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SELECTION METHOD FOR INTERPRETABILITY LOGIC IL WITH RESPECT TO VERBRUGGE SEMANTICS

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Abstract. Interpretability logic is a modal logic which formalizes the notion of relative interpretability between first-order arithmetical theories. Veltman semantics is the basic semantics for interpretability logic. Verbrugge semantics is a generalization of Veltman semantics. Selection is one of the methods to establish finite model property of a logical system, as a step towards showing that the system is decidable. In this paper we show that selection method can be applied to Verbrugge models, by adapting techniques used for Kripke models to this more complex setting.

1. Introduction

Finite model property is an important feature of many modal logics, leading to their decidability, somewhat surprisingly, considering their expressive power. Finite model property of a logical system means that, for any formula F, if F is satisfiable, then F is satisfied in a finite model. If the system is sound and complete, decidability is established, roughly, by simultaneous enumeration of theorems of the system and (up to isomorphism) finite models and testing in each step whether the current theorem equals F and whether $\neg F$ is satisfied at the current finite model, until one of these two questions is answered affirmatively. By finite model property, this procedure terminates in finitely many steps and thus decides if F is valid.

Interpretability logic is a modal logic, which extends provability logic with a binary modality $\rhd$, used to express relative interpretability between arithmetical theories. In this paper we focus on interpretability logic as a system of modal logic, so for details on arithmetical aspects we refer the reader to e.g. [8]. In modal logic, usual methods to establish finite model property are filtration and selection, so it is natural to consider these methods for interpretability logic.

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While filtration achieves finiteness by identifying worlds using some equivalence relation, selection takes finitely many worlds from the starting model, disregarding others. In both cases some care is needed to define structures on equivalence classes or selected worlds, respectively, so that F is still satisfied in the obtained finite structure. This is done arguably more naturally with selection, in sense that it is intuitively clear why we select worlds which we select. On the other hand, selection always results in a tree-like structure, which may be quite different from the starting model, while filtration keeps some structural resemblance to the starting model, which is sometimes important, in particular when one needs to establish finite model property w.r.t. a certain subclass of models. For a detailed presentation of both methods for basic modal logic see [1].

Filtration of Verbrugge models[1] is studied in [7] (cf. also [4]), and used to prove finite model property and decidability of various systems of interpretability logic in [5] and [6]. In this paper, we study selection of Verbrugge models. In Section 2, we overview some basic definitions and results. In Section 3, we overview basic facts on n-w-bisimulation, an equivalence between Verbrugge models which is used later to show that selection indeed achieves finite model property. In Section 4, we describe the first phase of the selection process: cutting the model to a finite height. In Section 5 we prove the main result. The proof contains the second phase: selecting finitely many worlds and proving that the obtained structure has the desired property.

2. Preliminaries

In this section we give an overview of basic notions of Verbrugge semantics, i.e., Verbrugge frames and Verbrugge models. We also introduce various notions needed in the rest of this paper.

The syntax of interpretability logic IL is given by $F ::= p \mid \bot \mid F_1 \to F_2 \mid F_1 \rhd F_2$, where p ranges over a fixed set of propositional variables, generally assumed to be enumerable, unless explicitly stated otherwise, namely in results which only hold in the case of a finite alphabet. Other Boolean connectives can be defined as abbreviations as usual. Also, $\Box$ can be defined as an abbreviation, namely $\Box F \equiv \neg F \rhd \bot$, and then $\Diamond F \equiv \neg\Box\neg F$ as usual. We use usual conventions to avoid too many parentheses in longer formulas, with an additional convention that $\rhd$ binds more strongly than $\to$.

We use the definition of Verbrugge models from [9] (where they are called generalized Veltman models). In the following, we use notation $R[w] := \{x : wRx\}$, where R is a binary relation.

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[1] Verbrugge models were usually called generalized Veltman models in the literature until recently. A discussion in interpretability logics community lead to the new terminology, in honor of Rineke Verbrugge who developed the notion in an unpublished note.

Definition 2.1. *An ordered triple* $(W, R, \{S_w : w \in W\})$ *is called a* **Verbrugge frame** *if it satisfies the following conditions:*

(i) *W is a non-empty set and R is a transitive and reverse well-founded binary relation on W*
(ii) *For every $w \in W$, S_w is a subset of $R[w] \times (\mathcal{P}(R[w]) \setminus \{\emptyset\})$*
(iii) *The relation S_w is quasi-reflexive for every $w \in W$, i.e., wRu implies $uS_w\{u\}$*
(iv) *The relation S_w is quasi-transitive for every $w \in W$, i.e., if uS_wV and $(\forall v \in V)(vS_wZ_v)$, then $uS_w\left(\bigcup_{v\in V} Z_v\right)$*
(v) *If $wRuRv$ then $uS_w\{v\}$*
(vi) *If uS_wV and $V \subseteq Z \subseteq R[w]$ then uS_wZ.*

An ordered quadruple $(W, R, \{S_w : w \in W\}, \Vdash)$ is called a ***Verbrugge model*** *if it satisfies the following conditions:*

(i) *$(W, R, \{S_w : w \in W\})$ is a Verbrugge frame*
(ii) *$\Vdash$ is a forcing relation defined as usual in Boolean cases, and*

$$w \Vdash F \rhd G \text{ iff } \forall u((wRu \ \&\ u \Vdash F) \Rightarrow \exists V(uS_wV \& (\forall v \in V)(v \Vdash G))).$$

A **pointed Verbrugge model** *is a pair $(\mathfrak{M}, w)$, where $\mathfrak{M}$ is a Verbrugge model and w is a world in $\mathfrak{M}$.*

In the rest of this paper we will denote by $\mathfrak{M}$ and $\mathfrak{M}'$ Verbrugge models $(W, R, \{S_w : w \in W\}, \Vdash)$ and $(W', R', \{S'_w : w \in W'\}, \Vdash)$, respectively. We will use the symbol $\Vdash$ to denote forcing relations in all Verbrugge models, since the context will always prevent confusion.

To define the notion of n-modal equivalence, for a natural number n, we first need the following auxiliary definition:

Definition 2.2. *The* **modal depth** *is the function $d : Form \to \mathbb{N}$, where $Form$ is the set of* IL*-formulas, defined as follows:*

$$\begin{aligned} d(p) &= 0, \\ d(\bot) &= 0, \\ d(F_1 \to F_2) &= \max\{d(F_1), d(F_2)\}, \\ d(F_1 \rhd F_2) &= 1 + \max\{d(F_1), d(F_2)\}. \end{aligned}$$

In particular, using the aforementioned conventions, we observe that the modal depth of IL-formulas of the form $\Box F$ and $\Diamond F$ is $1+d(F)$. In other words, modal depth of an IL-formula is the maximum number of nested modalities.

Using the notion of modal depth of an IL-formula we can now define modal equivalence and n-modal equivalence of two worlds.

Definition 2.3. *We say that $w \in W$ and $w' \in W'$ are* ***modally equivalent****, and we write $w \equiv w'$, if for every* IL*-formula F, $\mathfrak{M}, w \Vdash F$ if and only if $\mathfrak{M}', w' \Vdash F$, i.e., if they satisfy exactly the same formulas. We say that*

*$w \in W$ and $w' \in W'$ are n-**modally equivalent**, and we write $w \equiv_n w'$, if the equivalence holds for all* **IL**-*formulas of modal depth up to n.*

The notion of n-modal equivalence is closely related with the notion of n-w-bisimulation which we will define in the next section.

3. Weak n-bisimulations for Verbrugge semantics

The notion of n-w-bisimulation between Verbrugge models was defined in [4].

Definition 3.1. *An n-**w-bisimulation** between Verbrugge models $\mathfrak{M}$ and $\mathfrak{M}'$ is a decreasing sequence of relations*

$$Z_n \subseteq Z_{n-1} \subseteq \cdots \subseteq Z_1 \subseteq Z_0 \subseteq W \times W'$$

that possesses the following properties:

(at) *If wZ_0w', then $w \Vdash p$ if and only if $w' \Vdash p'$, for all propositional letters p*

(n-w-forth) *For every $i \in \{1, \dots, n\}$, if wZ_iw' and wRu, then there exists a nonempty set $U' \subseteq R'[w']$ such that for all $u' \in U'$ we have $uZ_{i-1}u'$ and for each function $V' : U' \to \mathcal{P}(W')$ such that for all $u' \in U'$ we have $u'S'_{w'}V'(u')$, there exists a set V such that uS_wV and for all $v \in V$ there exists $v' \in \bigcup_{u' \in U'} V'(u')$ with $vZ_{i-1}v'$*

(n-w-back) *For every $i \in \{1, \dots, n\}$, if wZ_iw' and $w'R'u'$, then there exists a nonempty set $U \subseteq R[w]$ such that for all $u \in U$ we have $uZ_{i-1}u'$ and for each function $V : U \to \mathcal{P}(W)$ such that for all $u \in U$ we have $uS_wV(u)$, there exists a set V' such that $u'S'_{w'}V'$ and for all $v' \in V'$ there exists $v \in \bigcup_{u \in U} V(u)$ with $vZ_{i-1}v'$.*

*We say that w and w' are n-**w-bisimilar** and we write $\mathfrak{M}, w \leftrightsquigarrow_n \mathfrak{M}', w'$ if there is an n-w-bisimulation $Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_n$ such that wZ_nw'.*

*A **w-bisimulation** between two Verbrugge models $\mathfrak{M}$ and $\mathfrak{M}'$ is a single relation $Z \subseteq W \times W'$, that has the properties (at), (n-w-forth) and (n-w-back), with all Z_i being equal to Z. When Z is a w-bisimulation linking two worlds $w \in W$ and $w' \in W'$, we say that w and w' are **w-bisimilar** and we write $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w'$.*

It is much easier to think of the previous definition pictorially. Figure 1 illustrates the (n-w-forth) clause. We use • to depict given (or universally quantified) nodes, while by ∘ we depict nodes whose existence we demand. We use straight arrows to depict relations R, R', Z, and wavy arrows to depict relations S_w and $S'_{w'}$. Full lines depict given (or universally quantified) relations, while dashed ones depict relations whose existence follows from some conditions. We depict sets of nodes analogously.

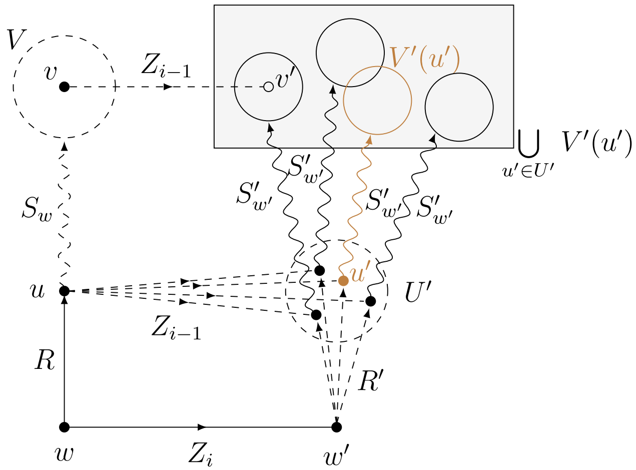

FIGURE 1. Illustration of the (n-w-forth) clause.

Various properties of w-bisimulations are stated in [4]. Here, we will just repeat some statements from Propositions 2.8 and 3.2 in [4], about properties of w-bisimulations and n-w-bisimulations which we will use in the selection method.

PROPOSITION 3.2. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be Verbrugge models, let $w \in W$ and $w' \in W'$, and $n \in \mathbb{N}$.*

(a) *If w and w' are n-w-bisimilar, then they are n-modally equivalent.*
(b) *If w and w' are w-bisimilar, then they are modally equivalent.*

4. OBTAINING A TREE MODEL WITH FINITE HEIGHT

In this section we will describe the first phase of the selection process. We will define the unravelling of a Verbrugge model in order to obtain the tree model property for logic of interpretability IL. Then we will define the height of a rooted Verbrugge model and consider a restriction of the unravelling of a Verbrugge model to obtain an important result: every IL-formula that is satisfiable on some Verbrugge model is satisfiable on some tree model with finite height.

Let F be an arbitrary satisfiable IL-formula. Then there exists an Verbrugge model $\mathfrak{M}$ and a world w in $\mathfrak{M}$ such that $\mathfrak{M}, w \Vdash F$. We say that a pointed Verbrugge model $(\mathfrak{M}, w)$ is *rooted* if for every world w' in $\mathfrak{M}$ there exists an R-path from w to w', i.e. there exists a sequence of worlds $w_0, w_1, \ldots, w_m$ in $\mathfrak{M}$ such that $w_0 = w$, $w_m = w'$ and for all $k < m$ we have $w_k R w_{k+1}$. We also say that the world w is the *root* of that model.

Before the definition of the unravelling of a Verbrugge model, we introduce some notation that will be used. Let W be a set. We denote by W^* the set of all finite words over the alphabet W. Also, we denote by π a function from W^* to W defined as follows:

$$\pi(w_0 w_1 \dots w_n) = w_n.$$

For $V^* \subseteq W^*$, we denote $\pi[V^*] = \{\pi(v^*) : v^* \in V^*\}$.

Now we give the definition of the unraveling of a Verbrugge model.

Definition 4.1. *Let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ be a Verbrugge model and $w_0 \in W$. The* ***unravelling*** *of $\mathfrak{M}$ from w_0 is a quadruple $(W^*, R^*, \{S^*_{w^*} : w^* \in W^*\}, \Vdash)$, where:*

(i) *W^* is the set of words of the form $w^* = w_0 w_1 \dots w_{n-2} w$, where $w_0 R w_1 \dots R w_{n-2} R w$ is any R-path in W starting from w_0,*
(ii) *$R^* \subseteq W^* \times W^*$ is the proper prefix relation,*
(iii) *$u^* S^*_{w^*} V^*$ if and only if $u^* \in R^*[w^*]$, $V^* \subseteq R^*[w^*]$ and $\pi(u^*) S_{\pi(w^*)} \pi[V^*]$,*
(iv) *$w^* \Vdash p$ if and only if $\pi(w^*) \Vdash p$, for any propositional variable p.*

In the following proposition we list the basic properties of the unravelling, all of which are easily verified from the definitions.

Proposition 4.2. *Let $\mathfrak{M}$ be a Verbrugge model, $w_0 \in W$ and $\mathfrak{M}^*$ the unravelling of $\mathfrak{M}$ from w_0. Then:*

(a) *$\mathfrak{M}^*$ is a Verbrugge model*
(b) *$w_0 \in W$ and $w_0 \in W^*$ are w-bisimilar*
(c) *(W^*, R^*) is a transitive tree rooted at w_0.*

By Proposition 4.2. we can now obtain a rooted Verbrugge model $(\mathfrak{M}^*, w^*)$ that is in fact a transitive tree such that $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}^*, w^*$, so by Proposition 3.2. we conclude $\mathfrak{M}, w \equiv \mathfrak{M}^*, w^*$. Now $\mathfrak{M}, w \Vdash F$ implies $\mathfrak{M}^*, w^* \Vdash F$. Thus we have obtained the following result:

> if an IL-formula is satisfiable on some Verbrugge model, then that IL-formula is satisfied at the root of some Verbrugge model which is a transitive tree.

Sometimes (see e.g. [1]) this property of a logic is called the tree model property. So, IL has the tree model property with respect to Verbrugge semantics. A related question of (finite) subtree model property for IL is considered in [2].

Now we define the notion of the height of a state in a rooted Verbrugge model $(\mathfrak{M}, w)$.

Definition 4.3. *Let $(\mathfrak{M}, w)$ be a rooted Verbrugge model. We recursively define the notion of the* **height of a world** *in $\mathfrak{M}$ as follows:*

(i) *the only element of height 0 is the root w,*

(ii) *the states of height* $n + 1$ *are those immediate* R*-successors of worlds of height* n *that have not yet been assigned a height smaller than* $n+1$.

We denote the height of a world w' *in* $\mathfrak{M}$ *by* $h(w')$*. The* **height of a rooted model** $(\mathfrak{M}, w)$ *is the maximum* n *such that there is a world of height* n *in* $\mathfrak{M}$*, if such a maximum exists; otherwise we say that the height of the rooted model* $(\mathfrak{M}, w)$ *is infinite.*

In order to obtain a finite submodel of a Verbrugge model, we will consider the restriction of a rooted Verbrugge model $(\mathfrak{M}, w)$ which will contain all the states that can be reached from the root w in at most k steps, for some natural number k, where each step is a move from the current world to its immediate R-successor.

Definition 4.4. *Let* $k \in \mathbb{N}$ *and let* $(\mathfrak{M}, w)$ *be a rooted Verbrugge model. The* **restriction of** $(\mathfrak{M}, w)$ **to** k*, denoted by* $(\mathfrak{M}, w) \restriction k$*, is defined as the submodel containing only states whose height is at most* k *and relations are restricted to those states. More precisely,* $(\mathfrak{M}, w) \restriction k = (W_k, R_k, \{S_v^{(k)} : v \in W_k\}, \Vdash)$*, where:*

- $W_k = \{v \in W : h(v) \leqslant k\}$
- $R_k = R \cap (W_k \times W_k)$
- $S_v^{(k)} = S_v \cap (R_k[v] \times (\mathcal{P}(R_k[v] \setminus \{\emptyset\})))$*, for all* $v \in W_k$
- $v \Vdash p$ *in the restriction if and only if* $v \Vdash p$ *in the starting model, for all* $v \in W_k$ *and all propositional variables* p.

It is easy to check that the restriction of a rooted Verbrugge model to some k is a Verbrugge model.

The following proposition will be used to show that for a given IL-formula F of modal depth k that is satisfiable at the root of some pointed Verbrugge model $(\mathfrak{M}, w_0)$, the restriction of $(\mathfrak{M}^*, w_0)$ to k contains all the worlds we need to satisfy F.

Proposition 4.5. *Let* $(\mathfrak{M}, w_0)$ *be a pointed Verbrugge model and let* $k \in \mathbb{N}$*. Let* $\mathfrak{M}^*$ *be the unravelling of* $\mathfrak{M}$ *from* w_0*. Then for every world* w^* *in* $\mathfrak{M}^* \restriction k$ *we have*

$$\mathfrak{M}^* \restriction k, w^* \leftrightsquigarrow_l \mathfrak{M}^*, w^*,$$

where $l = k - h(w^*)$.

Proof. Let $\mathfrak{M}^* \restriction k = (W', R', \{S'_{w'} : w' \in W', \Vdash)$ and let w^* be a world in $\mathfrak{M}^* \restriction k$. Define relations $Z_l \subseteq Z_{l-1} \subseteq \cdots \subseteq Z_0 \subseteq W' \times W^*$ as follows:

$$Z_i := \{(v', v^*) : v' \in \mathfrak{M}^* \restriction (k - i), \pi(v') = \pi(v^*)\}.$$

It follows that $Z_l \subseteq Z_{l-1} \subseteq \cdots \subseteq Z_0$ and $w^* Z_l w^*$. In the rest of the proof, we will show that the sequence of relations (Z_i) satisfies the conditions (at) and (l-w-forth) from the definition of l-w-bisimulation. The condition (l-w-back) is proved analogously.

Let $v' \in W'$ and $v^* \in W^*$ be worlds such that $v' Z_0 v^*$. From the definition of Z_0 we have $\pi(v') = \pi(v^*)$. Now for every propositional variable p from the definition of unravelling and its restriction we obtain the following equivalences: $\mathfrak{M}^* \upharpoonright k, v' \Vdash p$ if and only if $\mathfrak{M}^*, v' \Vdash p$ if and only if $\mathfrak{M}, \pi(v') \Vdash p$ if and only if $\mathfrak{M}^*, v^* \Vdash p$. So we have proved the condition (at) for the sequence of relations (Z_i) is satisfied.

We will now prove the (l-w-forth) condition holds. Let $i \in \{1, 2, \dots, l\}$, $x', u' \in W'$ and $x^* \in W^*$ such that $x' Z_i x^*$ and $x' R' u'$. From $x' Z_i x^*$ and the definition of Z_i we have that $\pi(x') = \pi(x^*)$ and $x' \in \mathfrak{M}^* \upharpoonright (k - i)$. Since R' is a restriction of the relation R^* on W', from $x' R' u'$ we have $x' R^* u'$, so by the definition of R^* and the transitivity of R we have $\pi(x') R \pi(u')$. From $x^* \in W^*$ and the definition of unravelling it follows that

$$x^* = w_0 w_1 \dots \pi(x^*),$$

where $w_0 R w_1 \dots R \pi(x^*)$ is an R–path in W from w_0. We define

$$u^* = w_0 w_1 \dots \pi(x^*) \pi(u').$$

From $\pi(x^*) = \pi(x') R \pi(u')$ it follows $u^* \in W^*$, $\pi(u^*) = \pi(u')$ and $x^* R^* u^*$. Now put $U^* = \{u^*\}$. Clearly $U^* \subseteq R^*[w^*]$ and $u' Z_{i-1} u_1^*$ for every $u_1^* \in U^*$.

Let $V^* : U^* \to \mathcal{P}(W^*)$ be an arbitrary function such that for every $u_1^* \in U^*$ we have $u_1^* S_{x^*}^* V^*(u_1^*)$. From the definition of the set U^* we have $u^* S_{x^*}^* V^*(u^*)$. First we define $V \subseteq W$ as the set $V = \pi[V^*(u^*)]$. From $x' \in W' \subseteq W^*$ and the definition of unravelling it follows that

$$x' = w_0 v_1 \dots \pi(x'),$$

where $w_0 R v_1 \dots R \pi(x')$ is an R–path in W from w_0. From $u^* S_{x^*}^* V^*(u^*)$ we have $\pi(u^*) S_{\pi(x^*)}\ \pi[V^*(u^*)]$ which implies $\pi[V^*(u^*)] \subseteq R(\pi(x^*))$. Now $V = \pi[V^*(u^*)]$ implies $V \subseteq R(\pi(x^*))$. We define the set V' in the following way:

$$V' = \big\{ w_0 v_1 \dots \pi(x') v \ \big|\ v \in V \big\}.$$

From $V \subseteq R(\pi(x^*)) = R(\pi(x'))$ it follows that the set V' consists of the immediate R^*-successors of the world x'. Now $\pi(u') = \pi(u^*)$, $\pi(x') = \pi(x^*)$, $\pi[V^*(u^*)] = V = \pi[V']$, $\pi(u^*) S_{\pi(x^*)} \pi[V^*(u^*)]$ and $V' \subseteq R'[x']$ by the definition of unravelling imply that $u' S_{x'}^* V'$. From $x' \in \mathfrak{M}^* \upharpoonright (k - i)$ it follows $V' \subseteq \mathfrak{M}^* \upharpoonright (k - i + 1) = \mathfrak{M}^* \upharpoonright (k - (i - 1))$ so by $u' S_{x'}^* V'$ we obtain $u' S'_{x'} V'$.

Let $v' \in V'$ be an arbitrary world. By the definition of the set V' we have $v' = w_0 v_1 \dots \pi(x') v$ for some $v \in V$. From $v \in V$ and $V = \pi[V^*]$ it follows that there is $v^* \in V^*(u^*)$ such that $\pi(v^*) = v$. So, we have $\pi(v^*) = v = \pi(v')$ and from $v' \in V'$ and $V' \subseteq \mathfrak{M}^* \upharpoonright (k - (i-1))$ we obtain $v' \in \mathfrak{M}^* \upharpoonright (k - (i-1))$. This means that for the world $v^* \in V^*(u^*) = \bigcup_{u_1^* \in U^*} V^*(u_1^*)$ we have $v' Z_{i-1} v^*$. Thus the condition (l-w-forth) is proved. ∎

Note that for every pointed Verbrugge model $(\mathfrak{M}, w)$, model $(\mathfrak{M}^*, w) \upharpoonright k$ has finite height, but is not necessarily finite since it can have infinitely many (finite) R^*-paths from the root w. In the next section we will use the selection method to fix that in order to obtain a finite Verbrugge model and thus conclude the proof by selection that interpretability logic has the finite model property with respect to Verbrugge semantics.

5. The selection method

In the previous section we proved that any satisfiable IL-formula is satisfiable in a Verbrugge model that is a tree of finite height.

Before the main theorem, let us introduce some notation. Let $\mathfrak{M}$ be a Verbrugge model, let $w \in W$ be an arbitrary world and F an IL-formula. Denote by *Prop* the (finite) set of all propositional variables that occur in F. Then it can be proved that for $n \in \mathbb{N}$ there exist only finitely many mutually non-equivalent IL-formulas G whose modal depth is less than or equal to n, whose variables are from the set *Prop* and for which $\mathfrak{M}, w \Vdash G$ holds. For details on this in the case of basic modal logic see e.g. [3]. The case of IL is analogous. We denote the conjunction of these finitely many formulas by χ^n_w. It follows that for each $n \in \mathbb{N}$ we have that $\mathfrak{M}, w \Vdash \chi^n_w$. Note that the definition of the formula χ^n_w implies that $d(\chi^n_w) \leqslant n$.

Now we are ready to prove the finite model property for interpretability logic IL with respect to Verbrugge semantics by selection.

Theorem 5.1. *Let F be an IL formula. If F is satisfiable in a Verbrugge model, then F is satisfiable in a finite Verbrugge model.*

Proof. Le F be an arbitrary satisfiable IL-formula. Then there exists a Verbrugge model $\mathfrak{M}$ and a world w_0 in that model such that $\mathfrak{M}, w_0 \Vdash F$. Let k be the modal depth of F and denote by *Prop* the (finite) set of all propositional variables that occur in F. Then by the tree model property of IL with respect to Verbrugge semantics there exists a rooted Verbrugge model $(\mathfrak{M}^*, w_0^*)$ which is a transitive tree such that $\mathfrak{M}^*, w_0^* \Vdash F$. By Proposition 4.5. for the Verbrugge model $\mathfrak{N} = (\mathfrak{M}^* \upharpoonright k)$ we have that $\mathfrak{M}^*, w_0^* \leftrightsquigarrow_k \mathfrak{N}, w_0^*$. Since $d(F) \leqslant k$ by Proposition 3.2. we have $\mathfrak{N}, w_0^* \Vdash F$.

This resulted in a model that is a transitive tree of finite height. But that tree can have infinite number of branches. All that remains is to adapt the process of selecting finitely many branches from Theorem 2.34. in [1] and show that thus obtained model, which we will denote by $\mathfrak{N}'$, satisfies $\mathfrak{N}, w_0^* \leftrightsquigarrow_k \mathfrak{N}', w_0^*$.

Let $\mathfrak{N} = (W, R, \{S_w : w \in W\}, \Vdash)$ be an arbitrary Verbrugge model. First we define a sequence of sets of worlds $(Set_i)_{i \in \mathbb{N}}$. Put $Set_0 = \{w_0^*\}$. Suppose that for some $i \in \mathbb{N}$ we have defined Set_i. Then we define Set_{i+1} as follows:

- For each $v \in Set_i$, consider the set of all IL-formulas of the form $\neg(G \rhd H)$ that are logically mutually non-equivalent, whose modal depth is less than or equal to k, whose set of propositional variables is a subset of *Prop* and for which $\mathfrak{N}, v \Vdash \neg(G \rhd H)$ holds. There are finitely many such formulas, so we can denote them by $F_0, F_1, \ldots, F_m$, for some $m \in \mathbb{N}$.

- For each $i \in \{0, 1, \ldots, m\}$, from $\mathfrak{N}, v \Vdash F_i \equiv \neg(G \rhd H)$ it follows that there exists some world $u \in W$ such that vRu and $\mathfrak{N}, u \Vdash G$ and for every $V \subseteq W$ such that uS_wV there exists $v' \in V$ such that $\mathfrak{N}, v' \nVdash H$.
- We build Set_{i+1} by placing in it one such world u for each $i \in \{0, 1, \ldots, m\}$ and we repeat this procedure for each world $v \in Set_i$.

It is easy to prove by induction that each Set_i is finite and $h(v) \geqslant i$ for all $v \in Set_i$. Since the height of the model $\mathfrak{N}$ is equal to k, it follows that $Set_i = \emptyset$ for all $i > k$.

We define the model $\mathfrak{N}' = (W', R', \{S'_w : w \in W'\}, \Vdash)$ as the submodel of $\mathfrak{N}$ whose set of worlds is $W' = \bigcup_{i \in \mathbb{N}} Set_i$. Due to the previous considerations, we have $W' = \bigcup_{i=0}^{k} Set_i$. As each Set_i is finite, it follows that the obtained model is finite. For each $i \in \{0, 1, \ldots, k\}$, we define the relation $Z_i \subseteq W \times W'$ as follows:

$$wZ_iw' \quad \text{if and only if} \quad \mathfrak{N}, w' \Vdash \chi^i_w.$$

Since for all u, v in $\mathfrak{N}$ and each $i \in \mathbb{N} \setminus \{0\}$ we have that $\mathfrak{N}, v \Vdash \chi^i_u$ implies $\mathfrak{N}, v \Vdash \chi^{i-1}_u$, we obtain $Z_k \subseteq Z_{k-1} \subseteq \ldots \subseteq Z_0$. Also, $\mathfrak{N}, w_0^* \Vdash \chi^k_{w_0^*}$ implies $w_0^* Z_k w_0^*$.

We will show that the sequence of relations $Z_0, Z_1, \ldots, Z_k$ is a k-w-bisimulation. It will follow that $\mathfrak{N}, w_0^* \leftrightsquigarrow_k \mathfrak{N}', w_0^*$, which implies $\mathfrak{N}, w_0^* \equiv_k \mathfrak{N}', w_0^*$. Since $\mathfrak{N}, w_0^* \Vdash F$ and $d(F) \leqslant k$, then we have $\mathfrak{N}', w_0^* \Vdash F$. Since $\mathfrak{N}'$ is a finite Verbrugge model, we obtain the desired claim.

First, let us show that the condition (at) from the definition of k-w-bisimulation holds. Let wZ_0w' and $p \in Prop$. The definition of Z_0 implies $\mathfrak{N}, w' \Vdash \chi^0_w$. This implies that $\mathfrak{N}, w \Vdash p$ holds if and only if $\mathfrak{N}, w' \Vdash p$ holds.

We now show that the condition (k-w-forth) from the definition of k-w-bisimulation holds. Let $i \in \{1, \ldots, k\}$. Let wZ_iw' and let $u \in W$ be such that wRu. From wRu and $\mathfrak{N}, u \Vdash \chi^{i-1}_u$ we have that $\mathfrak{N}, w \Vdash \Diamond\chi^{i-1}_u$. Recall that $\Diamond$ is an abbreviation, so this means $\mathfrak{N}, w \Vdash \neg(\chi^{i-1}_u \rhd \bot)$. From wZ_iw' and the definition of the relation Z_i, it follows that $\mathfrak{N}, w' \Vdash \chi^i_w$. Since the depth of $\neg(\chi^{i-1}_u \rhd \bot)$ is at most i, $\mathfrak{N}, w \Vdash \neg(\chi^{i-1}_u \rhd \bot)$ implies $\mathfrak{N}, w' \Vdash \neg(\chi^{i-1}_u \rhd \bot)$. The definition of W' implies that for some $j \in \mathbb{N}$ we have $w' \in Set_j$, so there exists some $u' \in Set_{j+1}$ (and thus $u' \in W'$) such that $\mathfrak{N}, u' \Vdash \chi^{i-1}_u$. So, $uZ_{i-1}u'$. Let U' be the set that consists of all $u' \in W'$ such that $w'R'u'$ and $\mathfrak{N}, u' \Vdash \chi^{i-1}_u$. The previous considerations imply $U' \neq \emptyset$.

Let $V' : U' \to \mathcal{P}(W')$ be an arbitrary function such that for every world $u' \in U'$ we have $u'S'_{w'}V'(u')$. Let $V'' = \bigcup_{u' \in U'} V'(u')$. Assume the opposite, i.e., for every set V such that uS_wV there exists a world $v \in V$ such that for every world $v' \in V''$ it does not hold $vZ_{i-1}v'$. The latter is by the definition of Z_{i-1} equivalent to $\mathfrak{N}, v' \Vdash \neg\chi^{i-1}_v$, which is equivalent to $\mathfrak{N}, v \Vdash \neg\chi^{i-1}_{v'}$.

Therefore, for every set V such that uS_wV there exists a world $v \in V$ such that $\mathfrak{N}, v \Vdash \neg \bigvee_{v' \in V''} \chi_{v'}^{i-1}$. Since W' is finite, and then also is $V'' \subseteq W'$, the latter disjunction is finite, and thereby the given formula is well-defined. So, we have $u \in W$ such that wRu and for any $V \subseteq W$ such that uS_wV there is $v \in V$ such that $\mathfrak{N}, v \nVdash \bigvee_{v' \in V''} \chi_{v'}^{i-1}$. Hence, $\mathfrak{N}, w \Vdash \neg \left(\chi_u^{i-1} \rhd \bigvee_{v' \in V''} \chi_{v'}^{i-1}\right)$. The depth of the previous formula is at most i, so from $\mathfrak{N}, w' \Vdash \chi_w^i$ we conclude that $\mathfrak{N}, w' \Vdash \neg \left(\chi_u^{i-1} \rhd \bigvee_{v' \in V''} \chi_{v'}^{i-1}\right)$. Since $w' \in Set_j$ for some $j \in \mathbb{N}$, there must be some $u'' \in Set_{j+1}$ such that $w'R'u''$, $\mathfrak{N}, u'' \Vdash \chi_u^{i-1}$ and for any X' such that $u'S'_{w'}X'$ there is $x' \in X'$ with $\mathfrak{N}, x' \nVdash \bigvee_{v' \in V''} \chi_{v'}^{i-1}$. Then in particular we have $u'' \in U'$, so the latter in particular holds for $X' = V'(u'')$. But this means that there exists a world $v'' \in V'(u'') \subseteq V''$ such that $\mathfrak{N}, v'' \Vdash \neg \bigvee_{v' \in V''} \chi_{v'}^{i-1}$, contradicting $\mathfrak{N}, v'' \Vdash \bigvee_{v' \in V''} \chi_{v'}^{i-1}$, which holds since $v'' \in V''$ and $\mathfrak{N}, v'' \Vdash \chi_{v''}^{i-1}$.

It remains to prove (k-w-back). Let $i \in \{1, \dots, k\}$. Assume wZ_iw' and $w'R'u'$. Then $w'Ru'$, which together with $\mathfrak{N}, u' \Vdash \chi_{u'}^{i-1}$ implies $\mathfrak{N}, w' \Vdash \Diamond\chi_{u'}^{i-1}$. This means $\mathfrak{N}, w' \Vdash \neg(\chi_{u'}^{i-1} \rhd \bot)$. From wZ_iw' and the definition of Z_i it follows $\mathfrak{N}, w' \Vdash \chi_w^i$, which is equivalent to $\mathfrak{N}, w \Vdash \chi_{w'}^i$. Since the depth of $\neg(\chi_{u'}^{i-1} \rhd \bot)$ is up to i, we have $\mathfrak{N}, w \Vdash \neg(\chi_{u'}^{i-1} \rhd \bot)$. Then there exists $u \in W$ such that wRu and $\mathfrak{N}, u \Vdash \chi_{u'}^{i-1}$, which is equivalent to $\mathfrak{N}, u' \Vdash \chi_u^{i-1}$. Therefore, $uZ_{i-1}u'$. We define U as the set of all worlds $u \in W$ such that wRu and $\mathfrak{N}, u \Vdash \chi_{u'}^{i-1}$. The previous considerations imply $U \neq \emptyset$.

Let $V : U \to \mathcal{P}(W)$ be an arbitrary function such that for all $u \in U$ we have $uS_wV(u)$. Put $V'' = \bigcup_{u \in U} V(u)$. Since U consists of all u such that wRu and $\mathfrak{N}, u \Vdash \chi_{u'}^{i-1}$, we obtain $\mathfrak{N}, w \Vdash \chi_{u'}^{i-1} \rhd \bigvee_{v \in V''} \chi_v^{i-1}$. Although the set $V'' \subseteq W$ is not necessarily finite, the previous disjunction is finite if we consider only mutually logically non-equivalent formulas χ_v^{i-1} whose modal depth is at most $i-1$. From wZ_iw' and the fact that the modal depth of $\chi_{u'}^{i-1} \rhd \bigvee_{v \in V''} \chi_v^{i-1}$ is at most i, we have $\mathfrak{N}, w' \Vdash \chi_{u'}^{i-1} \rhd \bigvee_{v \in V''} \chi_v^{i-1}$. Now, $\mathfrak{N}, u' \Vdash \chi_{u'}^{i-1}$ implies that there exists a set V' such that $u'S'_{w'}V'$ and for all $v' \in V'$ we have $\mathfrak{N}, v' \Vdash \bigvee_{v \in V''} \chi_v^{i-1}$. This implies that all $v' \in V'$ there is $v \in V''$ such that $\mathfrak{N}, v' \Vdash \chi_v^{i-1}$, i.e., $vZ_{i-1}v'$, which concludes the proof. $\square$

6. Conclusion and future work

Proving that a logical system has the finite model property using the selection technique leads to the so-called finite tree property (or in the present case, finite tree-like model property, since the obtained model is tree only with respect to the accessibility relation, while it has much richer structure than an ordinary tree). This may result in additional difficulties when we explore finite model property with respect to subclasses of models characteristic to particular principles of interpretability. It may be possible, though, to adapt the technique for a particular class, or to combine it with some other steps before or after the selection phase. We leave these considerations for future work.

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Metoda selekcije za logiku interpretabilnosti IL uz Verbruggeinu semantiku

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Sažetak. Logika interpretabilnosti je modalna logika koja formalizira pojam relativne interpretabilnosti između aritmetičkih teorija prvog reda. Veltmanova semantika je osnovna semantika za logiku intepretabilnosti. Verbruggeina semantika je generalizacija Veltmanove semantike. Ključan korak u dokazivanju odlučivosti nekog logičkog sistema je dokazivanje svojstva konačnih modela tog logičkog sistema. Jedna od metoda za dokazivanje tog svojstva je selekcija. Prilagodbom tehnika koje su korištene u slučaju Kripkeovih modela, u ovome radu pokazujemo da se metoda selekcije može primijeniti i u složenijem slučaju Verbruggeinih modela.

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