Taylor & Francis
Taylor & Francis Group

Automatika

Journal for Control, Measurement, Electronics, Computing and Communications

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/taut20

On solvability and optimal controls for impulsive stochastic integrodifferential varying-coefficient model

Dimplekumar Chalishajar, Ravikumar K., Ramkumar K. & S. Varshini

To cite this article: Dimplekumar Chalishajar, Ravikumar K., Ramkumar K. & S. Varshini (2024) On solvability and optimal controls for impulsive stochastic integrodifferential varying-coefficient model, Automatika, 65:3, 1271-1283, DOI: 10.1080/00051144.2024.2361212

To link to this article: https://doi.org/10.1080/00051144.2024.2361212

Published online: 11 Jun 2024.

AUTOMATIKA
2024, VOL. 65, NO. 3, 1271–1283
https://doi.org/10.1080/00051144.2024.2361212

RESEARCH ARTICLE

OPEN ACCESS Check for updates

On solvability and optimal controls for impulsive stochastic integrodifferential varying-coefficient model

Dimplekumar Chalishajar [a], Ravikumar K.[b], Ramkumar K.[b] and S. Varshini[b]

[a]Department of Applied Mathematics, Mallory Hall, Virginia Military Institute (VMI), Lexington, VA, USA; [b]Department of Mathematics, PSG College of Arts and Science, Coimbatore, India

ABSTRACT
This article concentrates in analyzing optimal controls for stochastic integrodifferential equation (SIDE) in Hilbert space. Necessary parameters are imposed to demonstrate the system that follows a unique variation of parameter formula using Leray Schauder Alternative. Subsequently, the existence of optimal control is investigated for the considered Lagrange control problem. The theoretical example with the mechanical example of ethanol fuelled engine are discussed to validate the results obtained.

ARTICLE HISTORY
Received 1 May 2023
Accepted 21 May 2024

KEYWORDS
Optimal control; impulsive stochastic system; time varying-coefficient model; the Leray Schauder alternative

2010 AMS SUBJECT CLASSIFICATIONS
35R10; 60G22; 60H20; 93B05

1. Introduction

Several writers study a family of regression and extended regression models that allow coefficient fluctuations as smooth functions of other variables. This class of models combines generalized additive models and dynamic generalized linear models into one cohesive system. When it comes to the proportional hazards model for survival data, this technique offers a unique way of modelling departures from the proportionate risks assumption. Over the past few decades, efforts have been made to increase the flexibility of linear regression models. Generalized additive models, which are smooth, non-parametric functions that can partially or completely replace the linear and parametric functions of regressors, have been the subject of research. Here, we have models with linear regressors that appear to be separate generalizations, but really have coefficients that can smoothly change in response to the values of other variables, or what we would call "effect modifiers". The random variable Y is dependent on a parameter η for its distribution. Additionally, there are predictors $X_l, X_2, \ldots, X_p$ and $R_1, R_2, \ldots, R_p$. A model with variable coefficients has the form

$$\eta = \beta_0 + X_1\beta_1(R_1) + \cdots + +X_p\beta_p(R_p) \qquad (1)$$

The model (1) states that the (unspecified) functions $\beta_l(\,), \beta_2(\,), \ldots, \beta_p(\,)$ cause $R_1, R_2, \ldots, R_p$ to alter the coefficients of the $X_l, X_2, \ldots, X_p$. A unique kind of interaction between each R_j and X_j is implied by the reliance of $\beta_j(\,)$ on R_j. At times, R_j can be easily confused with the variables X_j; in other scenarios, it might be a unique variable like "time".

As the linear predictor in the generalized linear model, η is connected to the mean $\mu = EY$ by the link function $\eta = g(\mu)$. Model (1) takes the form of the Gaussian model in its simplest example, where $g(\mu) = \mu$ and Y is normally distributed with mean μ.

$$Y = X_1\beta_1(r_1) + \cdots + +X_p\beta_p(r_p) + \epsilon, \qquad (2)$$

where $var(\epsilon) = \sigma^2$ and $E(\epsilon) = 0$. Several well-liked models include the log-linear models, in which $\eta = \log \mu$ and Y has a Poisson distribution, and the linear logistic model, in which $g(\mu) = \log\{\frac{\mu}{1-\mu}\}$ and Y a binomial variate. Generalized additive models are an extension of generalized linear models, where the linear predictor is replaced with an additive sum of smooth functions. As we will see, the varying-coefficient model has specific instances such as the generalized additive model and the dynamic generalized linear model.

Examples of Varying-coefficient Model

(1) That term is linear in X if $\beta_j(R_j) = \beta_j$ (the constant function). Model (1) is the standard linear model, often known as the extended linear model if every term is linear. In the case of $X_j = c$ (let's assume $c = 1$), the jth term is just $\beta_j(R_j)$, an ambiguous

CONTACT Dimplekumar Chalishajar dipu17370@gmail.com, chalishajardn@vmi.edu Department of Applied Mathematics, Mallory Hall, Virginia Military Institute (VMI), Lexington, VA 24450, USA

function R_j. If every term in the model has the same form as (1) or is linear, then (1) represents a generalized additive model.

(2) A linear function $\beta_j(R_j) = \beta_j R_j$ yields a product interaction of the type $\beta_j X_j(R_j)$.

(3) For the purposes of simplicity, let us assume that the model is a single-term normal linear model and that X_j is the modifying variable R_j. Thus, we obtain

$$Y = X\beta(X) + \epsilon.$$

This model has been studied extensively by researchers and is often used for smoothing or nonparametric regression of Y against X.

(4) We can have vector or scalar values for each R_j. We will assume for most of the study that the R_j are scalar; expansions to the vector-valued situation will also be examined.

Balasubramaniam and Tamilalagan [1] considered the following impulsive fractional stochastic integrodifferential system and investigated its solvability and optimal control properties.

$$\begin{aligned} {}^c\mathcal{D}_t^\alpha \mathsf{x}(t) &= \mathcal{A}\mathsf{x}(t) + \mathcal{J}_t^{1-\alpha}\Big[\mathcal{B}(t)\mathsf{u}(t) \\ &\quad + \mathfrak{f}(t, \mathsf{x}(t), \mathsf{x}(a_1(t)), \mathsf{x}(a_2(t)), \dots, \mathsf{x}(a_m(t))) \\ &\quad + \mathcal{J}_t^{1-\alpha}\left(\int_0^t \mathfrak{g}(s, \mathsf{x}(s), \mathsf{x}(b_1(s)), \mathsf{x}(b_2(s)), \right. \\ &\quad \left. \dots \mathsf{x}(b_n(s)))\, d\omega(s)\right)\Big], \quad t \in \mathcal{J},\ t \neq t_k \\ \triangle\, \mathsf{x}(t_k) &= \mathcal{I}_k \mathsf{x}(t_k), \quad k = 1, 2, \dots q,\ \mathsf{x}(0) = \mathsf{x}_0. \end{aligned} \quad (3)$$

In [2], the authors have established the existence results of Hilfer fractional integrodifferential equation of the form:

$$\begin{aligned} &\mathcal{D}_{0_+}^{\psi,\mu}[\mathsf{x}(t) + \mathcal{F}(t, \nu(t))] + \mathcal{A}\mathsf{x}(t) \\ &\quad = \int_0^t \mathcal{G}(s, \eta(s))\, d\omega(s), \quad t \in \mathcal{J} := [0, \mathfrak{b}], \\ &\mathcal{I}_{0_+}^{(1-\psi)(1-\mu)}\mathsf{x}(0) - \mathfrak{g}(\mathsf{x}) = \mathsf{x}_0, \end{aligned} \quad (4)$$

where $(t, \nu(t)) = (t, \mathsf{x}(t), \mathsf{x}(b_1(t)), \dots, \mathsf{x}(b_m(t)))$ and $(t, \eta(t)) = (t, \mathsf{x}(t), \mathsf{x}(a_1(t)), \dots, \mathsf{x}(a_n(t)))$. $\mathcal{D}_{0_+}^{\psi,\mu}$ is the Hilfer fractional derivative, $0 \leq \nu \leq 1, 0 < \mu < 1, -\mathcal{A}$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\mathcal{S}(t), t \geq 0$, on a separable Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

Recently, Hamdy M. Ahmed [3] considered semilinear neutral fractional stochastic integrodifferential system with non-local condition of the form:

$$\begin{aligned} &{}^c\mathcal{D}^\alpha[\mathsf{x}(t) + \mathcal{F}(t, \mathsf{x}(t), \mathsf{x}(b_1(t)), \dots, \mathsf{x}(b_m(t)))] \\ &\quad = \mathcal{A}\mathsf{x}(t) + \int_0^t \mathcal{G}(s, \mathsf{x}(s), \mathsf{x}(a_1(s)), \\ &\quad \dots, \mathsf{x}(a_n(s)))\, d\omega(s), \\ &\mathsf{x}(0) = \mathsf{x}_0 + \mathfrak{g}(\mathsf{x}) \quad t \in \mathcal{J} := [0, \mathfrak{b}]. \end{aligned} \quad (5)$$

Motivated by above works, we are concerned in considering impulsive SIDEs of the form:

$$\begin{aligned} d[\vartheta(t)] &= \Big[\mathscr{A}\vartheta(t) + \int_0^t \Theta(t-\varsigma)\vartheta(\varsigma)\, d\varsigma \\ &\quad + \mathscr{B}(t)\mathrm{u}(t) + \mathfrak{l}(t, \vartheta(t), \vartheta(\mathscr{A}_1(t)), \\ &\quad \vartheta(\mathscr{A}_2(t)), \dots \vartheta(\mathscr{A}_{\mathrm{m}}(t)))\Big]\, dt \\ &\quad + \int_0^t \sigma(\varsigma, \vartheta(\varsigma), \vartheta(\mathscr{B}_1(\varsigma)), \vartheta(\mathscr{B}_2(\varsigma)), \\ &\quad \dots \vartheta(\mathscr{B}_{\mathrm{n}}(\varsigma)))\, d\omega(\varsigma), \\ \Delta\vartheta(t_k) &= \mathcal{I}_k(\vartheta(t_k)), \quad k = 1, 2, \dots q, \\ \vartheta(0) &= \vartheta_0. \end{aligned} \quad (6)$$

Here $\mathscr{A} : \mathcal{D}(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ is the infinitesimal generator of a C_0-semigroup $(\mathscr{R}(t))_{t \geq 0}$ on a separable Hilbert space $\mathscr{H}$ with domain $\mathcal{D}(\mathscr{A})$. $\vartheta(.)$ takes values in $\mathscr{H}$ with inner product $\langle ., . \rangle_{\mathscr{H}}$ and $\|.\|_{\mathscr{H}}$. In this case, $(\Theta(t))_{t \geq 0}$ is a closed linear operator on $\mathscr{H}$ with domain $\mathcal{D}(\Theta) \subset \mathcal{D}(\mathscr{A})$. Similarly, $\mathcal{K}$ is separable Hilbert space with $\langle ., . \rangle_{\mathcal{K}}$ and norm $\|.\|_{\mathcal{K}}$. Assume ω is the given $\mathcal{K}$-valued Wiener process with nuclear covariance operator of finite trace $\mathcal{Q} \geq 0$ described on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}; \mathbb{P})$. u is a predefined control function that accepts values from another separable reflexive Hilbert space $\mathcal{U}$. $\mathscr{B}$ is a bounded linear operator that transforms $\mathcal{U}$ into $\mathscr{H}$ and $\Delta\vartheta(t_k) = \vartheta(t_k^+) - \vartheta(t_k)$ constitutes the jump in the state ϑ at t_k, $k = 1, 2, \dots, q$. Let $P\mathbb{C}(\mathscr{J}, \mathfrak{L}_2(\Omega, \mathscr{H})) = \{\Xi : \mathscr{J} \to \mathfrak{L}_2(\Omega, \mathscr{H}) / \Xi \in \mathbb{C}((t_k, t_{k+1}], \mathscr{H}),\ k = 1, 2, \dots q\}$, $\Xi(t_k^+)$, $\Xi(t_k^-)$ exist and $\Xi(t_k^-) = \Xi(t_k)$, provided

$$\|\Xi\|_{P\mathbb{C}} = \sup_{t \in \mathscr{J}} \left(\|\Xi(t)\|_{\mathfrak{L}_2}^2\right)^{1/2}.$$

Eventually, $(P\mathbb{C}(\mathscr{J}, \mathfrak{L}_2(\Omega, \mathscr{H})), \|.\|_{P\mathbb{C}})$ is a Banach space. $\mathfrak{l} : \mathscr{J} \times \mathscr{H}^{\mathrm{m}+1} \to \mathscr{H}$ and $\sigma : \mathscr{J} \times \mathscr{H}^{\mathrm{n}+1} \to \mathfrak{L}(\mathcal{K}, \mathscr{H})$ are the suitable maps used in this article.

Randomness has to be incorporated into mathematical models of real-world phenomena because random effects and noise have caused many real-world phenomena, such expanding population, heat conduction in materials with memory, stock prices and so forth, to fluctuate in recent years. Stochastic Differential Equations (SDEs) are differential equations that assume unpredictability. Because SDEs allow for the abstract representation of many issues, they are employed in a wide range of fields, including as engineering, finance and economics. Books [4,5] and publications [6,7] provide additional fundamental information on SDEs. Further information on the qualitative characteristics of mild solutions to different SIDEs and the fixed point approach may be found in [8–10] and the references therein.

When stochastic differential equations describe the system dynamics and performance index, an optimum control issue becomes a stochastic optimal control problem. Sathiyaraj et al. [11] recently shown optimum control and controllability for fractional SDEs with Poisson jumps. However, there aren't many papers in the literature that discuss optimum control problems. [12–17]. Using the Lagrange multiplier method and the fractional variational principle, Agrawal [13] provided comprehensive information for fractional optimum control problems, accounting for fractional derivatives in the Riemann–Liouville sense. Using resolvent operators, Tamaligan et al. [18] examined the solvability and best controls for FSDE driven by Poisson jumps. Tang and Liu [19] discovered recently that the robustness of the feedback optimum control is not ensured by the regularity of the solution to the backward stochastic Riccati equations. They prove the equivalence between the solvability of the associated backward stochastic Riccati equations and the existence of the resilient optimum feedback control strategy operators, under suitable regularity requirements. In order to construct the online H_∞ optimization problems for a class of nonlinear systems without taking the system dynamics into account, Shuping He et al. [20] created a novel policy iterative technique. Additionally, using a unique policy iteration (PI) approach, Shuping He et al. [21] investigated the online adaptive optimum controller design for a class of nonlinear systems. Without utilizing the system internal parameters, the optimal law for controller design is solved through the appropriate algebraic Riccati equation (ARE) by employing the neural network linear differential inclusion (LDI) approach to linearize the nonlinear components in each iteration. This paper's model is more sophisticated than [21] since it incorporates a stochastic term with a time-varying coefficient. Using a successive approximation technique, Ramkumar et al. [22] examined the optimum management of a neutral FSDE with a Caputo fractional derivative. See [23–28] for a list of more articles that discuss the solvability and optimum control for fractional SDEs.

Novelty of the work:

(1) Thus far, the literature has not addressed the optimal controllability for an impulsive stochastic time-varying-coefficient model.
(2) The Leray Schauder Alternative confirms the existence and solvability of the mild solution of (6).
(3) A comprehensive analysis of the 88 observations on the exhaust from an ethanol-fuelled engine in mechanical engineering is conducted to demonstrate the practical implementation of the stated hypothesis.

This paper's outline is: Section 2 establishes the concepts and preliminary steps needed to solve the aforementioned system. Section 3 proves the existence results of the expressed system (6). The system's existence results are established in Section 4. The illustrations are included in Section 5 to validate our findings.

2. Preliminaries and notations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space with increasing sub σ-algebra $\{\mathcal{F}_t, t \in \mathscr{J}\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. Let $\mathbb{C}(\mathscr{J}, \mathfrak{L}_2(\Omega, \mathscr{H}))$ be the Banach space of all continuous maps from $\mathscr{J}$ into $\mathfrak{L}_2(\Omega, \mathscr{H})$ fulfilling $\sup_{t\in\mathscr{J}} \mathbb{E}\|\vartheta(t)\|^2 < \infty$.

Definition 2.1: A one parameter family $\{\mathscr{R}(t) : t \geq 0\}$ of bounded linear operators is called resolvent operator for

$$\frac{d\vartheta}{dt} = \mathfrak{A}\left[\vartheta(t) + \int_0^t \Theta(t-\tau)\vartheta(t)\,dt\right], \qquad (7)$$

if

(i) $\mathscr{R}(0) = I$, $\|\mathscr{R}(t)\| \leq Ne^{\beta t}$ for β and $N \geq 1$.
(ii) For $\vartheta \in \mathscr{H}$, $\mathscr{R}(t)\vartheta$ is strongly continuously for $t \in \mathscr{J}$.
(iii) For $t \in \mathscr{J}$, $\mathscr{R}(t) \in \mathfrak{L}(\mathscr{H})$. $\forall \vartheta \in \mathscr{H}$, $\mathscr{R}(\cdot)\vartheta \in \mathbb{C}^1(\mathscr{J}, \mathscr{H}) \cap \mathbb{C}(\mathscr{J}, \mathcal{K})$ and

$$\begin{aligned}\frac{d}{dt}\mathscr{R}(t)\vartheta &= \mathfrak{A}\left[\mathscr{R}(t) + \int_0^t \Theta(t-\tau)\mathscr{R}(t)\vartheta\,dt\right]\\ &= \mathscr{R}(t)\mathfrak{A}\vartheta\\ &\quad + \int_0^t \mathscr{R}(t-\tau)\mathfrak{A}\Theta(\tau)\vartheta\,dt,\\ &\quad t \in \mathscr{J}.\end{aligned}$$

For more background on the resolvent operator, we refer to [29–31].

Definition 2.2 ([32]): Let E be a Banach space, $\Omega \in E$ a closed convex subset, $U \subset \Omega$ an open set (with respect to the topology of Ω) and such that $\theta \in U$. Assume that $F : \bar{U} \to \Omega$ is weakly sequence compact. If $F\bar{U}$ is relatively weakly compact then, either

(i) F has a fixed point, or
(ii) there is a point $u \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $u = \lambda F u$,

where θ be the zero vector of E. $\bar{U}$ and $\partial_\Omega U$ denote the closure and the boundary of U in Ω, respectively.

Remark 2.1: We know that a strongly continuous operator is weakly sequential compact (WSC). The converse is not true in general (even if E is reflexive). Leray–Schauder alternative is useful to derive WSC operators. In our study, we assume that the infinitesimal generator $\mathscr{A}$ does not generate the compact semigroup. We use MNC to prove the existence of the mild solution. Leray Schauder alternative fits well in this situation.

Definition 2.3: A stochastic process $\vartheta(t) \in P\mathbb{C}(\mathscr{J}, \mathfrak{L}_2(\Omega, \mathscr{H}))$ follows the variation of constant formula for the system (6) whenever

(i) $\vartheta(t)$ is $\mathcal{F}_t$-adapted , $t \in [0, \mathscr{B}]$
(ii) On $\in [0, \mathscr{B}], \vartheta(t) \in \mathscr{H}$ possesses a cadlag path a.s
(iii)

$$\begin{aligned}\vartheta(t) &= \mathscr{R}(t)\vartheta_0 + \sum_{k=1}^{q} \mathscr{R}(t-t_k)\mathcal{I}(\vartheta(t_k)) \\ &+ \int_0^t \mathscr{R}(t-\varsigma)\mathscr{B}(\varsigma)\mathrm{u}(\varsigma) \\ &+ \int_0^t \mathscr{R}(t-\varsigma)\mathfrak{l}(\varsigma, \vartheta(\varsigma), \\ &\vartheta(\mathscr{A}_1(\varsigma)), \vartheta(\mathscr{A}_2(\varsigma)), \\ &\ldots \vartheta(\mathscr{A}_{\mathrm{m}}(\varsigma)))\,\mathrm{d}\varsigma \\ &+ \int_0^t \mathscr{R}(t-\varsigma)\left(\int_0^{\varsigma} \sigma\,(s, \vartheta(s), \right. \\ &\vartheta(\mathscr{B}_1(s)), \vartheta(\mathscr{B}_2(s)), \\ &\left.\ldots \vartheta(\mathscr{B}_{\mathrm{n}}(s)))\,\mathrm{d}\omega(s)\right)\mathrm{d}\varsigma. \qquad (8)\end{aligned}$$

Let us have the following hypotheses:

(A1) The Resolvent operator $\mathscr{R}(.)$ is exponentially stable, i.e. $\exists$ a constant $\mathfrak{N} \geq 1 \ni \|\mathscr{R}(t)\| \leq \mathfrak{N}\ \forall\, t \geq 0$.
(A2) The map $\mathfrak{l} : \mathscr{J} \times \mathscr{H}^{\mathrm{m}+1} \to \mathscr{H}$ is a continuous function and $\exists \mathfrak{N}_{\mathfrak{l}}, \tilde{\mathfrak{N}}_{\mathfrak{l}} > 0 \ni$

$$\left\| \mathfrak{l}(\varsigma_1, \vartheta_0, \vartheta_1, \ldots \vartheta_{\mathrm{m}}) - \int_0^t \mathfrak{l}_2(\varsigma_2, \varpi_0, \varpi_1, \ldots \varpi_{\mathrm{m}}) \right\|^2 \leq \mathfrak{N}_{\mathfrak{l}} \left(\|\varsigma_1 - \varsigma_2\|^2 + \max_{i=0,1,2,\ldots \mathrm{m}} \|\vartheta_i - \varpi_i\|^2 \right)$$

for $0 \leq \varsigma_1, \varsigma_2 \leq \mathscr{B},\ \vartheta_i, \varpi_i \in \mathscr{H},\ i = 0, 1, 2, \ldots \mathrm{m}$ and

$$\|\mathfrak{l}(t, \vartheta_0, \vartheta_1, \ldots \vartheta_{\mathrm{m}})\|^2 \leq \tilde{\mathfrak{N}}_{\mathfrak{l}} \left(\max_{i=0,1,2,\ldots \mathrm{m}} \|\vartheta_i\|^2 + 1 \right)$$

holds for $(t, \vartheta_0, \vartheta_1, \ldots \vartheta_{\mathrm{m}}) \in \mathscr{J} \times \mathscr{H}^{\mathrm{m}+1}$.
(A3) The map $\sigma : \mathscr{J} \times \mathscr{H}^{\mathrm{n}+1} \to \mathfrak{L}(\mathcal{K}, \mathscr{H})$ satisfies the requirements:

(i) For each $t \in \mathscr{J}$ the function $\sigma(t, .) : \mathscr{H}^{\mathrm{n}+1} \to \mathfrak{L}(\mathcal{K}, \mathscr{H})$ is continuous and the function $\sigma(., \vartheta_0, \vartheta_1, \ldots \vartheta_{\mathrm{n}}) : \mathscr{J} \to \mathfrak{L}(\mathcal{K}, \mathscr{H})$ is $\mathcal{F}_t$-measurable for each $(\vartheta_0, \vartheta_1, \ldots \vartheta_{\mathrm{n}}) \in \mathscr{H}^{\mathrm{n}+1}$.
(ii) For each positive $\mathfrak{r} \in \mathbb{N}$, $\exists$ a positive function $\mathfrak{p}_{\mathfrak{r}}(.) \in \mathfrak{L}^1(\mathscr{J}) \ni$

$$\sup_{\|\vartheta_0\|^2, \|\vartheta_1\|^2, \ldots \|\vartheta_n\|^2 \leq \mathfrak{r}} \times \int_0^t \mathbb{E}\,\|\sigma(\varsigma, \vartheta_0, \vartheta_1, \ldots \vartheta_n)\|_{\mathcal{Q}}^2\,\mathrm{d}\varsigma \leq \mathfrak{p}_{\mathfrak{r}}(t)$$

and $\lim_{\mathfrak{r}\to\infty} \inf \frac{1}{\mathfrak{r}} \int_0^{\mathscr{B}} \mathfrak{p}_{\mathfrak{r}}(\varsigma)\,\mathrm{d}\varsigma = \Upsilon < \infty$.
(iii) The function $\sigma : \mathscr{J} \times \mathscr{H}^{\mathrm{n}+1} \to \mathfrak{L}(\mathcal{K}, \mathscr{H})$ satisfies (A3)(i) and $\exists \mathfrak{N}_{\sigma} > 0 \ni$

$$\begin{aligned}&\|\sigma(\varsigma_1, \vartheta_0, \vartheta_1, \ldots \vartheta_{\mathrm{m}}) - \sigma(\varsigma_2, \varpi_0, \varpi_1, \ldots \varpi_{\mathrm{m}})\|_{\mathcal{Q}}^2 \\ &\leq \mathfrak{N}_{\sigma} \left(\|\varsigma_1 - \varsigma_2\|^2 + \max_{i=0,1,2,\ldots \mathrm{n}} \|\vartheta_i - \varpi_i\|^2 \right)\end{aligned}$$

for $0 \leq \varsigma_1, \varsigma_2 \leq \mathscr{B}, \vartheta_i, \varpi_i \in \mathscr{H}, i = 0, 1, 2, \ldots \mathrm{n}$.

(A4) $\mathcal{I}_k : \mathscr{H} \to \mathscr{H},\ k = 1, 2, \ldots q$ and $\exists \mathfrak{N}_k, \tilde{\mathfrak{N}}_k \geq 0 \ni$

$$\mathbb{E}\,\|\mathcal{I}_k(\vartheta) - \mathcal{I}_k(\varpi)\|^2 \leq \mathfrak{N}_k \mathbb{E}\,\|\vartheta - \varpi\|^2$$

and $\mathbb{E}\|\mathcal{I}_k(\vartheta)\|^2 \leq \tilde{\mathfrak{N}}_k \mathbb{E}\|\vartheta\|^2$ for any $\vartheta, \varpi \in \mathscr{H}$.
(A5) Let $\mathrm{u} \in \mathcal{U}$ be the control function and $\mathscr{B} \in \mathfrak{L}_{\infty}(\mathscr{J}, \mathfrak{L}(\mathcal{U}, \mathscr{H}))$. $\|\mathscr{B}\|$ being norm of the operator $\mathscr{B}$.

Set the admissible set

$$\mathcal{U}_{ad} = \left\{ \mathfrak{v} : \mathscr{J} \times \Omega \to \mathscr{H} \mid \mathfrak{v} \text{ is } \mathcal{F}_t\text{–adapted and } \mathbb{E} \int_0^{\mathscr{B}} \|\mathfrak{v}(t)\|^2\,\mathrm{d}t < \infty \right\}.$$

3. Existence of mild solution

Theorem 3.1: *Assume that hypotheses* (A1)–(A5) *are satisfied then the system* (6) *has at least one mild solution on* $\mathscr{J}$*, given that*

$$5\mathfrak{N}^2 q \sum_{k=1}^{q} \tilde{\mathfrak{N}}_k + 5\mathfrak{N}^2 \mathscr{B}^2 \tilde{\mathfrak{N}}_{\mathfrak{l}} + \mathfrak{N}^2 \mathscr{B} Tr(\mathcal{Q})\Upsilon < 1. \quad (9)$$

Proof: Define the map $\Psi : P\mathbb{C}(\mathscr{J}, \mathfrak{L}_2(\Omega, \mathscr{H})) \to P\mathbb{C}(\mathscr{J}, \mathfrak{L}_2(\Omega, \mathscr{H}))$ by

$$\begin{aligned}(\Psi\vartheta)(t) &= \mathscr{R}(t)\vartheta_0 + \sum_{k=1}^{q} \mathscr{R}(t-t_k)\mathcal{I}(\vartheta(t_k)) \\ &+ \int_0^t \mathscr{R}(t-\varsigma)\mathscr{B}(\varsigma)\mathrm{u}(\varsigma) \\ &+ \int_0^t \mathscr{R}(t-\varsigma)\mathfrak{l}(\varsigma, \vartheta(\varsigma), \vartheta(\mathscr{A}_1(\varsigma)), \\ &\vartheta(\mathscr{A}_2(\varsigma)), \ldots \vartheta(\mathscr{A}_\mathrm{m}(\varsigma)))\,\mathrm{d}\varsigma \\ &+ \int_0^t \mathscr{R}(t-\varsigma) \\ &\times \left(\int_0^\varsigma \sigma\,(s, \vartheta(s), \vartheta(\mathscr{B}_1(s)), \right. \\ &\left. \vartheta(\mathscr{B}_2(s)), \ldots \vartheta(\mathscr{B}_\mathrm{n}(s)))\,\mathrm{d}\omega(s)\right) \mathrm{d}\varsigma. \end{aligned} \tag{10}$$

It is adequate to demonstrate Ψ seems to have a fixed point in $P\mathbb{C}(\mathscr{J}, \mathfrak{L}_2(\Omega, \mathscr{H}))$.

Let $\mathbb{B}_\mathfrak{r} = \{\vartheta \in P\mathbb{C}(\mathscr{J}, \mathfrak{L}_2(\Omega, \mathscr{H})) : \mathbb{E}\|\vartheta(t)\|^2_{P\mathbb{C}} \le \mathfrak{r},\ t \in \mathscr{J}\}$.

Step 1:

For each integer $\mathfrak{r}$, let $\mathbb{B}_\mathfrak{r} = \{\vartheta \in P\mathbb{C}(\mathscr{J}, \mathfrak{L}_2(\Omega, \mathscr{H})) : \mathbb{E}\|\vartheta(t)\|^2_{P\mathbb{C}} \le \mathfrak{r},\ t \in \mathscr{J}\}$.

We affirm that $\Psi\mathbb{B}_\mathfrak{r} \subseteq \mathbb{B}_\mathfrak{r}$. For each $\vartheta \in \mathbb{B}_\mathfrak{r}$, we have

$$\begin{aligned}\mathfrak{r} &\le \mathbb{E}\,\|(\Psi\vartheta)(t)\|^2 \le 5\mathbb{E}\,\left\|\mathscr{R}(t)\vartheta_0\right\|^2 \\ &+ 5\mathbb{E}\left\|\sum_{k=1}^{q}\mathscr{R}(t-t_k)\mathcal{I}(\vartheta(t_k))\right\|^2 \\ &+ 5\mathbb{E}\left\|\int_0^t \mathscr{R}(t-\varsigma)\mathscr{B}(\varsigma)\mathrm{u}(\varsigma)\right\|^2 \\ &+ 5\mathbb{E}\left\|\int_0^t \mathscr{R}(t-\varsigma)\mathfrak{l}(\varsigma, \vartheta(\varsigma), \vartheta(\mathscr{A}_1(\varsigma)), \right. \\ &\left. \vartheta(\mathscr{A}_2(\varsigma)), \ldots \vartheta(\mathscr{A}_\mathrm{m}(\varsigma)))\,\mathrm{d}\varsigma\right\|^2 \\ &+ 5\mathbb{E}\left\|\int_0^t \mathscr{R}(t-\varsigma)\left(\int_0^\varsigma \sigma\,(s, \vartheta(s), \vartheta(\mathscr{B}_1(s)), \right.\right. \\ &\left.\left. \vartheta(\mathscr{B}_2(s)), \ldots \vartheta(\mathscr{B}_\mathrm{n}(s)))\,\mathrm{d}\omega(s)\right) \mathrm{d}\varsigma\right\|^2 \\ &\le \sum_{i=5}^{k}\mathscr{W}_i. \end{aligned}$$

$$\mathscr{W}_1 = \mathbb{E}\,\left\|\mathscr{R}(t)\vartheta_0\right\|^2 \le \mathfrak{N}^2\mathbb{E}\|\vartheta_0\|^2$$

$$\begin{aligned}\mathscr{W}_2 &= \mathbb{E}\left\|\sum_{k=1}^{q}\mathscr{R}(t-t_k)\mathcal{I}(\vartheta(t_k))\right\|^2 \\ &\le \mathfrak{N}^2 q\sum_{k=1}^{q}\tilde{\mathfrak{N}}\mathbb{E}\|\vartheta\|^2 \le \mathfrak{N}^2 q\sum_{k=1}^{q}\tilde{\mathfrak{N}}r. \end{aligned}$$

$$\begin{aligned}\mathscr{W}_3 &= \mathbb{E}\left\|\int_0^t \mathscr{R}(t-\varsigma)\mathscr{B}(\varsigma)\mathrm{u}(\varsigma)\right\|^2 \\ &\le \mathfrak{N}^2\|\mathscr{B}\|^2\|\mathrm{u}\|^2_{\mathfrak{L}_p(\mathscr{J},\mathcal{U})} \\ \mathscr{W}_4 &= \mathbb{E}\left\|\int_0^t \mathscr{R}(t-\varsigma)\mathfrak{l}(\varsigma, \vartheta(\varsigma), \vartheta(\mathscr{A}_1(\varsigma)), \right. \\ &\left. \vartheta(\mathscr{A}_2(\varsigma)), \ldots \vartheta(\mathscr{A}_\mathrm{m}(\varsigma)))\,\mathrm{d}\varsigma\right\|^2 \\ &\le \mathfrak{N}^2\mathscr{B}^2\tilde{\mathfrak{N}}_\mathfrak{l}\left(\int_0^\mathrm{t}\mathbb{E}\|\vartheta(s)\|^2 + 1\right) \\ &\le \mathfrak{N}^2\mathscr{B}^2\tilde{\mathfrak{N}}_\mathfrak{l}(r+1) \\ \mathscr{W}_5 &= \mathbb{E}\left\|\int_0^t \mathscr{R}(t-\varsigma)\left(\int_0^\varsigma \sigma\,(s, \vartheta(s), \vartheta(\mathscr{B}_1(s)), \right.\right. \\ &\left.\left. \vartheta(\mathscr{B}_2(s)), \ldots \vartheta(\mathscr{B}_\mathrm{n}(s)))\,\mathrm{d}\omega(s)\right) \mathrm{d}\varsigma\right\|^2 \\ &\le \mathfrak{N}^2 Tr(\mathcal{Q})\mathscr{B}\Upsilon. \end{aligned}$$

We assume that there exists a positive number $\mathfrak{r}$ such that $\Psi\mathbb{B}_\mathfrak{r} \subseteq \mathbb{B}_\mathfrak{r}$. If it is not true, then for each positive number $\mathfrak{r}$, there is a function $\vartheta_\mathfrak{r}(\cdot) \in \mathbb{B}_\mathfrak{r}$ but $\Psi\mathbb{B}_\mathfrak{r} \neq \mathbb{B}_\mathfrak{r}$, but $\|\Psi\vartheta_\mathfrak{r}(t)\| > \mathfrak{r}$ for some $t(\mathfrak{r}) \in \mathscr{J}$, where $t(\mathfrak{r})$ denotes that t is dependent on r.

$$\begin{aligned}r \le 5\Bigg[&\mathfrak{N}^2\mathbb{E}\|\vartheta_0\|^2 + \mathfrak{N}^2 q\sum_{k=1}^{q}\tilde{\mathfrak{N}}r \\ &+ \mathfrak{N}^2\|\mathscr{B}\|^2\|\mathrm{u}\|^2_{\mathfrak{L}_p(\mathscr{J},\mathcal{U})} \\ &+ \mathfrak{N}^2\mathscr{B}^2\tilde{\mathfrak{N}}_\mathfrak{l}(r+1) + \mathfrak{N}^2 Tr(\mathcal{Q})\mathscr{B}\Upsilon\Bigg]. \end{aligned}$$

Dividing r throughout and let $r \to \infty$,

$$1 \le 5\mathfrak{N}^2 q\sum_{k=1}^{q}\tilde{\mathfrak{N}}_k + 5\mathfrak{N}^2\mathscr{B}^2\tilde{\mathfrak{N}}_\mathfrak{l}.$$

which contradicts our assumption (9).

Step 2: To prove Ψ is continuous, let $\{\vartheta_{\hat{n}}\}$ be a sequence $\ni \vartheta_{\hat{n}} \to \vartheta$ in $P\mathbb{C}(\mathscr{J}, \mathfrak{L}_2(\Omega, \mathscr{H}))$ as $\hat{n} \to \infty$ then for $t \in (t_k, t_{k+1}]$, we get

$$\begin{aligned}&\mathbb{E}\|(\Psi\vartheta_{\hat{n}})(t) - (\Psi\vartheta)(t)\|^2 \\ &\le 3\mathbb{E}\left\|\sum_{k=1}^{q}\mathscr{R}(t-t_k)[\mathcal{I}_k(\vartheta_{\hat{n}}(t_k)) - \mathcal{I}_k(\vartheta(t_k))]\right\|^2 \\ &+ 3\mathbb{E}\left\|\int_0^t \mathscr{R}(t-\varsigma)\,[\mathfrak{l}(\varsigma, \vartheta_{\hat{n}}(\varsigma), \vartheta_{\hat{n}}(\mathscr{A}_1(\varsigma)), \right. \\ &\ldots\vartheta_{\hat{n}}(\mathscr{A}_\mathrm{m}(\varsigma))) - \mathfrak{l}(\varsigma, \vartheta(\varsigma), \vartheta(\mathscr{A}_1(\varsigma)), \\ &\left.\ldots\vartheta(\mathscr{A}_\mathrm{m}(\varsigma)))]\,\mathrm{d}\varsigma\right\|^2 \\ &+ 3\mathbb{E}\left\|\int_0^t \mathscr{R}(t-\varsigma)\left(\int_0^\varsigma [\sigma(s, \vartheta_{\hat{n}}(s), \vartheta_{\hat{n}}(\mathscr{B}_1(s)), \right.\right. \end{aligned}$$

$$+\dots\vartheta_{\hat{n}}(\mathscr{B}_n(s))) - \sigma(s,\vartheta(s),\vartheta(\mathscr{B}_1(s)),$$
$$\dots\vartheta(\mathscr{B}_n(s)))]\,\mathrm{d}\omega(s)\Big)\,\mathrm{d}\varsigma\Big\|^2$$
$$\leq 3\mathfrak{N}^2 q\sum_{k=1}^{q}\mathfrak{N}_k\mathbb{E}\,\|\vartheta_{\hat{n}}-\vartheta\|^2$$
$$+3\mathfrak{N}^2\mathscr{B}\int_0^t\mathfrak{N}_{\mathfrak{l}}\sup_{0\leq\varsigma\leq\mathscr{B}}\mathbb{E}\,\|\vartheta_{\hat{n}}(\varsigma)-\vartheta(\varsigma)\|^2\,\mathrm{d}\varsigma$$
$$+3\mathfrak{N}^2\mathscr{B}Tr(\mathcal{Q})\mathfrak{N}_\sigma$$
$$\times\int_0^t\int_0^\varsigma\sup_{0\leq s\leq\mathscr{B}}\mathbb{E}\,\|\vartheta_{\hat{n}}(s)-\vartheta(s)\|^2\,\mathrm{d}s\,\mathrm{d}\varsigma.$$

Obviously, $\mathbb{E}\|(\Psi\vartheta_{\hat{n}})(t)-(\Psi\vartheta)(t)\|^2\to 0$ as $\hat{n}\to\infty$. Thus Ψ is continuous.

Step 3:

To prove Ψ is equicontinuous on $\mathbb{B}_r$, let $0\leq s_1\leq s_2\leq\mathscr{B}$, for $\vartheta\in\mathbb{B}_r$, we have

$$\mathbb{E}\|(\Psi\vartheta)(s_2)-(\Psi\vartheta)(s_1)\|^2$$
$$\leq 8\mathbb{E}\left\|[\mathscr{R}_{s_2}-\mathscr{R}_{s_1}]\vartheta_0\right\|^2$$
$$+8\mathbb{E}\left\|\sum_{k=1}^{q}[\mathscr{R}(s_2-t_k)-\mathscr{R}(s_1-t_k)]\mathcal{I}_k(\vartheta(t_k))\right\|^2$$
$$+8\mathbb{E}\left\|\int_0^{s_1}[\mathscr{R}(s_2-\varsigma)-\mathscr{R}(s_1-\varsigma)]\right.$$
$$\left.\times\,\mathscr{B}(\varsigma)\mathrm{u}(\varsigma)\,\mathrm{d}\varsigma\right\|^2$$
$$+8\mathbb{E}\left\|\int_{s_1}^{s_2}\mathscr{R}(s_2-\varsigma)\mathscr{B}(\varsigma)\mathrm{u}(\varsigma)\,\mathrm{d}\varsigma\right\|^2$$
$$+8\mathbb{E}\left\|\int_0^{s_1}[\mathscr{R}(s_2-\varsigma)-\mathscr{R}(s_1-\varsigma)]\mathfrak{l}(\varsigma,\vartheta(\varsigma),\right.$$
$$\left.\vartheta(\mathscr{A}_1(\varsigma)),\dots\vartheta(\mathscr{A}_\mathrm{m}(\varsigma)))\,\mathrm{d}\varsigma\right\|^2$$
$$+8\mathbb{E}\left\|\int_{s_1}^{s_2}\mathscr{R}(s_2-\varsigma)\mathfrak{l}(\varsigma,\vartheta(\varsigma),\vartheta(\mathscr{A}_1(\varsigma)),\right.$$
$$\left.\dots\vartheta(\mathscr{A}_\mathrm{m}(\varsigma)))\,\mathrm{d}\varsigma\right\|^2$$
$$+8\mathbb{E}\left\|\int_0^{s_1}[\mathscr{R}(s_2-\varsigma)-\mathscr{R}(s_1-\varsigma)]\right.$$
$$\times\Big(\int_0^\varsigma\sigma(s,\vartheta(s),\vartheta(\mathscr{B}_1(s)),$$
$$\left.+\dots\vartheta(\mathscr{B}_n(s)))\,\mathrm{d}\omega(s)\Big)\,\mathrm{d}\varsigma\right\|^2$$
$$+8\mathbb{E}\left\|\int_{s_1}^{s_2}\mathscr{R}(s_2-\varsigma)\right.$$
$$\times\Big(\int_0^\varsigma\sigma(s,\vartheta(s),\vartheta(\mathscr{B}_1(s)),$$
$$\left.\dots\vartheta(\mathscr{B}_n(s)))\,\mathrm{d}\omega(s)\Big)\,\mathrm{d}\varsigma\right\|^2$$
$$\leq 8\left\|\mathscr{R}(s_2)-\mathscr{R}(s_1)\right\|^2\mathbb{E}\|\vartheta_0\|^2$$
$$+8qr\sum_{k=1}^{q}\tilde{\mathfrak{N}}_{\mathfrak{l}}\left\|\mathscr{R}(s_2-t_k-\mathscr{R}(s_1-t_k))\right\|^2$$
$$+8\|\mathscr{B}\|^2(s_1)^{\frac{2p-2}{p}}\|\mathrm{u}\|^2_{\mathfrak{L}_p(\mathscr{J},\mathcal{U})}$$
$$\times\sup_{\varsigma\in[0,s_1-\epsilon]}\left\|\mathscr{R}(s_2-\varsigma)-\mathscr{R}(s_1-\varsigma)\right\|^2$$
$$\times\|\mathrm{u}\|_{\mathfrak{L}_p(\mathscr{J},\mathcal{U})}$$
$$+8\mathfrak{N}^2\|\mathscr{B}\|^2(s_2-s_1)^{\frac{2p-2}{p}}\|\mathrm{u}\|^2_{\mathfrak{L}_p(\mathscr{J},\mathcal{U})}$$
$$+8(s_2-s_1)\mathfrak{N}^2Tr(\mathcal{Q})\int_{s_1}^{s_2}h_r(\varsigma)$$
$$+8\mathfrak{N}^2(s_2-s_1)^2\tilde{\mathfrak{N}}_{\mathfrak{l}}(1+r)$$
$$+8s_1\tilde{\mathfrak{N}}_{\mathfrak{l}}(1+r)$$
$$\times\int_0^{s_1}\left\|\mathscr{R}(s_2-\varsigma)-\mathscr{R}(s_1-\varsigma)\right\|^2\,\mathrm{d}\varsigma$$
$$+8s_1Tr(\mathcal{Q})\int_0^{s_1}\left\|\mathscr{R}(s_2-\varsigma)-\mathscr{R}(s_1-\varsigma)\right\|^2$$
$$\times h_r(\varsigma)\,\mathrm{d}\varsigma;$$

as $s_2\to s_1$ and $\epsilon\to 0$, we get $\mathbb{E}\|(\Psi\vartheta)(s_2)-(\Psi\vartheta)(s_1)\|^2\to 0$ implying Ψ to be continuous.

Step 4:

To show $(\Psi\vartheta)(t)$ is compact for $0\leq t\leq\mathscr{B}$, firstly we need to prove $(\Psi\vartheta)(0)$ is relatively compact in $\mathbb{B}_r$. For $0<\epsilon<\mathscr{B}$, $\vartheta\in\mathbb{B}_r$,

$$(\Psi^\epsilon\vartheta)(t)=\mathscr{R}(t)\vartheta_0+\sum_{k=1}^{q}\mathscr{R}(t-t_k)\mathcal{I}(\vartheta(t_k))$$
$$+\int_0^{t-\epsilon}\mathscr{R}(t-\varsigma)\mathscr{B}(\varsigma)\mathrm{u}(\varsigma)$$
$$+\int_0^{t-\epsilon}\mathscr{R}(t-\varsigma)\mathfrak{l}(\varsigma,\vartheta(\varsigma),\vartheta(\mathscr{A}_1(\varsigma)),$$
$$\vartheta(\mathscr{A}_2(\varsigma)),\dots\vartheta(\mathscr{A}_\mathrm{m}(\varsigma)))\,\mathrm{d}\varsigma$$
$$+\int_0^{t-\epsilon}\mathscr{R}(t-\varsigma)\Big(\int_0^\varsigma\sigma(s,\vartheta(s),$$
$$\vartheta(\mathscr{B}_1(s)),\vartheta(\mathscr{B}_2(s)),$$
$$\dots\vartheta(\mathscr{B}_\mathrm{n}(s)))\,\mathrm{d}\omega(s)\Big)\,\mathrm{d}\varsigma$$
$$=\mathscr{R}(t)\vartheta_0+\sum_{k=1}^{q}\mathscr{R}(t-t_k)\mathcal{I}(\vartheta(t_k))$$
$$+\mathscr{R}(\epsilon)\int_0^{t-\epsilon}\mathscr{R}(t-\epsilon-\varsigma)\mathscr{B}(\varsigma)\mathrm{u}(\varsigma)$$
$$+\mathscr{R}(\epsilon)\int_0^{t-\epsilon}\mathscr{R}(t-\epsilon-\varsigma)\mathfrak{l}(\varsigma,\vartheta(\varsigma),$$

$$\vartheta(\mathscr{A}_1(\varsigma)),\vartheta(\mathscr{A}_2(\varsigma)),\ldots \vartheta(\mathscr{A}_{\mathrm{m}}(\varsigma)))\,\mathrm{d}\varsigma + \mathscr{R}(\epsilon)\int_0^{t-\epsilon}\mathscr{R}(t-\epsilon-\varsigma) \times \left(\int_0^{\varsigma}\sigma\,(s,\vartheta(s),\vartheta(\mathscr{B}_1(s)),\vartheta(\mathscr{B}_2(s)), \ldots\vartheta(\mathscr{B}_{\mathrm{n}}(s)))\,\mathrm{d}\omega(s)\right)\mathrm{d}\varsigma.$$

The compactness condition of $\mathscr{R}(\epsilon)(\epsilon>0)$ yields $\{(\Psi^{\epsilon}\vartheta)(t):\vartheta\in\mathscr{B}_r\}$ which is relatively compact in $\mathscr{H}$ $\forall\epsilon\in(0,t)$. By each $\vartheta\in\mathbb{B}_r$, we get

$$\begin{aligned}
&\mathbb{E}\|(\Psi\vartheta)(t)-(\Psi\vartheta^{\epsilon})(t)\|^2\\
&\le 6\mathbb{E}\left\|\int_0^{t-\epsilon}[\mathscr{R}(t-\epsilon-\varsigma)-\mathscr{R}(t-\varsigma)] \times\mathscr{B}(\varsigma)\mathrm{u}(\varsigma)\,\mathrm{d}\varsigma\right\|^2\\
&+6\mathbb{E}\left\|\int_{t-\epsilon}^{t}\mathscr{R}(t-\varsigma)\mathscr{B}(\varsigma)\mathrm{u}(\varsigma)\,\mathrm{d}\varsigma\right\|^2\\
&+6\mathbb{E}\left\|\int_0^{t-\epsilon}[\mathscr{R}(\epsilon)\mathscr{R}(t-\epsilon-\varsigma)-\mathscr{R}(t-\varsigma)] \times\mathfrak{l}(\varsigma,\vartheta(\varsigma),\vartheta(\mathscr{A}_1(\varsigma)),\vartheta(\mathscr{A}_2(\varsigma)), \ldots\vartheta(\mathscr{A}_{\mathrm{m}}(\varsigma)))\,\mathrm{d}\varsigma\right\|^2\\
&+6\mathbb{E}\left\|\int_{t-\epsilon}^{t}\mathscr{R}(t-\varsigma)\mathfrak{l}(\varsigma,\vartheta(\varsigma),\vartheta(\mathscr{A}_1(\varsigma)), \vartheta(\mathscr{A}_2(\varsigma)),\ldots\vartheta(\mathscr{A}_{\mathrm{m}}(\varsigma)))\,\mathrm{d}\varsigma\right\|^2\\
&+6\mathbb{E}\left\|\int_0^{t-\epsilon}[\mathscr{R}(\epsilon)\mathscr{R}(t-\epsilon-\varsigma)-\mathscr{R}(t-\varsigma)] \times\int_0^{\varsigma}\sigma\,(s,\vartheta(s),\vartheta(\mathscr{B}_1(s)),\vartheta(\mathscr{B}_2(s)), \ldots\vartheta(\mathscr{B}_{\mathrm{n}}(s)))\,\mathrm{d}\omega(s)\,\mathrm{d}\varsigma\right\|^2\\
&+6\mathbb{E}\left\|\int_{t-\epsilon}^{t}\mathscr{R}(t-\varsigma)\int_0^{t}\sigma\,(s,\vartheta(s),\vartheta(\mathscr{B}_1(s)), \vartheta(\mathscr{B}_2(s)),\ldots\vartheta(\mathscr{B}_{\mathrm{n}}(s)))\,\mathrm{d}\omega(s)\,\mathrm{d}\varsigma\right\|^2\\
&\le 6(t-\epsilon)\|\mathscr{B}\|^2\int_0^{t-\epsilon}\|\mathscr{R}(\epsilon)\mathscr{R}(t-\epsilon-\varsigma) -\mathscr{R}(t-\varsigma)\|^2\,\mathbb{E}\|\mathrm{u}(\varsigma)\|^2\,\mathrm{d}\varsigma\\
&+6\mathfrak{N}^2\|\mathscr{B}\|^2\epsilon^{\frac{2p-2}{p}}\|\mathrm{u}\|_{\mathfrak{L}_p(\mathscr{J},\mathcal{U})}\\
&+6\epsilon^2\mathfrak{N}^2(1+r)+6(t-\epsilon)(1+r) \int_0^{t-\epsilon}\|\mathscr{R}(\epsilon)\mathscr{R}(t-\epsilon-\varsigma)-\mathscr{R}(t-\varsigma)\|^2\,\mathrm{d}\varsigma\\
&+6(t-\epsilon)Tr(\mathcal{Q})\int_0^{t-\epsilon}\|\mathscr{R}(\epsilon)\mathscr{R}(t-\epsilon-\varsigma) -\mathscr{R}(t-\varsigma)\|^2\,h_r(\varsigma)\,\mathrm{d}\varsigma\\
&+6\mathfrak{N}^2\epsilon\,Tr(\mathcal{Q})\int_{t-\epsilon}^{t}h_r(\varsigma)\,\mathrm{d}\varsigma.
\end{aligned}$$

For $\varsigma\in[0,t-\delta]$ it is known that $\mathscr{R}(\epsilon)\mathscr{R}(t-\epsilon-\varsigma)-\mathscr{R}(t-\varsigma)\to 0$ as $\epsilon\to 0$. This implies that there are relatively compact sets to the set $\{(\Psi\vartheta)(t):\vartheta\in\mathbb{B}_r\}$. Hence $\Psi(t)$ is also relatively compact in $\mathbb{B}_r$. Hence Ψ has a fixed point $\vartheta(.)$ on $\mathbb{B}_r$. Thus all conditions of Leray Schauder alternative are satisfied, as a result, the system (6) possesses a mild solution. ■

4. Existence of optimal control

The Lagrange Problem $(\mathcal{P})$ can be considered as follows: Considering, $(\vartheta^0,\mathrm{u}^0)\in P\mathbb{C}(\mathscr{J},\mathfrak{L}_2(\Omega,\mathscr{H}))\times\mathcal{U}_{ad}\ni$

$$\begin{aligned}&\Im(\vartheta^0,\mathrm{u}^0)\le\Im(\vartheta^{\mathrm{u}},\mathrm{u}),\\ &\forall\,(\vartheta,\mathrm{u})\in P\mathbb{C}(\mathscr{J},\mathfrak{L}_2(\Omega,\mathscr{H}))\times\mathcal{U}_{ad},\end{aligned}\tag{11}$$

where

$$\Im(\mathrm{u})=\mathbb{E}\left\{\int_0^{\mathscr{B}}L(t,\vartheta^{\mathrm{u}}(t),\mathrm{u}(t))\,\mathrm{d}t\right\}\tag{12}$$

and ϑ^{u} represents the mild solution of (6) which corresponds to the control $\mathrm{u}\in\mathcal{U}_{ad}$. Regarding the existence of solutions to $(\mathcal{P})$, let us assume the following:

(A6)
- (i) The functional $L:\mathscr{J}\times\mathscr{H}\times\mathcal{U}\to\mathbb{R}\cup\{\infty\}$ is $\mathcal{F}_t$-measurable.
- (ii) On $\mathscr{H}\times\mathcal{U}$, $L(t,.,.)$ is sequentially lower semicontinuous.
- (iii) $L(t,\vartheta,.)$ is convex on $\mathcal{U}$ for each $\vartheta\in\mathscr{H}$ and almost all $t\in\mathscr{J}$.
- (iv) There exist constants $\mathfrak{d}\ge 0$, $\mathfrak{f}>0$, $\mu\ge 0$ and $\mu\in\mathfrak{L}'(\mathscr{J},\mathbb{R})\ni$

$$L(t,\vartheta,\mathrm{u})\ge\mu(t)+\mathfrak{d}\mathbb{E}\,\|\vartheta\|^p_{\mathscr{H}}+\mathfrak{f}\,\|\mathrm{u}\|^p_{\mathcal{U}}.$$

Theorem 4.1: *The Lagrange problem $(\mathcal{P})$ permits at least one optimum pair if assumptions* (A1)–(A6) *are true and $\mathscr{B}$ is a strongly continuous operator, (i .e.) $\exists$ an admissible control pair $(\vartheta^0,\mathrm{u}^0)\in P\mathbb{C}(\mathscr{J},\mathfrak{L}_2(\Omega,\mathscr{H}))\times\mathcal{U}_{ad}\ni$*

$$\begin{aligned}&\Im(\vartheta^0,\mathrm{u}^0)=\mathbb{E}\left(\int_0^{\mathscr{B}}\Im(t,\vartheta^0(t),\mathrm{u}^0(t))\,\mathrm{d}t\right)\le\Im(\vartheta^{\mathrm{u}},\mathrm{u}),\\ &\forall\,(\vartheta^{\mathrm{u}},\mathrm{u})\in P\mathbb{C}(\mathscr{J},\mathfrak{L}_2(\Omega,\mathscr{H}))\times\mathcal{U}_{ad}.\end{aligned}\tag{13}$$

Proof: If $\inf\{\Im(\vartheta^{\mathrm{u}},\mathrm{u})\mid(\vartheta^{\mathrm{u}},\mathrm{u})\in P\mathbb{C}(\mathscr{J},\mathfrak{L}_2(\Omega,\mathscr{H}))\times\mathcal{U}_{ad}\}=+\infty$, there is nothing to demonstrate. Without losing generality, we suppose $\inf\{\Im(\vartheta^{\mathrm{u}},\mathrm{u})\mid(\vartheta^{\mathrm{u}},\mathrm{u})$

$\in P\mathbb{C}(\mathscr{J},\mathfrak{L}_2(\Omega,\mathscr{H}))\times\mathcal{U}_{ad}\}=\rho<+\infty$. Applying (A6), we have $\rho>-\infty$. A minimizing sequence of feasible pair $\{(\vartheta^{\hat{m}},\mathrm{u}^{\hat{m}})\}\subset\mathcal{P}_{ad}$ exists according to the definition of infimum, where $\mathcal{P}_{ad}=\{(\vartheta,\mathrm{u}):\vartheta$ is a mild solution of system (1) corresponding to $\mathrm{u}\in\mathcal{U}_{ad}\}$ $\ni\Im(\vartheta^{\hat{m}},\mathrm{u}^{\hat{m}})\to\rho$ as $\hat{m}\to+\infty$. As $\{\mathrm{u}^{\hat{m}}\}\subseteq\mathcal{U}_{ad}$, $\hat{m}=1,2,\ldots,\mathrm{u}^{\hat{m}}$ is a bounded subset of the separable reflexive Banach space $\mathfrak{L}^p(\mathscr{J},\mathcal{U})$, $\exists$ a subsequence, relabelled as $\mathrm{u}^{\hat{m}}$ and $\mathrm{u}^0\in\mathfrak{L}^p(\mathscr{J},\mathcal{U})\ni\mathrm{u}^{\hat{m}}\to\mathrm{u}^0$ weakly in $\mathfrak{L}^p(\mathscr{J},\mathcal{U})$. Since $\mathcal{U}_{ad}$ is closed and convex. Through Marzur lemma [33], $\mathrm{u}^0\in\mathcal{U}_{ad}$. Let $\{\mathrm{u}^{\hat{m}}\}$ denotes the sequence of solutions of the system corresponding to $\mathrm{u}^{\hat{m}}$, ϑ^0 is the mild solution that accords to u^0. $\vartheta^{\hat{m}},\vartheta^0$ fulfil the integral equations:

$$\begin{aligned}\vartheta^{\hat{m}}(t)&=\mathscr{R}(t)\vartheta_0+\sum_{k=1}^{q}\mathscr{R}(t-t_k)\mathcal{I}(\vartheta^{\hat{m}}(t_k))\\&\quad+\int_0^t\mathscr{R}(t-\varsigma)\mathscr{B}(\varsigma)\mathrm{u}^{\hat{m}}(\varsigma)\\&\quad+\int_0^t\mathscr{R}(t-\varsigma)\mathfrak{l}\Big(\varsigma,\vartheta^{\hat{m}}(\varsigma),\vartheta^{\hat{m}}(\mathscr{A}_1(\varsigma)),\\&\quad\vartheta(\mathscr{A}_2(\varsigma)),\ldots\vartheta^{\hat{m}}(\mathscr{A}_{\mathrm{m}}(\varsigma))\Big)\,\mathrm{d}\varsigma\\&\quad+\int_0^t\mathscr{R}(t-\varsigma)\Big(\int_0^{\varsigma}\sigma\Big(s,\vartheta^{\hat{m}}(s),\vartheta^{\hat{m}}\\&\quad(\mathscr{B}_1(s)),\vartheta^{\hat{m}}(\mathscr{B}_2(s)),\\&\quad\ldots\vartheta^{\hat{m}}(\mathscr{B}_{\mathrm{n}}(s))\Big)\,\mathrm{d}\omega(s)\Big)\,\mathrm{d}\varsigma\end{aligned}\tag{14}$$

and

$$\begin{aligned}\vartheta^{0}(t)&=\mathscr{R}(t)\vartheta_0+\sum_{k=1}^{q}\mathscr{R}(t-t_k)\mathcal{I}(\vartheta^{0}(t_k))\\&\quad+\int_0^t\mathscr{R}(t-\varsigma)\mathscr{B}(\varsigma)\mathrm{u}^{0}(\varsigma)\\&\quad+\int_0^t\mathscr{R}(t-\varsigma)\mathfrak{l}\left(\varsigma,\vartheta^{0}(\varsigma),\vartheta^{0}(\mathscr{A}_1(\varsigma)),\right.\\&\quad\left.\vartheta(\mathscr{A}_2(\varsigma)),\ldots\vartheta^{0}(\mathscr{A}_{\mathrm{m}}(\varsigma))\right)\mathrm{d}\varsigma\\&\quad+\int_0^t\mathscr{R}(t-\varsigma)\Big(\int_0^{\varsigma}\sigma\left(s,\vartheta^{0}(s),\vartheta^{0}(\mathscr{B}_1(s)),\right.\\&\quad\left.\vartheta^{0}(\mathscr{B}_2(s)),\ldots\vartheta^{0}(\mathscr{B}_{\mathrm{n}}(s))\right)\mathrm{d}\omega(s)\Big)\,\mathrm{d}\varsigma.\end{aligned}\tag{15}$$

From the boundedness of $\{\mathrm{u}^{\hat{m}}\}$, $\{\mathrm{u}^0\}$, and Theorem 3.1, $\exists$ a constant $\delta\ni\mathbb{E}\|\vartheta^{\hat{m}}\|^2\le\delta,\mathbb{E}\|\vartheta^0\|^2\le\delta$.

$$\begin{aligned}&\mathbb{E}\|\vartheta^{\hat{m}}(t)-\vartheta^0(t)\|^2\\&\le4\mathbb{E}\left\|\sum_{k=1}^{q}\mathscr{R}(t-t_k)[\mathcal{I}_k(\vartheta^{\hat{m}}(t_k))-\mathcal{I}_k(\vartheta^0(t_k))]\right\|^2\\&\quad+4\mathbb{E}\left\|\int_0^t\mathscr{R}(t-\varsigma)\right.\\&\quad\left.\times[\mathscr{B}(\varsigma)\mathrm{u}^{\hat{m}}(\varsigma)-\mathscr{B}(\varsigma)\mathrm{u}^0(\varsigma)]\right\|^2\\&\quad+4\mathbb{E}\left\|\int_0^t\mathscr{R}(t-\varsigma)\right.\\&\quad\times\Big[\mathfrak{l}\Big(\varsigma,\vartheta^{\hat{m}}(\varsigma),\vartheta^{\hat{m}}(\mathscr{A}_1(\varsigma)),\\&\quad\vartheta(\mathscr{A}_2(\varsigma)),\ldots\vartheta^{\hat{m}}(\mathscr{A}_{\mathrm{m}}(\varsigma))\Big)\\&\quad-\mathfrak{l}\left(\varsigma,\vartheta^0(\varsigma),\vartheta^0(\mathscr{A}_1(\varsigma)),\vartheta(\mathscr{A}_2(\varsigma)),\right.\\&\quad\left.\left.\ldots\vartheta^0(\mathscr{A}_{\mathrm{m}}(\varsigma))\right)\Big]\,\mathrm{d}s\right\|^2\\&\quad+4\mathbb{E}\left\|\int_0^t\mathscr{R}(t-\varsigma)\right.\\&\quad\times\Big[\int_0^s\Big(\sigma\Big(s,\vartheta^{\hat{m}}(s),\vartheta^{\hat{m}}(\mathscr{B}_1(s)),\\&\quad\vartheta^{\hat{m}}(\mathscr{B}_2(s)),\ldots\vartheta^{\hat{m}}(\mathscr{B}_{\mathrm{n}}(s))\Big)\\&\quad-\sigma\left(s,\vartheta^0(s),\vartheta^0(\mathscr{B}_1(s)),\vartheta^0(\mathscr{B}_2(s)),\right.\\&\quad\left.\left.\ldots\vartheta^0(\mathscr{B}_{\mathrm{n}}(s))\right)\Big)\,\mathrm{d}\omega(\varsigma)\Big]\,\mathrm{d}\varsigma\right\|^2\\&\le4\mathfrak{N}^2q\sum_{k=1}^{q}\mathbb{E}\left\|\mathcal{I}_k(\vartheta^{\hat{m}}(t_k))-\mathcal{I}_k(\vartheta^0(t_k))\right\|^2\\&\quad+4\mathfrak{N}^2\Big[\Big(\int_0^t\mathbb{E}\left\|\mathscr{B}(\varsigma)\mathrm{u}^{\hat{m}}(\varsigma)\right.\\&\quad\left.-\mathscr{B}(\varsigma)\mathrm{u}^0(\varsigma)\right\|^p\mathrm{d}\varsigma\Big)^{1/p}\Big(\int_0^t\mathrm{d}\varsigma\Big)^{1-1/p}\Big]^2\\&\quad+4\mathfrak{N}^2\mathscr{B}\int_0^t\mathbb{E}\left\|\Big[\mathfrak{l}\Big(\varsigma,\vartheta^{\hat{m}}(\varsigma),\vartheta^{\hat{m}}(\mathscr{A}_1(\varsigma)),\right.\\&\quad\vartheta(\mathscr{A}_2(\varsigma)),\ldots\vartheta^{\hat{m}}(\mathscr{A}_{\mathrm{m}}(\varsigma))\Big)\\&\quad-\mathfrak{l}\left(\varsigma,\vartheta^0(\varsigma),\vartheta^0(\mathscr{A}_1(\varsigma)),\vartheta(\mathscr{A}_2(\varsigma)),\right.\\&\quad\left.\left.\ldots\vartheta^0(\mathscr{A}_{\mathrm{m}}(\varsigma))\right)\Big]\right\|^2\mathrm{d}\varsigma+4\mathfrak{N}^2\mathscr{B}Tr(\mathcal{Q})\\&\quad\times\int_0^t\int_0^{\varsigma}\mathbb{E}\Big(\sigma\Big(s,\vartheta^{\hat{m}}(s),\vartheta^{\hat{m}}(\mathscr{B}_1(s)),\\&\quad\vartheta^{\hat{m}}(\mathscr{B}_2(s)),\ldots\vartheta^{\hat{m}}(\mathscr{B}_{\mathrm{n}}(s))\Big)\\&\quad-\sigma\left(s,\vartheta^0(s),\vartheta^0(\mathscr{B}_1(s)),\vartheta^0(\mathscr{B}_2(s)),\right.\\&\quad\left.\ldots\vartheta^0(\mathscr{B}_{\mathrm{n}}(s))\right)\Big)\\&\le4\mathfrak{N}^2q\sum_{k=1}^{q}\mathfrak{N}_k\mathbb{E}\left\|\vartheta^{\hat{m}}-\vartheta^0\right\|^2\\&\quad+4\mathfrak{N}^2\mathscr{B}^{\frac{2p-2}{p}}\mathbb{E}\left\|\mathscr{B}\mathrm{u}^{\hat{m}}-\mathscr{B}\mathrm{u}^0\right\|^2_{\mathfrak{L}_p(\mathscr{J},\mathcal{U})}\\&\quad+4\mathfrak{N}^2\mathscr{B}\int_0^t\mathfrak{N}_{\mathfrak{l}}\mathbb{E}\left\|\vartheta^{\hat{m}}-\vartheta^0\right\|^2\mathrm{d}\varsigma\end{aligned}$$

$$+4\mathfrak{N}^2\mathscr{B}^2 Tr(\mathcal{Q})\int_0^t \mathfrak{N}_\sigma \mathbb{E}\left\|\vartheta^{\hat{m}}-\vartheta^0\right\|^2 \mathrm{d}\varsigma.$$

Now,

$$\begin{aligned}\sup_{t\in\mathscr{J}} &\mathbb{E}\|\vartheta^{\hat{m}}(t)-\vartheta^0(t)\|^2\\ &\leq 4\mathfrak{N}^2 q\sum_{k=1}^{q}\mathfrak{N}_k \sup_{t\in\mathscr{J}}\mathbb{E}\left\|\vartheta^{\hat{m}}-\vartheta^0\right\|^2\\ &\quad+4\mathfrak{N}^2\mathscr{B}^{\frac{2p-2}{p}}\mathbb{E}\left\|\mathscr{B}\mathrm{u}^{\hat{m}}-\mathscr{B}\mathrm{u}^0\right\|^2_{\mathfrak{L}_p(\mathscr{J},\mathcal{U})}\\ &\quad+4\mathfrak{N}^2\mathscr{B}^2\sup_{t\in\mathscr{J}}\mathfrak{N}_{\mathfrak{l}}\mathbb{E}\left\|\vartheta^{\hat{m}}-\vartheta^0\right\|^2\\ &\quad+4\mathfrak{N}^2\mathscr{B}^3 Tr(\mathcal{Q})\sup_{t\in\mathscr{J}}\mathfrak{N}_\sigma\mathbb{E}\left\|\vartheta^{\hat{m}}-\vartheta^0\right\|^2\\ &\leq \mathrm{W}^*\mathbb{E}\left\|\mathscr{B}\mathrm{u}^{\hat{m}}-\mathscr{B}\mathrm{u}^0\right\|^2_{\mathfrak{L}_p(\mathscr{J},\mathcal{U})},\quad t\in\mathscr{J}',\end{aligned}$$

where

$$\mathrm{W}^*=\frac{4\mathfrak{N}^2\mathscr{B}^{\frac{2p-2}{p}}}{1-\left[4\mathfrak{N}^2 q\sum_{k=1}^{q}\mathfrak{N}_k+4\mathfrak{N}^2\mathscr{B}^2\mathfrak{N}_{\mathfrak{l}}+4\mathfrak{N}^2\mathscr{B}^3 Tr(\mathcal{Q})\mathfrak{N}_\sigma\right]}$$

is a constant. Since $\mathscr{B}$ is strongly continuous, $\|\mathscr{B}\mathrm{u}^{\hat{m}}-\mathscr{B}\mathrm{u}^0\|^2_{\mathfrak{L}_p(\mathscr{J},\mathcal{U})}\xrightarrow{w} 0$ as $\hat{m}\to\infty$.

Then we have, $\mathbb{E}\|\vartheta^{\hat{m}}-\vartheta^0\|^2\xrightarrow{w} 0$ as $\hat{m}\to\infty$ yields $\vartheta^{\hat{m}}\xrightarrow{w}\vartheta^0$ as $\hat{m}\to\infty$. (H6) implies Balder's hypotheses [34]. Henceforth, $(\vartheta,\mathrm{u})\to\mathbb{E}(\int_0^{\mathscr{B}} L(t,\vartheta(t),\mathrm{u}(t)))$ satisfies the assumptions in the weak topology of $\mathfrak{L}^p(\mathscr{J},\mathcal{U})\subset\mathfrak{L}^1(\mathscr{J},\mathcal{U})$ and strong topology of $\mathfrak{L}^1(\mathscr{J},\mathcal{U})$. As a result, on $\mathfrak{L}^p(\mathscr{J},\mathcal{U})$ $\mathcal{I}$ is weakly lower semicontinuous and by (A6)(iv), $\mathfrak{I}>-\infty$, $\mathcal{I}$ reaches its infimum at $\mathrm{u}^0\in\mathcal{U}_{ad}$, (i.e.)

$$\begin{aligned}\rho&=\lim_{\hat{m}\to\infty}\mathbb{E}\left(\int_0^{\mathscr{B}} L(t,\vartheta^{\hat{m}}(t),\mathrm{u}^{\hat{m}}(t))\,\mathrm{d}t\right)\\ &\geq\mathbb{E}\left(\int_0^{\mathscr{B}} L(t,\vartheta^{\hat{m}}(t),\mathrm{u}^{\hat{m}}(t))\,\mathrm{d}t\right)=\mathcal{I}(\vartheta^0,\mathrm{u}^0)\geq\rho.\end{aligned}$$

Hence the proof of optimal controllability. ■

5. Illustrations

5.1. Example 1

Consider,

$$\begin{aligned}\frac{\partial}{\partial t}\vartheta(t,\varpi)&=\frac{\partial^2}{\partial\varpi^2}\vartheta(t,\varpi)\\ &\quad+\int_0^t\alpha(t-s)\frac{\partial^2}{\partial\varpi^2}\vartheta(s,\varpi)\,\mathrm{d}s\\ &\quad+\int_0^1 e(\varpi,s)\mathrm{u}(s,t)\,\mathrm{d}s\\ &\quad+\int_0^1\mu(s,\varpi)\vartheta(t\sin t,s)\,\mathrm{d}s\\ &\quad+\int_0^1\mathfrak{g}(s,\vartheta(s\sin s,\varpi))\,\mathrm{d}\omega(s),\\ &\quad t\in[0,1],t\neq t_k,\ k=1,2,\dots\mathrm{q},\\ &\quad\varpi\in[0,1]\\ \triangle\vartheta(t_k,\varpi)&=\mathfrak{I}_k(\vartheta(t_k,\varpi)),\quad k=1,2,\dots\mathrm{q},\\ \vartheta(t,0)&=\vartheta(t,1)=0,\quad t\in\mathscr{J},\\ \vartheta(0,\varpi)&=\vartheta_0(\varpi),\quad\varpi\in[0,1].\end{aligned}\tag{16}$$

Let $\mathscr{H}=\mathcal{U}=\mathfrak{L}_2[0,1]$, $\omega(t)$ is a standard cylindrical Wiener process in $\mathscr{H}$ defined on $(\Omega,\mathcal{F},\{\mathcal{F}\}_{t\geq0},\mathbb{P})$. Let $\mathscr{A}:\mathcal{D}(\mathscr{A})\subseteq\mathscr{H}\to\mathscr{H}$ expressed by $\mathscr{A}\tau=\tau''$ with the domain $\mathcal{D}(\mathscr{A})=\{\tau\in\mathscr{H}:\tau,\tau'$ are absolutely continuous, $\tau''\in\mathscr{H},\tau(0)=\tau(1)=0\}$, $\mathscr{A}\tau=\sum_{j=1}^{\infty}-j^2\langle\tau,e_j\rangle e_j$, the normalized eigenfunctions are $e_j(s)=\sqrt{\frac{2}{\pi}}\sin(js)$, $j=1,2,\dots$. $\mathscr{R}(t)\tau=\sum_{j=1}^{\infty}e^{-j^2t}\langle\tau,e_j\rangle e_j$ $\forall\tau\in\mathscr{H}$. Thus $\{\mathscr{R}(t)\}_{t\geq0}$ becomes uniformly bounded compact semigroup. Also,

$$\left\|\mathscr{R}(t)\right\|\leq 1,\quad t>0.$$

Let us consider the control function to be $\mathrm{u}:\Pi\vartheta([0,1])\to\mathbb{R}\ni\mathrm{u}\in\mathfrak{L}^2(\Pi\vartheta[0,1])$, which implies $t\to\mathrm{u}(t)$ is measurable. The set $\mathscr{A}:=\{\mathrm{u}\in\mathcal{U}:\|\mathrm{u}\|_{\mathcal{U}}\leq\kappa\}$, $\kappa\in\mathfrak{L}_2(\mathfrak{J},\mathbb{R}^+)$. To exhibit 16 in the abstract form of (6), we incorporate

$$\begin{aligned}\vartheta(t)(\varpi)&=\vartheta(t,\varpi),\quad t\in[0,1];\\ \mathscr{B}(t)\mathrm{u}(t)(\varpi)&=\int_0^1 e(\varpi,s)\mathrm{u}(s,t)\,\mathrm{d}s\\ \mathfrak{l}(t,\xi)(\varpi)&=\int_0^1\mu(s,\varpi)\vartheta(s)\,\mathrm{d}s;\\ \sigma(t,\xi)(\varpi)&=\sigma(t,\vartheta(t\sin t,\varpi))\\ \phi(t)(\varpi)&=\phi_0(\varpi),\quad\varpi\in[0,1].\end{aligned}$$

Furthermore, we assume the functions

(i) $e:[0,1]\times[0,1]\to\mathbb{R}$ being continuous.
(ii) $\int_0^1\int_0^1\mu^2(s,\varpi)\,\mathrm{d}s\,\mathrm{d}\varpi<\infty$, μ being measurable.
(iii) $\frac{\partial}{\partial\varpi}\mu(s,\varpi)$ is measurable, $\mu(s,0)=\mu(s,1)=0$, and

$$\left(\int_0^1\int_0^1\left(\frac{\partial}{\partial\varpi}\mu(s,\varpi)\right)^2\mathrm{d}s\,\mathrm{d}\varpi\right)^{1/2}<\infty.$$

(iv) For the map $\sigma:\mathscr{J}\times\mathbb{R}\to\mathbb{R}$, the succeeding circumstances exist:
  (a) $\sigma(t,.)$ is continuous for $t\in\mathscr{J}$,
  (b) $\forall\vartheta\in\mathscr{H},\sigma(.,\vartheta)$ is measurable,
  (c) $\sigma_1,\sigma_2\in\mathfrak{L}^1(\mathscr{J})\ni\ \forall\ (t\in\vartheta)\in\mathscr{J}\times\mathscr{H}$, $\|\sigma(t,\vartheta)\|\leq\sigma_1(t)\|\vartheta\|+\sigma_2(t)$.

Take into consideration, the cost function

$$\mathfrak{J}(\mathrm{u}) := \mathbb{E}\left\{\int_0^t L(t,\vartheta^{\mathrm{u}}(t),\mathrm{u}(t))\,\mathrm{d}t\right\},$$

where $L(t,\vartheta^{\mathrm{u}}(t),\mathrm{u}(t))(\varpi) := \int_0^1\int_0^1 |\vartheta^{\mathrm{u}}(t,\varpi)|^2\,\mathrm{d}\varpi\,\mathrm{d}t$. It is clear that the assumptions of Theorem 3.1 are met, implying there is at least one optimal pair. Hence justified.

5.2. Example 2: ethanol fuelled engine

An analysis of 88 observations on the exhaust from an engine running on ethanol was conducted for this mechanical engineering example. The concentration of nitric oxide and nitrogen dioxide, normalized by the engine's workload, is the response variable, represented by NO_x. The engine's compression ratio C and the equivalency ratio E, which measures the fuel-air mixture, are the two predictors. The information is displayed in Figure 1. Plotting NO_x vs E and C is displayed in Figure 1(a,b).

The basic model $NO_x \approx E^2$ is suggested by the strong quadratic-like influence of E and the seemingly little effect of C. NO_x vs E is depicted in Figure 1(c), where C levels are classified as low, medium and high. This implies that C may be interacting with E. The structure of this interaction is seen in Figure 2. The fitted linear regressions of NO_x on C in four non-overlapping ranges of E are displayed by the broken lines. A linear model in C appears to fit well inside each range of E. However, the intercept and slope of the line both change as E does. This prompts us to think of a model of the kind

$$NO_x = \beta_0(E) + \beta_1(E)C + \epsilon. \qquad (17)$$

Despite the plots' suggestion that $\beta_0(E) \approx E^2$, we will fit $\beta_0(E)$ and $\beta_1(E)$ flexibly and leave them both undetermined. Cleveland et al. [35] examined this model, which is an illustration of a varying-coefficient model. The term $\mathfrak{l}(t,\vartheta(t),\vartheta(\mathscr{A}_1(t)),\vartheta(\mathscr{A}_2(t)),\ldots\vartheta(\mathscr{A}_{\mathrm{m}}(t)))$ of (1) is comparable to the term in Equation (17). By integrating an additional engine with the suggested engine, we arrive at model (6), from which the best control of exhaust ethanol engines may be studied.

6. Conclusions

This article is devoted to studying the optimal controls for stochastic integrodifferential equation (SIDE) in Hilbert space. Necessary parameters are imposed to demonstrate the system that follows a unique variation of parameter formula using Leray Schauder alternative. Subsequently, the existence of optimal control is investigated for the considered Lagrange control problem. We

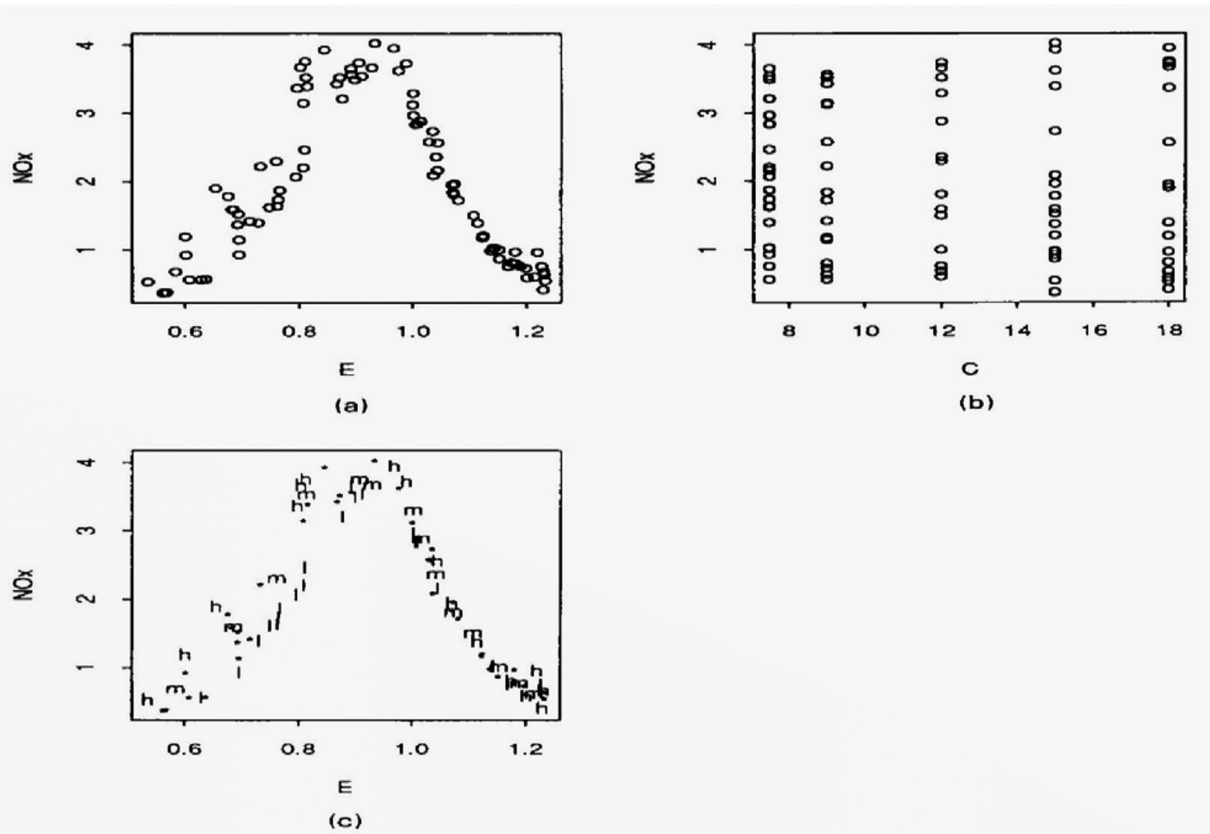

Figure 1. Using certain values of C coded as low (l), medium (m), or high (h) (intermediate values are coded with $*$), (a) NO_x versus E; (b) NO_x versus C and (c) NO_x versus E.

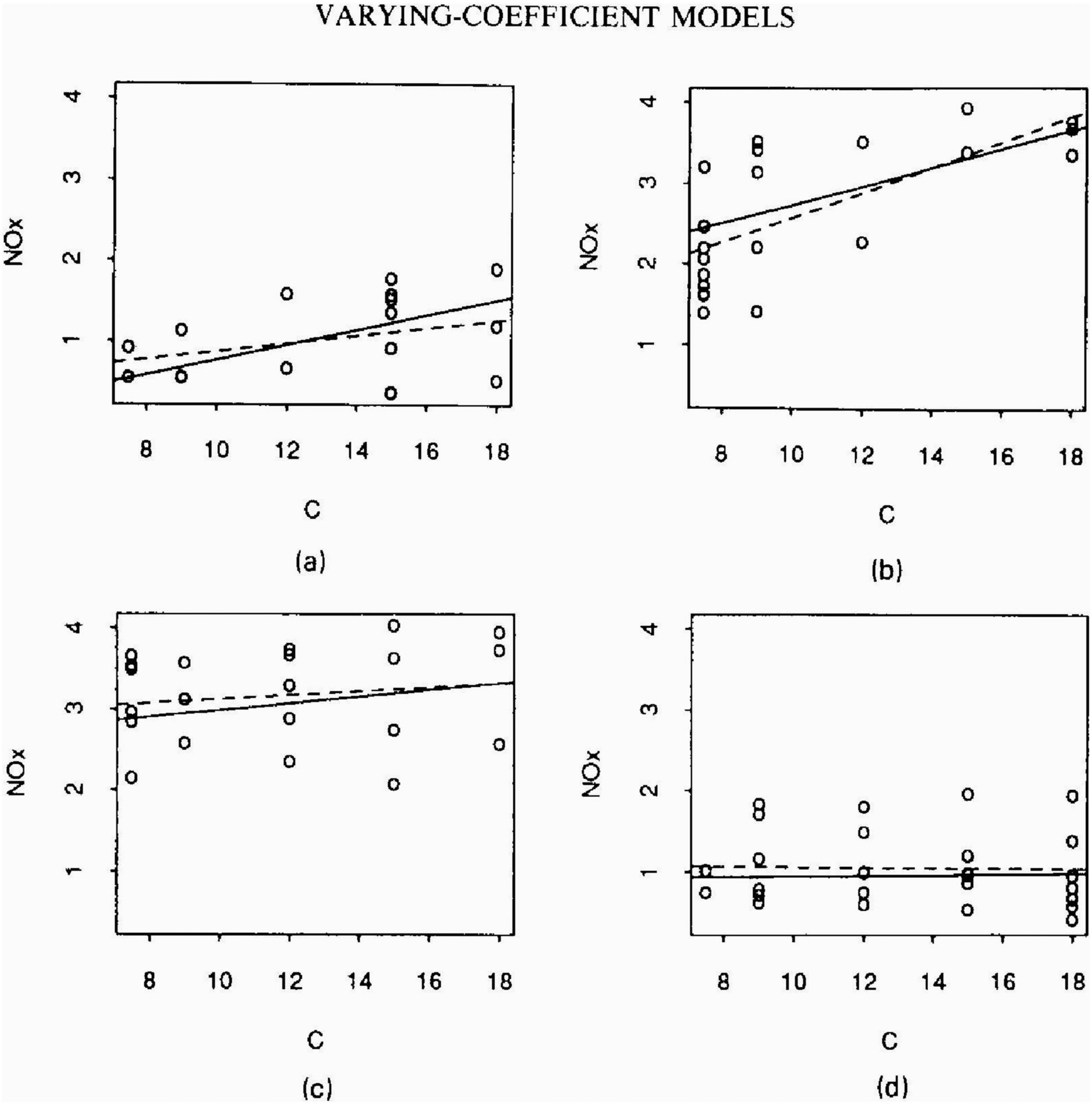

Figure 2. NO_x vs C for (a) low, (b) medium, (c) high and (d) very high values of E: – -, fitted linear regression; – - fitted lines from the varying-coefficient model, chosen at the median value of E for the panel's data.

explored a class of regression and generalized regression models in which the coefficients are allowed to vary as smooth functions of other variables. General algorithms are presented for estimating the models flexibly. This class of models ties together generalized additive models and dynamic generalized linear models into one common framework. When applied to the proportional hazards model for survival data, this approach provides a new way of modelling departures from the proportional hazards assumption.

There are some directions in which this work could be extended.

(1) The effect modifier R might be vector valued, in which case a multidimensional smoother would be used in the estimation procedure for the function $\beta(R)$. The conditionally parametric models of Cleveland et al. [35] automatically allow for this case when all the terms are modelled conditionally on the same R.
(2) One would look for directions in the effect-modifier space that result in large changes in the coefficients.
(3) Model (6) can be generalized to Hilfer fractional order model with the integral boundary conditions.
(4) Proposed model (6) is w.r.t. time varying-coefficient, one can extend the same model with state varying-coefficient with the proper arguments and with the corresponding real-life applications.
(5) Second order system in the frame work of (6) can be studied using sine and cosine operators. Numerical simulation will be interesting as well to justify the theory [36].

Acknowledgments

The authors would like to thank the referees and the editor for their careful comments and valuable suggestions to improve this manuscript. All authors contributed equally in this research work.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Ethical statement

No human and/or animal study is involved in this research, so this is not applicable.

Data availability statement

Authors have not used any scientific data to be determined for this research.

ORCID

Dimplekumar Chalishajar http://orcid.org/0000-0002-6146-5544

References

[1] Balasubramaniam P, Tamilalagan P. The solvability and optimal controls for impulsive fractional stochastic integro-differential equations via resolvent operators. J Optim Theory Appl. 2017;174:139–155. doi: 10.1007/s10957-016-0865-6
[2] Ahmed HM, El-Borai MM. Hilfer fractional stochastic integro-differential equations. Appl Math Comput. 2018;331:182–189.
[3] Ahmed HM. Semilinear neutral fractional stochastic integro-differential equations with nonlocal-conditions. J Theor Probab. 2015;28:667–680. doi: 10.1007/s10959-013-0520-1
[4] Mao X. Stochastic differential equations and applications. Woodhead publisher/Glasgow: Elsevier; 2007.
[5] Oksendal B. Stochastic differential equations: an introduction with applications. Springer-Verlag, Heidelberg, NY: Springer Science & Business Media; 2013.
[6] Da Prato G, Zabczyk J. Stochastic equations in infinite dimensions. Cambridge: Cambridge University Press; 1992.
[7] Yang X, Zhu Q. pth moment exponential stability of stochastic partial differential equations with Poisson jumps. Asian J Control. 2014;16(5):1482–1491. doi: 10.1002/asjc.v16.5
[8] Caraballo T, Diop MA, Mane A. Controllability for neutral stochastic functional integrodifferential equations with infinite delay. Appl Math Nonlinear Sci. 2016;1(2):493–506. doi: 10.21042/AMNS.2016.2.00039
[9] Diop M, Gbaguidi AA, Ogouyandjou C, et al. Existence results for impulsive stochastic neutral integrodifferential equations with state-dependent delay. Trans Razmadze Math Inst. 2019;173(06):17–31.
[10] Diop MA, Ezzinbi K, Mamadou PL, et al. Existence results for some partial stochastic integrodifferential equations with nonlocal conditions in Hilbert spaces. Res Math. 2022;9(1). 2043019. doi: 10.1080/27658449.2022.2043019
[11] Sathiyaraj T, Wang J, Balasubramaniam P. Controllability and optimal control for a class of time-delayed fractional stochastic integro-differential systems. Appl Math Optim. 2021;84:2527–2554. doi: 10.1007/s00245-020-09716-w
[12] Al-Hussein A. Necessary conditions for optimal control of stochastic evolution equations in Hilbert spaces. Appl Math Optim. 2011;63(3):385–400. doi: 10.1007/s00245-010-9125-6
[13] Agrawal OP. A general formulation and solution scheme for fractional optimal control problems. Nonlinear Dyn. 2004;38(1):323–337. doi: 10.1007/s11071-004-3764-6
[14] Agrawal OP, Defterli O, Baleanu D. Fractional optimal control problems with several state and control variables. J Vibr Control. 2010;16:1967–1976. doi: 10.1177/1077546309353361
[15] Fan Z, Mophou G. Existence of optimal controls for a semilinear composite fractional relaxation equation. Rep Math Phys. 2014;73:311–323. doi: 10.1016/S0034-4877(14)60047-1
[16] Wang J, Zhou Y. A class of fractional evolution equations and optimal controls. Nonlinear Anal Real World Appl. 2011;12(1):262–272. doi: 10.1016/j.nonrwa.2010.06.013
[17] Liu X, Liu Z, Han J. The solvability and optimal controls for some fractional impulsive equation. In: Abstract and applied analysis; 2013.
[18] Tamilalagan P, Balasubramaniam P. The solvability and optimal controls for fractional stochastic differential equations driven by Poisson jumps via resolvent operators. Appl Math Optim. 2018;77(3):443–462. doi: 10.1007/s00245-016-9380-2
[19] Tang C, Liu J. The equivalence conditions of optimal feedback control-Strategy operators for zero-sum linear quadratic stochastic differential game with random coefficients. Symmetry. 2023;15(9):1726. doi: 10.3390/sym15091726
[20] He S, Fang H, Zhang M, et al. Online policy iterative-based H_∞ optimization algorithm for a class of nonlinear systems. Inf Sci (Ny). 2019;495:1–13. doi: 10.1016/j.ins.2019.04.027
[21] He S, Fang H, Zhang M, et al. Adaptive optimal control for a class of nonlinear systems: the online policy iteration approach. IEEE Trans Neural Netw Learn Syst. 2020;31(2):549–558.
[22] Ramkumar K, Ravikumar K, Varshini S. Fractional neutral stochastic differential equations with caputo fractional derivative: fractional Brownian motion, poisson jumps, and optimal control. Stoch Anal Appl. 2020;39(1):157–176. doi: 10.1080/07362994.2020.1789476
[23] Wang J, Zhou Y, Medved M. On the solvability and optimal controls of fractional integrodifferential evolution systems with infinite delay. J Optim Theory Appl. 2012;152(1):31–50. doi: 10.1007/s10957-011-9892-5
[24] Harrat A, Nieto JJ, Debbouche A. Solvability and optimal controls of impulsive Hilfer fractional delay evolution inclusions with Clarke subdifferential. J Comput Appl Math. 2018;344:725–737. doi: 10.1016/j.cam.2018.05.031
[25] Li X, Liu Z. The solvability and optimal controls of impulsive fractional semilinear differential equations. Taiwan J Math. 2015;19(2):433–453.
[26] Diop A, Diop MA, Ezzinbi K, et al. Optimal controls problems for some impulsive stochastic integro-differential equations with state-dependent delay. Stochastic Int J Probab Stochastic Processes. 2022;94(8):433–453.
[27] Ren Y, Cheng X, Sakthivel R. On time-dependent stochastic evolution equations driven by fractional

Brownian motion in a Hilbert space with finite delay. Math Methods Appl Sci. 2014;37(14):2177–2184. doi: 10.1002/mma.v37.14
[28] Redjil A, Gherbal HB, Kebiri O. Existence of relaxed stochastic optimal control for G-SDEs with controlled jumps. Stoch Anal Appl. 2023;41(1):115–133. doi: 10.1080/07362994.2021.1991809.
[29] Diop MA, Rathinasamy S, Ndiaye AA. Neutral stochastic integrodifferential equations driven by a fractional Brownian motion with impulsive effects and time-varying delays. Mediterr J Math. 2016;13(5):2425–2442. doi: 10.1007/s00009-015-0632-1
[30] Diop MA, Ezzinbi K, Issaka LM, et al. Stability for some impulsive neutral stochastic functional integro-differential equations driven by fractional Brownian motion. Cogent Math Stat. 2020;7(1):1782120. doi: 10.1080/25742558.2020.1782120
[31] Grimmer RC. Resolvent operators for integral equations in a Banach space. Trans Am Math Soc. 1982;273(1): 333–349. doi: 10.1090/tran/1982-273-01.
[32] Amar A. Nonlinear Leray-Schauder alternatives and application to nonlinear problem arising in the theory of growing cell population. Open Math. 2011;9(4): 851–865.
[33] Podlubny I. Fractional differential equations. San Diego: Academic; 1998.
[34] Balder EJ. Necessary and sufficient conditions for L_1-strong weak lower semicontinuity of integral functionals. Nonlinear Anal. 1987;11:1399–1404. doi: 10.1016/0362-546X(87)90092-7
[35] Cleveland WS, Grosse E, Shyu WM. Local regression models. In: Chambers JM, Hastie T, editors. Statistical models in S. Pacific Grove: Wadsworth and Brooks/Cole; 1991. pp. 47–62.
[36] Chalishajar D, Chalishajar H. Trajectory controllability of second order nonlinear integro-differential system: an analytical and a numerical estimation. Differ Equations Dyn Syst. 2015;23:467–481. doi: 10.1007/s12591-014-0220-z