2. *$(X, G)$ is $1^{\ominus}$-minimal.*

*3.* $(X,G)$ *is* 1*-minimal.*
*4.* $(X,G)$ *is* 1*-backward minimal.*

Note that so far, we have not presented an example of a closed relation $G$ on $[0,1]$ such that $([0,1],G)$ is 1-minimal. Also, note that all the closed relations $G$ on $[0,1]$ that are presented in our examples, contain a vertical or a horizontal line. Example 7.14 is an example of a closed relation $G$ on $[0,1]$ such that $([0,1],G)$ is 1-minimal and $G$ does not contain a vertical or a horizontal line. We use Theorem 7.13 in its construction.

THEOREM 7.13. *Let* $(X,G)$ *be a CR-relation such that* $p_1(G)=p_2(G)=X$ *and let* $\sigma_G : \star_{i=1}^{\infty}G^{-1} \to \star_{i=1}^{\infty}G^{-1}$ *be the shift map*

$$\sigma_G(x_1,x_2,x_3,\ldots)=(x_2,x_3,\ldots)$$

*for each* $(x_1,x_2,x_3,\ldots)$. *If* $(\star_{i=1}^{\infty}G^{-1},\sigma_G)$ *is minimal, then* $(X,G)$ *is* 1*-minimal.*

PROOF. We show that $(X,G)$ is 1-backward minimal. Let $A$ be a non-empty closed subset of $X$ such that $A$ is 1-backward invariant. Also, let

$$B=\Big(\prod_{i=1}^{\infty}A\Big)\cap\Big(\star_{i=1}^{\infty} G^{-1}\Big).$$

Since $A$ is 1-backward invariant, $B$ is non-empty. Note, that $B$ is also a closed subset of $\star_{i=1}^{\infty}G^{-1}$ such that $\sigma_G(B)\subseteq B$. Since $(\star_{i=1}^{\infty}G^{-1},\sigma_G)$ is minimal, it follows that $B=\star_{i=1}^{\infty}G^{-1}$. Therefore,

$$\star_{i=1}^{\infty}G^{-1}\subseteq\prod_{i=1}^{\infty}A.$$

Since $p_1(G)=p_2(G)=X$, it follows that

$$X=\pi_1(\star_{i=1}^{\infty}G^{-1})=\pi_1(B)\subseteq\pi_1(\prod_{i=1}^{\infty}A)=A.$$

Therefore, $(X,G)$ is 1-backward minimal. By Theorem 7.11, $(X,G)$ is 1-minimal. □

EXAMPLE 7.14. Let $\lambda$ be an irrational number in $(0,1)$ and let $G$ be the union of the following line segments in $[0,1]\times[0,1]$:

1. the line segment from $(0,\lambda)$ to $(1-\lambda,1)$ and
2. the line segment from $(1-\lambda,0)$ to $(1,\lambda)$,

see Figure 3.

Then $(\star_{i=1}^{\infty}G^{-1},\sigma_G)$ is minimal; this follows from the proof of [KK, Theorem 3.4, page 103]. By Theorem 7.13, $([0,1],G)$ is 1-minimal.

In the following example, we demonstrate that there is a $2^{\oplus}$-minimal CR-dynamical system $(X,G)$ which is not $2^{\ominus}$-minimal.

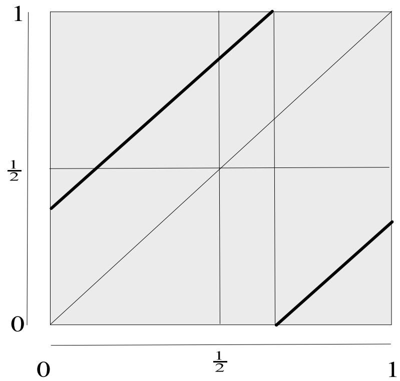


FIGURE 3. The relation $G$ from Example 7.14

EXAMPLE 7.15. Let $X = [0, 1]$ and and let $G = A \cup B \cup C$, where $A$ is a line segment from $(0, \frac{1}{2})$ to $(1, \frac{1}{2})$, $B$ is the line segment from $(0, 0)$ to $(1, 1)$, and $C$ is defined as follows.

Let $d_1 = \frac{1}{2}$, let $d_{10} = \frac{1}{2^2}$ and $d_{11} = \frac{3}{2^2}$, and let $d_{100} = \frac{1}{2^3}$, $d_{101} = \frac{3}{2^3}$, $d_{110} = \frac{5}{2^3}$ and $d_{111} = \frac{7}{2^3}$. Let $n$ be a positive integer and suppose that for any

$$\mathbf{s} \in \{s_1 s_2 s_3 \ldots s_n \mid s_1 = 1, s_2, s_3, s_4, \ldots, s_n \in \{0, 1\}\},$$

we have already defined $d_{\mathbf{s}}$ to be $d_{\mathbf{s}} = \frac{k}{2^n}$ for some $k \in \{1, 3, 5, 7, \ldots, 2^n - 1\}$. Then we define $d_{\mathbf{s}0}$ and $d_{\mathbf{s}1}$ as follows. If $k = 1$ then $d_{\mathbf{s}0} = \frac{1}{2^{n+1}}$ and $d_{\mathbf{s}1} = \frac{3}{2^{n+1}}$, if $k = 3$ then $d_{\mathbf{s}0} = \frac{5}{2^{n+1}}$ and $d_{\mathbf{s}1} = \frac{7}{2^{n+1}}$, ..., and if $k = 2^n - 1$ then $d_{\mathbf{s}0} = \frac{2^{n+1}-3}{2^{n+1}}$ and $d_{\mathbf{s}1} = \frac{2^{n+1}-1}{2^{n+1}}$.

For each positive integer $n$, let

$$\mathcal{S}_n = \{s_1 s_2 s_3 \ldots s_n \mid s_1 = 1, s_2, s_3, s_4, \ldots, s_n \in \{0, 1\}\}$$

and let $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$. Then we define the set $C$ as

$$C = \bigcup_{\mathbf{s} \in \mathcal{S}} \Big(\{d_{\mathbf{s}}\} \times \{d_{\mathbf{s}0}, d_{\mathbf{s}1}\}\Big),$$

see Figure 4, where the construction of the set $C$ is presented – in particular, together with the sets $A$ and $B$, the set $\bigcup_{\mathbf{s} \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3} \Big(\{d_{\mathbf{s}}\} \times \{d_{\mathbf{s}0}, d_{\mathbf{s}1}\}\Big)$ is also pictured in the figure.

Then $(X, G)$ is $2^{\oplus}$-minimal (since for any $x \in [0, 1]$, $(x, \frac{1}{2}) \in G$ and therefore, there is $\mathbf{x} \in T_G^+(x)$ such that $\mathrm{Cl}(\mathcal{O}_G^{\oplus}(\mathbf{x})) = X$) but it is not $2^{\ominus}$-minimal (note that $T_G^-(1) = \{(1, 1, 1, 1, \ldots)\}$ and, therefore, for any $\mathbf{x} \in T_G^-(1)$, $\mathrm{Cl}(\mathcal{O}_G^{\ominus}(\mathbf{x})) \neq X$).

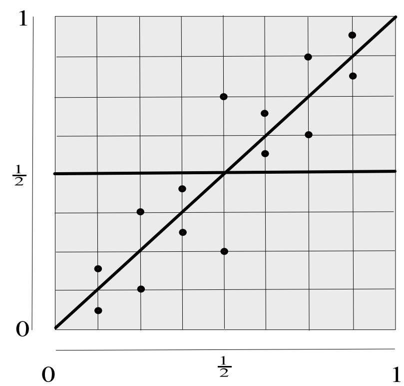


FIGURE 4. The construction of the set $C$

Note that if $(X, G)$ is the CR-dynamical system from Example 7.15, then $(X, G^{-1})$ is an example of a $2^{\ominus}$-minimal CR-dynamical system which is not $2^{\oplus}$-minimal.

## 8. Minimality and alpha limit sets

In this section we define an alpha limit set and (using such a set) introduce new types of minimality of CR-dynamical systems, all of them generalizing minimal dynamical systems.

DEFINITION 8.1. *Let $(X, f)$ be a dynamical system and let $\mathbf{x} \in \star_{i=1}^{\infty}\Gamma(f)^{-1}$. The set*

$$\alpha_f(\mathbf{x}) = \{x \in X \mid \text{ there is a subsequence of the sequence } \mathbf{x} \text{ with limit } x\}$$

*is called* the alpha limit set of $\mathbf{x}$.

The following is a well-known result.

THEOREM 8.2. *Let $(X, f)$ be a dynamical system. The following statements are equivalent.*

1. *$(X, f)$ is minimal.*
2. *For each $x \in X$, $T_f^-(x) \neq \emptyset$, and for each $\mathbf{x} \in \star_{i=1}^{\infty}\Gamma(f)^{-1}$,*

$$\alpha_f(\mathbf{x}) = X.$$

DEFINITION 8.3. *Let $(X, G)$ be a CR-dynamical system, let $x_0 \in X$ and let $\mathbf{x} \in T_G^-(x_0)$. The set*

$$\alpha_G(\mathbf{x}) = \{x \in X \mid \text{ there is a subsequence of the sequence } \mathbf{x} \text{ with limit } x\}$$

*is called* the alpha limit set of $\mathbf{x}$ *and we use* $\beta_G(x_0)$ *to denote the set*

$$\beta_G(x_0) = \bigcup_{\mathbf{x}\in T_G^-(x_0)} \alpha_G(\mathbf{x}).$$

Theorem 8.4. *Let $(X,G)$ be a CR-dynamical system. Then the following hold.*

1. *$(X,G)$ is $1^{\ominus}$-minimal if and only if for each $x\in X$, $T_G^-(x)\neq\emptyset$, and for each $\mathbf{x}\in\star_{i=1}^{\infty}G^{-1}$,*
$$\alpha_G(\mathbf{x}) = X.$$
2. *$(X,G)$ is $2^{\ominus}$-minimal if and only if for each $x\in X$ there is $\mathbf{x}\in T_G^-(x)$ such that*
$$\alpha_G(\mathbf{x}) = X.$$
3. *$(X,G)$ is $2^{\ominus}$-minimal if and only if for each $x\in X$,*
$$\beta_G(x) = X.$$

Proof. Suppose that for each $x\in X$, $T_G^-(x)\neq\emptyset$, and for each $\mathbf{x}\in\star_{i=1}^{\infty}G^{-1}$, $\alpha_G(\mathbf{x}) = X$. That holds if and only if for each $x\in X$, $T_{G^{-1}}^+(x)\neq\emptyset$, and for each $\mathbf{x}\in\star_{i=1}^{\infty}G^{-1}$, $\omega_{G^{-1}}(\mathbf{x}) = X$. And that holds, by Theorem 5.5 if and only if $(X,G^{-1})$ is $1^{\oplus}$-minimal. By Observation 7.7, $(X,G^{-1})$ is $1^{\oplus}$-minimal if and only if $(X,G)$ is $1^{\ominus}$-minimal.

Suppose that for each $x\in X$ there is $\mathbf{x}\in T_G^-(x)$ such that $\alpha_G(\mathbf{x}) = X$. That is equivalent to: for each $x\in X$ there is $\mathbf{x}\in T_{G^{-1}}^+(x)$ such that $\omega_{G^{-1}}(\mathbf{x}) = X$. Therefore again applying Theorem 5.5 and Observation 7.7 we prove 2.

Suppose that for each $x\in X$, $\beta_G(x) = X$. That holds if and only if for each $x\in X$, $\psi_{G^{-1}}(x) = X$. We now apply Theorem 5.5 and Observation 7.7 and prove 3. □

## 9. Preserving different types of minimality by topological conjugation

The main results of this section are obtained in Theorem 9.4 saying that any kind of minimality of a dynamical system is preserved by a topological conjugation.

Definition 9.1. *Let $X$ and $Y$ be metric spaces, and let $f: X\to X$ and $g: Y\to Y$ be functions. If there is a homeomorphism $\varphi: X\to Y$ such that*

$$\varphi\circ f = g\circ\varphi,$$

*then we say that* $f$ and $g$ are topological conjugates.

The following is a well-known result.

THEOREM 9.2. *Let $(X, f)$ and $(Y, g)$ be dynamical systems. If $f$ and $g$ are topological conjugates, then*

$$(X, f) \text{ is minimal } \iff (Y, g) \text{ is minimal.}$$

The following definition generalizes the notion of topological conjugacy of continuous functions to the topological conjugacy of closed relations. See [BEK] for details.

DEFINITION 9.3. *Let $(X, G)$ and $(Y, H)$ be CR-dynamical systems. We say that* $G$ and $H$ are topological conjugates *if there is a homeomorphism $\varphi : X \to Y$ such that for each $(x, y) \in X \times X$, the following holds*

$$(x, y) \in G \iff (\varphi(x), \varphi(y)) \in H.$$

Theorem 9.4 is the main result of this section. Since the proof is straight forward, we leave it to the reader. Therefore, this is a good place to finish the paper.

THEOREM 9.4. *Let $(X, G)$ and $(Y, H)$ be CR-dynamical systems and suppose that $G$ and $H$ are topological conjugates. Then the following hold.*

1. *Let $k \in \{1, \infty, 1^{\oplus}, 2^{\oplus}, 1^{\ominus}, 2^{\ominus}\}$. Then*
$$(X, G) \text{ is } k\text{-minimal} \iff (Y, H) \text{ is } k\text{-minimal.}$$
2. *Let $k \in \{1, \infty\}$. Then*
$$(X, G) \text{ is } k\text{-backward minimal} \iff (Y, H) \text{ is } k\text{-backward minimal.}$$

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## MINIMALNI DINAMIČKI SUSTAVI SA ZATVORENIM RELACIJAMA

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Sažetak. Uvodimo dinamičke sustave $(X, G)$ sa zatvorenim relacijama $G$ na kompaktnim metričkim prostorima $X$ te razmatramo različite tipove minimalnosti tih dinamičkih sustava, od kojih svi poopćuju minimalne dinamičke sustave $(X, f)$ s neprekidnim funkcijama $f$ na kompaktnim metričkim prostorima $X$.