

CONICS IN CUBIC STRUCTURE

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ABSTRACT. By using ternary relation, in this paper we introduce the concept of a conic in a general cubic structure, and study the properties of the conics in cubic structures of ranks 1, 2, and 3. By means of points of a conic, we construct different configurations of points and conics, lines and conics, as well as some well-known configurations of points and lines.

1. INTRODUCTION

There is extensive literature on the properties of cubic curves in the real and complex Euclidean and projective planes, as well as in finite projective planes. Much of this literature relates to the geometry of a cubic curve itself. The study of these properties is carried out in a variety of ways, one commonly used method being the parametrization of the curve itself. In the case of a rational cubic curve, i.e., cubic with a singular point, this can be done using some elementary functions, and in the case of elliptic cubic curves, i.e., cubics without singular points, this is done using, for example, Weierstrass' doubly periodic elliptic function. In this paper, we will show that this can also be done using a very simple structure — a cubic structure for which the geometry on cubic curves is only one model, and there is a number of other models. From the literature on cubic geometry, we can recommend the classic books [1, 2, 4–6, 10, 11].

2. PRELIMINARIES

The cubic structure is defined in [8]. Let Q be a nonempty set, whose elements are called *points*, and let $[] \subseteq Q^3$ be a ternary relation on Q . Such a relation and the ordered pair $(Q, [])$ are called a *cubic relation* and a *cubic structure*, respectively, if the following properties are satisfied:

- C1. For any two points $a, b \in Q$ there is a unique point $c \in Q$ such that $[a, b, c]$, i.e., $(a, b, c) \in []$.

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- C2. The relation $[\]$ is totally symmetric, i.e., $[a, b, c]$ implies $[a, c, b]$, $[b, a, c]$, $[b, c, a]$, $[c, a, b]$, and $[c, b, a]$.
- C3. $[a, b, c]$, $[d, e, f]$, $[g, h, i]$, $[a, d, g]$, and $[b, e, h]$ imply $[c, f, i]$, which can be clearly written in the form of the following table:

$$\begin{array}{cc} a & b \\ d & e \\ g & h \end{array} \begin{array}{|c} c \\ f \\ i \end{array}.$$

In [8], many examples of cubic structures are listed, the best known of which is the one in which Q is the set of all non-singular points of a cubic curve in the plane. The statement $[a, b, c]$ means that the points $a, b, c \in Q$ are collinear. Therefore, in a general cubic structure $(Q, [\])$ we will also say that the points a, b, c form a *line* if the statement $[a, b, c]$ is valid, and otherwise we will say that we have a *triangle* (a, b, c) .

In [9], the *tangential* of a point a is defined as the point a' with the property that the statement $[a, a, a']$ holds. The tangential a'' of the tangential a' of a point a is called its *second tangential*. For any point x , we will always denote the tangential and the second tangential by x' and x'' . It can be proved that $[a, b, c]$ implies $[a', b', c']$. Different points having the same tangential are called *associated points*, and there are always 2^m of them for some $m \in \mathbf{N} \cup \{0\}$, where the number m is called the *rank* of the observed cubic structures. In the case of a cubic structure, when observing collinear triplets of non-singular points on a cubic curve in the complex plane, the ranks 0, 1, or 2 occur depending on whether the cubic has a cusp, an ordinary double point, or has no singular points. That is why we are especially interested in the case $m = 2$, for which several theorems have been proved in [9]. If the statements $[a, b, c]$, $[a, e, f]$, $[d, b, f]$, and $[d, e, c]$ are valid, we will say that $\{a, d; b, e; c, f\}$ is a *quadrilateral* with pairs of *opposite vertices* a, d ; b, e ; c, f . The triangles (a, b, f) , (a, e, c) , (d, b, c) , and (d, e, f) are then referred to as *triads*. According to [7], an *inflection point* is defined as a point that is its own tangential, i.e., for which $[i, i, i]$. The following statements, which we will express in the form of several lemmas, are proved in the two papers mentioned above.

LEMMA 2.1. *If the points a and b are inflection points and $[a, b, c]$ holds true, then the point c is also an inflection point.*

LEMMA 2.2. *The opposite vertices of a quadrilateral have common tangentials and the obtained three tangentials belong to one line.*

LEMMA 2.3. *The triangle (a, b, c) is a triad if and only if $[a', b', c']$.*

LEMMA 2.4. *If the points a_1, a_2, a_3 , and a_4 in a cubic structure of rank 2 are associated points having the common tangential a' , then there are points p, q , and r such that $[a_1, a_2, p]$, $[a_3, a_4, p]$, $[a_1, a_3, q]$, $[a_2, a_4, q]$,*

$[a_1, a_4, r]$, and $[a_2, a_3, r]$, the points a' , p , q , and r are associated points, and their common tangential is a'' .

3. CONICS IN A CUBIC STRUCTURE

Given six points a, b, c, d, e, f in a cubic structure, we say that they form a *conic*, and we denote it $[a, b, c, d, e, f]$, provided there are points g, h , and i such that $[a, b, g]$, $[c, d, h]$, $[e, f, i]$, and $[g, h, i]$. Due to the property C1, the points g, h , and i such that $[a, b, g]$, $[c, d, h]$, and $[e, f, i]$, are uniquely determined. As a result, the following theorem holds true.

THEOREM 3.1. *If $[a, b, g]$, $[c, d, h]$, and $[e, f, i]$, then the statements $[a, b, c, d, e, f]$ and $[g, h, i]$ are equivalent.*

COROLLARY 3.2. *If $[a, b, i]$, $[c, d, i]$, and $[e, f, i]$, then the conic $[a, b, c, d, e, f]$ exists if and only if i is an inflection point.*

COROLLARY 3.3. *If i is an inflection point and if $[a, b, i]$, $[c, d, i]$, $[e, f, i]$, and $[a, c, e]$, then $[b, d, f]$ also holds true.*

Theorem 3.1 can be strengthened, i.e.:

THEOREM 3.4. *Each of the five statements $[a, b, g]$, $[c, d, h]$, $[e, f, i]$, $[g, h, i]$, and $[a, b, c, d, e, f]$ follows from the remaining four.*

PROOF. For example, let us prove the theorem for $[c, d, h]$. Let j be a point such that $[c, d, j]$ holds true. From $[a, b, g]$, $[c, d, j]$, $[e, f, i]$, and $[a, b, c, d, e, f]$, by Theorem 3.1, we get $[g, j, i]$, which, together with $[g, h, i]$, implies $j = h$, so the statement $[c, d, j]$ becomes $[c, d, h]$. Similarly, one can show that each of $[a, b, g]$, $[e, f, i]$, and $[g, h, i]$ follows from the other four statements. \square

THEOREM 3.5. *Every two points of a conic are mutually equivalent, i.e., $[a, b, c, d, e, f]$ implies $[u, v, w, x, y, z]$, where (u, v, w, x, y, z) is any permutation of (a, b, c, d, e, f) .*

PROOF. From the property C2 and the definition of the conic $[a, b, c, d, e, f]$, it is obvious that the two points in each pair $\{a, b\}$, $\{c, d\}$, and $\{e, f\}$ are equivalent, and that these pairs are equivalent by themselves. Therefore, to prove the equivalence of all six points in the symbol $[a, b, c, d, e, f]$, it is enough to prove that $[a, b, c, e, d, f]$ follows from $[a, b, c, d, e, f]$. The assumption $[a, b, c, d, e, f]$ means that there are points g, h , and i such that the statements $[a, b, g]$, $[c, d, h]$, $[e, f, i]$ and $[g, h, i]$ hold. Now, let j and k be points such that $[c, e, j]$ and $[d, f, k]$. The symmetry C2 and the table

$$\begin{array}{cc} h & i \\ c & e \\ d & f \end{array} \boxed{\begin{array}{c} g \\ j \\ k \end{array}},$$

imply $[g, j, k]$. Consequently, $[a, b, c, e, d, f]$ holds because there exist points g, j , and k such that $[a, b, g]$, $[c, e, j]$, $[d, f, k]$ and $[g, j, k]$. \square

THEOREM 3.6. *A conic is uniquely determined by any five of its points.*

PROOF. According to Theorem 3.1, it is enough to prove that for every five points a, b, c, d , and e there is a unique point f such that $[a, b, c, d, e, f]$. Due to the property C1, there exist unique points g and h such that $[a, b, g]$ and $[c, d, h]$, and there exists a unique point i such that $[g, h, i]$, and, finally, there exists a unique point f such that $[e, i, f]$. \square

THEOREM 3.7. *If $[a, b, c]$ holds then $[a, b, c, d, e, f]$ and $[d, e, f]$ are equivalent statements.*

PROOF. Let $[d, e, f]$ hold true and let h be the point such that $[c, d, h]$ holds. Then we have statements $[a, b, c]$, $[c, d, h]$, $[e, f, d]$, and $[c, h, d]$, so by the definition of a conic, we have $[a, b, c, d, e, f]$. Conversely, let $[a, b, c, d, e, f]$ be a conic. Then there are points g, h , and i such that $[a, b, g]$, $[c, d, h]$, $[e, f, i]$ and $[g, h, i]$. However, from $[a, b, g]$ and $[a, b, c]$, by C1, we get $g = c$, so we have $[c, h, i]$, which, together with $[c, h, d]$, gives the equality $i = d$. Hence, $[e, f, i]$ becomes the required statement $[e, f, d]$. \square

The assertion of Theorem 3.7 can be interpreted as saying that two lines form a conic. This, of course, has different meanings in various examples of cubic structures. We shall say that a conic is *genuine* if it does not consist of two lines.

THEOREM 3.8. *If $[a, b, p]$ and $[c, d, p]$ hold, then for any two points e and f such that $[a, b, c, d, e, f]$, it follows that $[e, f, p']$.*

PROOF. By the definition of the conic $[a, b, c, d, e, f]$, from $[a, b, p]$, $[c, d, p]$, and $[e, f, q]$ follows $[p, p, q]$, i.e., $q = p'$. \square

THEOREM 3.9. *From $[a_1, a_2, a_3, a_4, a_5, a_6]$, $[b_1, b_2, b_3, b_4, b_5, b_6]$ and $[a_i, b_i, c_i]$, $i = 1, 2, 3, 4, 5, 6$, follows $[c_1, c_2, c_3, c_4, c_5, c_6]$.*

PROOF. Let $[a_1, a_2, d_1]$, $[a_3, a_4, d_2]$, $[a_5, a_6, d_3]$, $[b_1, b_2, e_1]$, $[b_3, b_4, e_2]$, $[b_5, b_6, e_3]$, $[c_1, c_2, f_1]$, $[c_3, c_4, f_2]$, and $[c_5, c_6, f_3]$. Then, by the definition of a conic, the statements $[d_1, d_2, d_3]$ and $[e_1, e_2, e_3]$ are valid. Let us prove $[f_1, f_2, f_3]$. From the table

$$\begin{array}{cc} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{array} \boxed{\begin{array}{c} d_1 \\ e_1 \\ f_1 \end{array}}$$

we get $[d_1, e_1, f_1]$, and similarly for the statements $[d_2, e_2, f_2]$ and $[d_3, e_3, f_3]$. Finally, from the table

$$\begin{array}{cc} d_1 & e_1 \\ d_2 & e_2 \\ d_3 & e_3 \end{array} \boxed{\begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array}}$$

follows $[f_1, f_2, f_3]$. □

THEOREM 3.10. $[a_1, a_2, a_3, a_4, a_5, a_6]$, $[b_1, b_2, a_3, a_4, a_5, a_6]$, $[a_1, a_2, b_3, b_4, a_5, a_6]$, and $[a_1, a_2, a_3, a_4, b_5, b_6]$ imply $[b_1, b_2, b_3, b_4, b_5, b_6]$.

PROOF. For $(i, j) \in \{(1, 2), (3, 4), (5, 6)\}$, let a_{ij} and b_{ij} be points such that $[a_i, a_j, a_{ij}]$ and $[b_i, b_j, b_{ij}]$. The assumptions of the theorem mean then that there are lines $[a_{12}, a_{34}, a_{56}]$, $[b_{12}, a_{34}, a_{56}]$, $[a_{12}, b_{34}, a_{56}]$, and $[a_{12}, a_{34}, b_{56}]$. From the table

$$\begin{array}{cc} a_{34} & a_{56} & \boxed{b_{12}} \\ a_{56} & a_{12} & \boxed{b_{34}} \\ a_{12} & a_{34} & \boxed{b_{56}} \end{array}$$

we get $[b_{12}, b_{34}, b_{56}]$, which proves the statement $[b_1, b_2, b_3, b_4, b_5, b_6]$. □

THEOREM 3.11. $[a, b, c, d, e, f]$ and $[e, f, p]$ imply $[a, b, c, d, p, p']$.

PROOF. If g and h are points such that $[a, b, g]$ and $[c, d, h]$ hold, then, due to $[e, f, p]$, by the definition of the conic $[a, b, c, d, e, f]$, we get $[g, h, p]$. Statements $[a, b, g]$, $[c, d, h]$, $[p, p', p]$ and $[g, h, p]$ prove $[a, b, c, d, p, p']$. □

$[a, b, c]$ implies $[a', b', c']$, and then, according to Theorem 3.7, we have $[a, a', b, b', c, c']$. However, the reverse of this statement is also valid, i.e., we have the following theorem:

THEOREM 3.12. *If a' , b' , and c' are the tangentials of the points a , b , and c , then $[a, a', b, b', c, c']$ implies $[a, b, c]$.*

PROOF. $[a, a', b, b', c, c']$, $[a, a', a]$, $[b, b', b]$, and $[c, c', c]$ imply, by Theorem 3.1, the statement $[a, b, c]$. □

THEOREM 3.13. $[a, b, c, d, e, f]$ implies $[a', b', c', d', e', f']$.

PROOF. Because of $[a, b, c, d, e, f]$, there are points u, v , and w such that $[a, b, u]$, $[c, d, v]$, $[e, f, w]$, and $[u, v, w]$. From these statements we get $[a', b', u']$, $[c', d', v']$, $[e', f', w']$, and $[u', v', w']$, thus $[a', b', c', d', e', f']$ holds. □

THEOREM 3.14. *If $[p, a, a_0]$, $[p, b, b_0]$, $[p, c, c_0]$, $[p, d, d_0]$, $[p, e, e_0]$, and $[a, b, c, d, e, p]$, then $[a_0, b_0, c_0, d_0, e_0, p'']$.*

PROOF. From $[a, b, c, d, e, p]$ and $[e, p, e_0]$, it follows that there are points u and v , such that $[a, b, u]$, $[c, d, v]$, and $[u, v, e_0]$. Let u_0, v_0 , and w_0 be points such that $[a_0, b_0, u_0]$, $[c_0, d_0, v_0]$, and $[e_0, p'', w_0]$. To prove $[a_0, b_0, c_0, d_0, e_0, p'']$, it is necessary to prove $[u_0, v_0, w_0]$. However, from the tables

$$\begin{array}{cc} a & b & \boxed{u} \\ a_0 & b_0 & \boxed{u_0} \\ p & p & \boxed{p'} \end{array} \quad \text{and} \quad \begin{array}{cc} c & d & \boxed{v} \\ c_0 & d_0 & \boxed{v_0} \\ p & p & \boxed{p'} \end{array},$$

$[u, u_0, p']$ and $[v, v_0, p']$ follow, and then from the table

$$\begin{array}{ccc} p' & u & \boxed{u_0} \\ p' & v & \boxed{v_0} \\ p'' & e_0 & \boxed{w_0} \end{array}$$

we obtain $[u_0, v_0, w_0]$. \square

THEOREM 3.15. *If $a, b, c, d, e,$ and f are distinct points, the first five of which are inflection points, and if $[a, b, c, d, e, f]$, then f is also an inflection point.*

PROOF. If any three of these six points are on a line, then, due to Theorem 3.7, the remaining three points are also on a line, so by Lemma 2.1, the assertion of the theorem follows. If no three of the observed six points are on the same line, then for points $g, h,$ and i such that $[a, b, g], [c, d, h],$ and $[e, f, i],$ because of $[a, b, c, d, e, f],$ it also follows that $[g, h, i].$ From $[a, b, g]$ and $[c, d, h]$ it follows that g and h are inflection points, then from $[g, h, i]$ it follows that the point i is an inflection point, and, finally, from $[e, f, i]$ we get that f is an inflection point (all this using Lemma 2.1). \square

THEOREM 3.16. $[a, b, c, d, e, f], [a, b, c, g, h, i],$ and $[d, e, f, j, k, l]$ imply $[g, h, i, j, k, l].$

PROOF. From $[a, b, c, d, e, f]$ it follows that there are points $m, n,$ and o such that $[a, b, m], [c, d, n], [e, f, o],$ and $[m, n, o].$ Then, from $[a, b, c, g, h, i],$ together with $[a, b, m],$ it follows that there are points p and q such that $[c, g, p], [h, i, q],$ and $[m, p, q]$ hold. Therefore, from $[e, f, d, j, k, l]$ and $[e, f, o],$ it follows that there are points r and s such that $[d, j, r], [k, l, s],$ and $[o, r, s].$ Let t and u be points such that $[g, j, t]$ and $[p, r, u].$ From the first of the two tables

$$\begin{array}{ccc} c & d & \boxed{n} \\ g & j & \boxed{t} \\ p & r & \boxed{u} \end{array} \quad \text{and} \quad \begin{array}{ccc} m & p & \boxed{q} \\ o & r & \boxed{s} \\ n & u & \boxed{t} \end{array},$$

it follows that $[n, t, u],$ and then from the second table follows $[q, s, t].$ This, together with $[h, i, q], [k, l, s],$ and $[g, j, t],$ proves $[h, i, k, l, g, j].$ \square

THEOREM 3.17. *If $a, b, c,$ and d are given points, then $[a, b, c, d, u, v]$ and $[u, v, w]$ imply that such a point w is unique.*

PROOF. $[a, b, c, d, u, v]$ implies the existence of points $m, n,$ and o such that $[a, b, m], [c, d, n], [u, v, o],$ and $[m, n, o].$ From the first of these two facts, such points m and n are unique, and then from $[m, n, o]$ it follows that the point o is also unique. But $[u, v, o]$ and $[u, v, w]$ imply $w = o.$ Hence, the point w is unique. \square

COROLLARY 3.18. *Let $[a, b, c, d, e, f]$ and $[e, f, i]$. Then the statements $[a, b, c, d, g, h]$ and $[g, h, i]$ are equivalent.*

THEOREM 3.19. *Let a, b, c , and m be the given points. If $[m, u, v]$ and $[a, b, c, u, v, d]$, then such a point d is unique.*

PROOF. Because of $[a, b, c, d, u, v]$ there are points p and q such that $[a, b, p]$, $[c, d, q]$, $[u, v, m]$, and $[p, q, m]$. Such a point p is unique, and then, because of $[p, q, m]$, the point q is also unique, and therefore $[c, d, q]$ implies that the point d is also unique. \square

THEOREM 3.20. *If the conics $[a, b, c, d, e, f]$, $[c, d, e, f, p, q]$, $[a, b, g, h, i, j]$, and $[g, h, i, j, r, s]$ exist, then there exists a point t such that $[p, q, t]$ and $[r, s, t]$.*

PROOF. By the definition of the four conics, there are points u, v, w, t, x, y , and z such that $[a, b, u]$, $[c, d, v]$, $[e, f, w]$, $[u, v, w]$; $[p, q, t]$, $[v, w, t]$; $[g, h, x]$, $[i, j, y]$, $[u, x, y]$; $[r, s, z]$, $[x, y, z]$. From the fourth and the sixth of these lines we conclude that $u = t$, and the ninth and the eleventh (the last) of these lines show that $u = z$, hence $[r, s, z]$ becomes $[r, s, t]$. \square

THEOREM 3.21. *If $[a, b, c, d, e, u]$, $[f, g, v]$, $[u, v, t]$, $[a, b, c, d, f, x]$, and $[e, g, y]$ hold, then $[x, y, t]$ also holds.*

PROOF. Let l, m , and n be points such that $[a, b, l]$, $[c, d, m]$, and $[e, u, n]$ hold. From $[a, b, c, d, e, u]$, according to the definition of a conic, follows $[l, m, n]$, and from $[a, b, c, d, f, x]$, $[a, b, l]$, $[c, d, m]$, and $[l, m, n]$ we get $[f, x, n]$. Finally, the table

$$\begin{array}{cc} n & f \\ e & g \\ u & v \end{array} \boxed{\begin{array}{c} x \\ y \\ t \end{array}}$$

implies $[x, y, t]$. \square

Splitting a given set of seven points into a quintuplet and a pair can be done in twenty-one ways. To each quintuplet add a point to form a conic, and to the corresponding pair add a point to form a line. These two new points define a line, and it follows from the previous theorem that the twenty-one lines obtained in this way have a common point.

THEOREM 3.22. *Let a, b, c, d, e , and f be six points of a cubic structure which are not on the same conic. If the points e_0 and f_0 are such that $[a, b, c, d, e, f_0]$ and $[a, b, c, d, f, e_0]$, then there is a point p satisfying $[e, f_0, p]$ and $[f, e_0, p]$.*

PROOF. Let p, q , and r be such points that $[e, f_0, p]$, $[a, b, q]$, and $[c, d, r]$. Then the definition of the conic $[e, f_0, a, b, c, d]$ implies $[p, q, r]$, and then, by the definition of the conic $[a, b, c, d, f, e_0]$, from $[a, b, q]$, $[c, d, r]$, and $[q, r, p]$ we get $[f, e_0, p]$. \square

THEOREM 3.23. *If $\{a_1, a_2; b_1, b_2; c_1, c_2\}$ is a quadrilateral, then the pairs of points $a_1, a_2; b_1, b_2; c_1, c_2$ have common tangentials $a, b,$ and c such that $[a, b, c]$, and there exist conics $[a, a_1, a_2, b, b_1, b_2], [a, a_1, a_2, c, c_1, c_2],$ and $[b, b_1, b_2, c, c_1, c_2].$*

PROOF. The first part of the theorem is dealt with in Lemma 2.2. The statements $[a, b, c], [a_1, b_1, c_1], [a_2, b_2, c_1],$ and $[c, c_1, c_1]$ prove $[a, b, a_1, b_1, a_2, b_2],$ i.e., $[a, a_1, a_2, b, b_1, b_2],$ and the other two statements are proved similarly. \square

THEOREM 3.24. *If (a, b, c) is a triad, then there exists the conic $[a, a, b, b, c, c].$*

PROOF. This follows from $[a, a, a'], [b, b, b'], [c, c, c'],$ and $[a', b', c'].$ \square

THEOREM 3.25. *If (a, b, c) is a triad and if there is a conic $[a, b, c, d, e, f],$ then (d, e, f) is also a triad.*

PROOF. From $[a, b, c, d, e, f]$ we get $[a', b', c', d', e', f'],$ according to Theorem 3.13. Then, due to $[a', b', c'],$ Theorem 3.7 implies $[d', e', f'].$ Because of the noncollinearity of points $a, b, c,$ from $[a, b, c, d, e, f],$ we obtain, by the same theorem, that the points d, e, f are also noncollinear. Therefore, by Lemma 2.3, (d, e, f) is a triad. \square

THEOREM 3.26. *If $\{a, d; b, e; c, f\}$ is a quadrilateral and two points p and q are given, then there is a point r such that $[a, b, f, p, q, r], [a, c, e, p, q, r], [b, c, d, p, q, r],$ and $[d, e, f, p, q, r].$*

PROOF. By Lemma 2.2, the points a and d have the same tangential $m,$ and from the given quadrilateral we get $[a, b, c], [a, e, f], [b, d, f],$ and $[c, d, e].$ Let $r, s, t,$ and u be such points that $[a, b, f, p, q, r], [p, q, s], [a, r, t],$ and $[d, r, u].$ From $[p, q, b, f, a, r], [p, q, s], [b, f, d],$ and $[a, r, t],$ we get $[s, d, t].$ The table

$$\begin{array}{ccc} d & t & s \\ m & a & a \\ d & r & u \end{array}$$

gives $[s, a, u].$ From $[p, q, s], [c, e, d], [a, r, t],$ and $[s, d, t]$ we now get $[p, q, c, e, a, r],$ and from $[p, q, s], [b, c, a], [d, r, u]$ and $[s, a, u]$ we get $[p, q, b, c, d, r].$ Finally, $[p, q, s], [e, f, a], [d, r, u],$ and $[s, a, u]$ imply $[p, q, e, f, d, r].$ \square

Under the conditions of Theorem 3.26, (a, b, f) is a triad, so, according to Theorem 3.25, (p, q, r) is also a triad. This leads to the following conclusion.

COROLLARY 3.27. *For every two distinct points p and q there is a point r such that (p, q, r) is a triad.*

THEOREM 3.28. *If $[a, b, c, d, e, f], [b, c, g], [c, a, h], [a, b, i], [e, f, j], [f, d, k],$ and $[d, e, l],$ then $[g, h, i, j, k, l].$*

PROOF. From $[a, b, c, d, e, f]$, $[a, b, i]$, and $[e, f, j]$ it follows that there is a point m such that $[c, d, m]$ and $[i, m, j]$, and hence $[c', d', m']$. Let n and o be such points that $[g, h, n]$ and $[k, l, o]$. Then, to prove the statement $[g, h, i, j, k, l]$, in addition to $[i, j, m]$, one must prove $[n, m, o]$. Let us look at the tables

$$\begin{array}{ccc} a & b & \boxed{i} \\ c & c & \boxed{c'} \\ h & g & \boxed{n} \end{array}, \quad \begin{array}{ccc} d & d & \boxed{d'} \\ e & f & \boxed{j} \\ l & k & \boxed{o} \end{array}, \quad \begin{array}{ccc} i & c' & \boxed{n} \\ m & m' & \boxed{m} \\ j & d' & \boxed{o} \end{array}.$$

First, from the first two tables we get $[i, c', n]$, $[d', j, o]$ and then the third table proves $[n, m, o]$. □

THEOREM 3.29. *If $[a_1, a_2, a_3, b_1, b_2, b_3]$ is a genuine conic and if c_1, c_2, c_3, d_1, d_2 , and d_3 are such points that $[a_2, a_3, c_1]$, $[a_3, a_1, c_2]$, $[a_1, a_2, c_3]$, $[b_2, b_3, d_1]$, $[b_3, b_1, d_2]$, and $[b_1, b_2, d_3]$, then the conic $[c_1, c_2, c_3, d_1, d_2, d_3]$ also exists.*

PROOF. Let e_1, e_2 , and e_3 be such points that $[a_i, b_i, e_i]$, $i = 1, 2, 3$. From $[a_2, a_3, b_2, b_3, a_1, b_1]$, $[a_2, a_3, c_1]$, $[b_2, b_3, d_1]$, and $[a_1, b_1, e_1]$ it follows, by the definition of a conic, that $[c_1, d_1, e_1]$. Similarly, from $[a_3, a_1, b_3, b_1, a_2, b_2]$ and $[a_1, a_2, b_1, b_2, a_3, b_3]$ one obtains $[c_2, d_2, e_2]$ and $[c_3, d_3, e_3]$. Finally, $[e_1, e_2, e_3]$ follows from the table

$$\begin{array}{ccc} a_1 & b_1 & \boxed{e_1} \\ a_2 & b_2 & \boxed{e_2} \\ c_3 & d_3 & \boxed{e_3} \end{array}$$

which, together with $[c_1, d_1, e_1]$, $[c_2, d_2, e_2]$, and $[c_3, d_3, e_3]$, yields $[c_1, d_1, c_2, d_2, c_3, d_3]$. □

The six points of a given genuine conic can be divided into a pair of triplets in ten ways, producing ten new conics as in the previous theorem. Each of these conics contains six points out of a total of fifteen points, which are obtained as the third point on lines through two points of the given genuine conic. Therefore, each of these fifteen points belongs to four of these new conics, i.e., these points and these conics form a configuration $(15, 10_6)$.

Let us examine this situation in more detail. We have a conic $[a_1, a_2, a_3, a_4, a_5, a_6]$. If we form two triangles $[a_1, a_2, a_3]$ and $[a_4, a_5, a_6]$ and denote this division by $(123, 456)$, we have ten such symbols in total. For $i \neq j$ let b_{ij} be a point such that $[a_i, a_j, b_{ij}]$ holds. Then, by Theorem 3.29, there exists a conic $[b_{23}, b_{13}, b_{12}, b_{56}, b_{46}, b_{45}]$. From the definition of the conic $[a_1, a_4, a_2, a_5, a_3, a_6]$ follows the existence of the line $[b_{14}, b_{25}, b_{36}]$, to which we assign the symbol $(14, 25, 36)$, and we have fifteen such symbols in total. In this way, the symbol $(123, 456)$ is associated with six symbols: $(14, 25, 36)$, $(14, 26, 35)$, $(15, 24, 36)$, $(15, 26, 34)$, $(16, 24, 35)$, and $(16, 25, 34)$, while the symbol $(14, 25, 36)$ is associated with four symbols: $(123, 456)$, $(126, 453)$,

(153, 426), and (156, 423). The same is true for other analogous symbols. We obtain a configuration $(15_4, 10_6)$ of lines and conics, where the association relation of these two types of symbols takes the place of incidence. We have fifteen lines like the line $[b_{14}, b_{25}, b_{36}]$, and we also have fifteen points of the form b_{ij} . At the same time, we have three points on each line, and each point lies on three such lines. For example, the point b_{12} lies on the lines with symbols $(12, 34, 56)$, $(12, 35, 46)$, and $(12, 36, 45)$. Thus, we obtained a configuration 15_3 of points and lines. On the web, this configuration is called the *Cremona-Richmond configuration*, and is also mentioned in [12].

THEOREM 3.30. *Let $[a, b, c, d, e, f]$ be a conic in a cubic structure of rank 1, and let $a_1, a_2; b_1, b_2; c_1, c_2; d_1, d_2; e_1, e_2; f_1, f_2$ be two by two points whose common tangentials are the points a, b, c, d, e , and f , respectively. Then there exists a total of 32 conics of the form $[a_i, b_j, c_k, d_m, e_n, f_p]$, where $i, j, k, m, n, p \in \{1, 2\}$. Each of the 12 points belongs to 16 such conics, i.e., these points and conics form a configuration $(12_{16}, 32_6)$.*

PROOF. For each choice of five points a_i, b_j, c_k, d_m , and e_n there is a unique point u such that $[a_i, b_j, c_k, d_m, e_n, u]$. By Theorem 3.13, we get $[a, b, c, d, e, u']$, where u' is the tangential of the point u . Since we have $[a, b, c, d, e, f]$, it follows that $u' = f$, and then $u = f_1$ or $u = f_2$. Now, if $[a_i, b_j, c_k, d_m, e_1, f_1]$, then we get $[a_i, b_j, c_k, d_m, e_2, f_2]$, and if $[a_i, b_j, c_k, d_1, e_1, f_1]$, then we also have $[a_i, b_j, c_k, d_1, e_2, f_2]$, $[a_i, b_j, c_k, d_2, e_1, f_2]$, and $[a_i, b_j, c_k, d_2, e_2, f_1]$. Thus, we can choose the indices from the set $\{1, 2\}$ such that the following statements hold true:

$$\begin{aligned} & [a_1, b_1, c_1, d_1, e_1, f_1], [a_1, b_1, c_1, d_1, e_2, f_2], [a_1, b_1, c_1, d_2, e_1, f_2], [a_1, b_1, c_1, d_2, e_2, f_1], \\ & [a_1, b_1, c_2, d_1, e_1, f_2], [a_1, b_1, c_2, d_1, e_2, f_1], [a_1, b_1, c_2, d_2, e_1, f_1], [a_1, b_1, c_2, d_2, e_2, f_2], \\ & [a_1, b_2, c_1, d_1, e_1, f_2], [a_1, b_2, c_1, d_1, e_2, f_1], [a_1, b_2, c_1, d_2, e_1, f_1], [a_1, b_2, c_1, d_2, e_2, f_2], \\ & [a_1, b_2, c_2, d_1, e_1, f_1], [a_1, b_2, c_2, d_1, e_2, f_2], [a_1, b_2, c_2, d_2, e_1, f_2], [a_1, b_2, c_2, d_2, e_2, f_1], \\ & [a_2, b_1, c_1, d_1, e_1, f_2], [a_2, b_1, c_1, d_1, e_2, f_1], [a_2, b_1, c_1, d_2, e_1, f_1], [a_2, b_1, c_1, d_2, e_2, f_2], \\ & [a_2, b_1, c_2, d_1, e_1, f_1], [a_2, b_1, c_2, d_1, e_2, f_2], [a_2, b_1, c_2, d_2, e_1, f_2], [a_2, b_1, c_2, d_2, e_2, f_1], \\ & [a_2, b_2, c_1, d_1, e_1, f_1], [a_2, b_2, c_1, d_1, e_2, f_2], [a_2, b_2, c_1, d_2, e_1, f_2], [a_2, b_2, c_1, d_2, e_2, f_1], \\ & [a_2, b_2, c_2, d_1, e_1, f_2], [a_2, b_2, c_2, d_1, e_2, f_1], [a_2, b_2, c_2, d_2, e_1, f_1], [a_2, b_2, c_2, d_2, e_2, f_2]. \end{aligned}$$

□

Notice that in each of the above 32 conics there is always an even number of indices 1 and of indices 2.

Analogous consideration in a cubic structure of rank 2 gives 24 points and $4^5 = 1024$ conics forming a configuration $(24_{256}, 1024_6)$, and in a structure of rank 3 it gives 48 points and $8^5 = 32768$ conics forming a configuration $(48_{4096}, 32768_6)$, etc.

THEOREM 3.31. *If in a cubic structure of rank 2 the points a_1, a_2, a_3 , and a_4 are associated and have the common tangential a' and the second tangential a'' , then $[a_1, a_2, a_3, a_4, b, c]$ and $[b, c, a'']$ are equivalent statements.*

PROOF. By Lemma 2.4 there is a point p such that $[a_1, a_2, p]$ and $[a_3, a_4, p]$, and the point p has the tangential a'' . Since $[a_1, a_2, p]$, $[a_3, a_4, p]$, and $[p, p, a'']$ are valid, the statements $[a_1, a_2, a_3, a_4, b, c]$ and $[b, c, a'']$ are equivalent. \square

COROLLARY 3.32. *If the point a' is the common tangential of different points a_1, a_2, a_3 , and a_4 in a cubic structure of rank 2, then the conic $[a_1, a_2, a_3, a_4, a', a']$ exists.*

THEOREM 3.33. *If in a cubic structure of rank 2 the points a_1, a_2, a_3 , and a_4 have the common tangential a' and if the conic $[a', a_1, a_2, a_3, b, c]$ exists, then also exists the line $[b, c, a_4]$.*

PROOF. According to Lemma 2.4, there is a point r such that $[a_1, a_4, r]$ and $[a_2, a_3, r]$. From $[a', a_1, a_2, a_3, b, c]$, $[a', a_1, a_1]$, $[a_2, a_3, r]$, and $[a_1, r, a_4]$, by Theorem 3.4, we get $[b, c, a_4]$. \square

THEOREM 3.34. *If in a cubic structure of rank 2 three distinct points a_1, a_2 , and a_3 have the common tangential a' and the second tangential a'' , and if $[a_1, a_2, a_3, a_5, b, c]$ and $[b, c, a'']$, then a_1, a_2, a_3 , and a_5 are associated points.*

PROOF. Let a_1, a_2, a_3 , and a_4 be associated points. Then, due to $[b, c, a'']$, according to the previous theorem, we get $[a_1, a_2, a_3, a_4, b, c]$, which, together with $[a_1, a_2, a_3, a_5, b, c]$, gives the equality $a_5 = a_4$. \square

In a cubic structure of rank 2 we have the Hesse configuration $(12_4, 16_3)$ of antecedent points $a_i, b_i, c_i, i = 1, 2, 3, 4$, of three points a', b', c' on a line. For example, consider two lines $[a_1, b_1, c_1]$ and $[a_2, b_2, c_1]$. Let x and y be points such that $[a', b', a_1, b_1, a_2, x]$ and $[a_2, x, y]$. Since we have $[a', a_1, a_1]$, $[b', b_1, b_1]$, and $[a_2, x, y]$, it follows from $[a', a_1, b', b_1, a_2, x]$ that $[a_1, b_1, x]$, which, together with $[a_1, b_1, c_1]$, gives $x = c_1$, and then, from $[a_2, c_1, y]$ and $[a_2, c_1, b_2]$, we get $y = b_1$. So, we have the conic $[a', b', a_1, b_1, a_2, b_2]$. The same is true for other combinations of indices and letters a, b, c , i.e., we have the following theorem:

THEOREM 3.35. *Let a_i, b_i , and $c_i, i = 1, 2, 3, 4$, be antecedent points of three points a', b' , and c' on a line. Then, the conic containing the points a', b' and two antecedent points of the point a' and an antecedent point of the point b' , contains another antecedent point of the point b' , which is such that there is an antecedent point of the point c' lying on both lines through an antecedent point of the point a' and an antecedent point of the point b' .*

In a cubic structure of rank 1, the previous theorem is analogous to the following one:

THEOREM 3.36 (see [3]). *If in a cubic structure of rank 1 the pairs of points a_1, a_2 and b_1, b_2 have tangentials a and b , then the conic $[a_1, b_1, a_2, b_2, a, b]$ exists.*

PROOF. Let c , d , and e be such points that $[a_1, b_1, c]$, $[a_2, b_2, d]$, and $[a_1, b_2, e]$. Obviously, $e \neq c, d$. The transition to the tangentials gives $[a, b, c']$, $[a, b, d']$, and $[a, b, e']$, so the points c' , d' , and e' coincide, i.e., the points c , d , and e have the same tangential, hence all three cannot be different. Therefore $c = d$. We now have $[a_1, b_1, c]$, $[a_2, b_2, c]$, $[a, b, c']$, and $[c, c, c']$, and thus $[a_1, b_1, a_2, b_2, a, b]$. \square

COROLLARY 3.37. *If, in addition to the assumptions of Theorem 3.36, $[a_1, a_2, b]$ holds true, then $[b_1, b_2, a]$.*

If point a is the tangential of the point b , then the point b is one of the points a_1, a_2 . Let it be a_1 . Then we have the line $[a_1, b, a]$, and then $[a_2, b_1, b_2]$ follows.

Regardless of rank, the following theorem corresponds to Theorem 3.36.

THEOREM 3.38. *Let the points a_1 and a_2 have the common tangential a , and the points b_1 and b_2 have the common tangential b . If there is a point c such that $[a_1, b_1, c]$ and $[a_2, b_2, c]$, then $[a_1, a_2, b_1, b_2, a, b]$.*

PROOF. From $[a_1, b_1, c]$ it follows that $[a, b, c']$, and from $[a_1, b_1, c]$, $[a_2, b_2, c]$, $[a, b, c']$ and $[c, c, c']$ we get $[a_1, a_2, b_1, b_2, a, b]$. \square

THEOREM 3.39. *$[a_1, a_2, a_3, a_4, a_5, a_6]$, $[b_1, b_2, b_3, b_4, b_5, b_6]$, $[c_1, c_2, c_3, c_4, c_5, c_6]$, $[d_1, d_2, d_3, d_4, d_5, d_6]$, $[e_1, e_2, e_3, e_4, e_5, e_6]$, $[f_1, f_2, f_3, f_4, f_5, f_6]$, and $[a_m, b_m, c_m, d_m, e_m, f_m]$, $m = 1, 2, 3, 4, 5$, imply $[a_6, b_6, c_6, d_6, e_6, f_6]$.*

PROOF. For $m = 1, 2, 3, 4, 5, 6$ let the points g_m, h_m , and i_m be such that $[a_m, b_m, g_m]$, $[c_m, d_m, h_m]$, and $[e_m, f_m, i_m]$. Furthermore, let the points $a_{mn}, b_{mn}, c_{mn}, d_{mn}, e_{mn}, f_{mn}, g_{mn}, h_{mn}$, and i_{mn} be such that for all $(m, n) \in \{(1, 2), (3, 4), (5, 6)\}$, $[a_m, a_n, a_{mn}]$, $[b_m, b_n, b_{mn}]$, $[c_m, c_n, c_{mn}]$, $[d_m, d_n, d_{mn}]$, $[e_m, e_n, e_{mn}]$, $[f_m, f_n, f_{mn}]$, $[g_m, g_n, g_{mn}]$, $[h_m, h_n, h_{mn}]$, and $[i_m, i_n, i_{mn}]$. From the assumptions of the theorem we get $[a_{12}, a_{34}, a_{56}]$, $[b_{12}, b_{34}, b_{56}]$, $[c_{12}, c_{34}, c_{56}]$, $[d_{12}, d_{34}, d_{56}]$, $[e_{12}, e_{34}, e_{56}]$, $[f_{12}, f_{34}, f_{56}]$, $[g_m, h_m, i_m]$, $m = 1, 2, 3, 4, 5$, and it remains to prove $[g_6, h_6, i_6]$. From the tables

$$\begin{array}{ccc} a_1 & a_2 & \boxed{a_{12}} \\ b_1 & b_2 & \boxed{b_{12}} \end{array}, \quad \begin{array}{ccc} a_3 & a_4 & \boxed{a_{34}} \\ b_3 & b_4 & \boxed{b_{34}} \end{array}, \quad \begin{array}{ccc} a_5 & a_6 & \boxed{a_{56}} \\ b_5 & b_6 & \boxed{b_{56}} \end{array},$$

$$\begin{array}{ccc} g_1 & g_2 & \boxed{g_{12}} \\ g_3 & g_4 & \boxed{g_{34}} \end{array}, \quad \begin{array}{ccc} g_5 & g_6 & \boxed{g_{56}} \end{array}$$

we get $[a_{12}, b_{12}, g_{12}]$, $[a_{34}, b_{34}, g_{134}]$, and $[a_{56}, b_{56}, g_{56}]$, and similarly, replacing a, b, g by c, d, h , respectively by e, f, i , one proves $[c_{12}, d_{12}, h_{12}]$, $[c_{34}, d_{34}, h_{34}]$, and $[c_{56}, d_{56}, h_{56}]$, respectively $[e_{12}, f_{12}, i_{12}]$, $[e_{34}, f_{34}, i_{34}]$, and $[e_{56}, f_{56}, i_{56}]$. From the tables

$$\begin{array}{ccc} a_{12} & b_{12} & \boxed{g_{12}} \\ a_{34} & b_{34} & \boxed{g_{34}} \\ a_{56} & b_{56} & \boxed{g_{56}} \end{array}, \quad \begin{array}{ccc} c_{12} & d_{12} & \boxed{h_{12}} \\ c_{34} & d_{34} & \boxed{h_{34}} \\ c_{56} & d_{56} & \boxed{h_{56}} \end{array}, \quad \begin{array}{ccc} e_{12} & f_{12} & \boxed{i_{12}} \\ e_{34} & f_{34} & \boxed{i_{34}} \\ e_{56} & f_{56} & \boxed{i_{56}} \end{array},$$

we get $[g_{12}, g_{34}, g_{56}]$, $[h_{12}, h_{34}, h_{56}]$, and $[i_{12}, i_{34}, i_{56}]$, and then from the tables

$$\begin{array}{ccc} g_1 & g_2 & \boxed{g_{12}} \\ h_1 & h_2 & \boxed{h_{12}} \\ i_1 & i_2 & \boxed{i_{12}} \end{array} \quad \text{and} \quad \begin{array}{ccc} g_3 & g_4 & \boxed{g_{34}} \\ h_3 & h_4 & \boxed{h_{34}} \\ i_3 & i_4 & \boxed{i_{34}} \end{array}$$

we have $[g_{12}, h_{12}, i_{12}]$ and $[g_{34}, h_{34}, i_{34}]$. Finally, let us look at the tables

$$\begin{array}{ccc} g_{12} & g_{34} & \boxed{g_{56}} \\ h_{12} & h_{34} & \boxed{h_{56}} \\ i_{12} & i_{34} & \boxed{i_{56}} \end{array} \quad \text{and} \quad \begin{array}{ccc} g_5 & g_6 & \boxed{g_6} \\ h_5 & h_6 & \boxed{h_6} \\ i_5 & i_6 & \boxed{i_6} \end{array}.$$

From the first one we get $[g_{56}, h_{56}, i_{56}]$, and then from the second one we get $[g_6, h_6, i_6]$. □

The statement of Theorem 3.39 can be concisely presented by the table

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & \boxed{a_6} \\ b_1 & b_2 & b_3 & b_4 & b_5 & \boxed{b_6} \\ c_1 & c_2 & c_3 & c_4 & c_5 & \boxed{c_6} \\ d_1 & d_2 & d_3 & d_4 & d_5 & \boxed{d_6} \\ e_1 & e_2 & e_3 & e_4 & e_5 & \boxed{e_6} \\ f_1 & f_2 & f_3 & f_4 & f_5 & \boxed{f_6} \end{array}.$$

If $a_6 = b_6 = c_6 = d_6 = e_6 = f_1 = f_2 = f_3 = f_4 = f_5 = i$, respectively $a_5 = b_5 = c_5 = d_5 = e_1 = e_2 = e_3 = e_4 = i$, and $a_6 = b_6 = c_6 = d_6 = f_1 = f_2 = f_3 = f_4 = j$, and if in the first case we suppose that i^* is the point such that $[i, i, i, i, i, i^*]$, and in the second case that u, v, w are points such that $[i, i, i, i, u, v]$ and $[j, j, j, j, v, w]$, then the tables

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & i & \boxed{a_5} \\ b_1 & b_2 & b_3 & b_4 & i & \boxed{b_5} \\ c_1 & c_2 & c_3 & c_4 & i & \boxed{c_5} \\ d_1 & d_2 & d_3 & d_4 & i & \boxed{d_5} \\ e_1 & e_2 & e_3 & e_4 & i & \boxed{e_5} \\ i & i & i & i & i^* & \boxed{i} \end{array} \quad \text{and} \quad \begin{array}{cccccc} a_1 & a_2 & a_3 & i & j & \boxed{a_4} \\ b_1 & b_2 & b_3 & i & j & \boxed{b_4} \\ c_1 & c_2 & c_3 & i & j & \boxed{c_4} \\ d_1 & d_2 & d_3 & i & j & \boxed{d_4} \\ i & i & i & u & v & \boxed{i} \\ j & j & j & v & w & \boxed{j} \end{array}$$

prove the following two corollaries.

COROLLARY 3.40. $[a_1, a_2, a_3, a_4, a_5, i]$, $[b_1, b_2, b_3, b_4, b_5, i]$, $[c_1, c_2, c_3, c_4, c_5, i]$, $[d_1, d_2, d_3, d_4, d_5, i]$, $[e_1, e_2, e_3, e_4, e_5, i]$ and $[a_m, b_m, c_m, d_m, e_m, i]$, $m=1, 2, 3, 4$, imply $[a_5, b_5, c_5, d_5, e_5, i]$.

COROLLARY 3.41. $[a_1, a_2, a_3, a_4, i, j]$, $[b_1, b_2, b_3, b_4, i, i, j]$, $[c_1, c_2, c_3, c_4, i, j]$, $[d_1, d_2, d_3, d_4, i, j]$ and $[a_m, b_m, c_m, d_m, i, j]$, $m=1, 2, 3$, imply $[a_4, b_4, c_4, d_4, i, j]$.

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Konike u kubnoj strukturi

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SAŽETAK. Koristeći ternarnu relaciju u radu uvodimo koncept konike u kubnoj strukturi i proučavamo svojstva konika u kubnim strukturama ranga 1, 2 i 3. Pomoću točaka konike, konstruiramo različite konfiguracije točaka i konika, pravaca i konika, kao i neke poznate konfiguracije točaka i pravaca.

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