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GENERALIZED HERMITE-HADAMARD INEQUALITIES FOR $(\alpha, \eta, \gamma, \delta) - p$ CONVEX FUNCTIONS

Muhammad Bilal, Silvestru Sever Dragomir and Asif Raza Khan

Abstract. In this article, we would like to introduce another generalized class of convex functions which we call as $(\alpha, \eta, \gamma, \delta) - p$ convex functions. This new class contains another two new classes namely, $(\alpha, \eta) - p$ convex functions of the 1^{st} and 2^{nd} kinds. Further, we also generalize some results related to famous Hermite-Hadamard type inequality stated in [2] for the aforementioned class of functions with distinct techniques. Hence various existed and new results would be captured as special case of our obtained results. Moreover, application to midpoint formula has also been established.

"All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove."

– Hardy

1. Introduction

Mathematical inequality is one of the finest field for researchers from the past few decades due to its various applications in different fields of daily life like management sciences, architecture, arts, industrial and pharmaceutical research and many more. The theory of Convex functions (some times called water holding functions) has valuable importance in the stated field due to its involvement in different fields like probability theory, Operations research, Graph theory and many more. From these inequalities Hermite-Hadamard type are the one's which are being given more attention by researchers these days. For further study, one can go through the following articles: [3]−[7], [17, 18, 22, 23] and [25].

Here, we highlighted that throughout this article we would use the convention $0^0 = 1$ and the following notations:

1. I is a real interval $[\varpi_1, \varpi_2]$,

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2. I° is interior of interval I,
3. $M_p = \dfrac{\varpi_2{}^p - \varpi_1{}^p}{p}$ and
4. $\beta_r(\varpi_1, \varpi_2) = \int_0^r u^{\varpi_1 - 1}(1-u)^{\varpi_2 - 1} du, \quad \varpi_1, \varpi_2 > 0$ is an incomplete Beta function.

We organized our article as: First section is devoted to background of our study and some preliminary results. In the next section, we are going to introduce another generalized class of convex function which we termed as $(\alpha, \eta, \gamma, \delta) - p$ convex functions. Here we estimate the left bound (absolute difference of the first and middle terms of Hermite-Hadamard dual inequality) of Hermite-Hadamard inequality for the class of differentiable $(\alpha, \eta, \gamma, \delta) - p$ convex functions using distinct techniques. These results would secure various results stated in the articles [2, 15, 16] and [21] as special cases. Third section is devoted to application to midpoint formula and the last section holds conclusion with some general remarks and future ideas for interested readers.

Now, we are going to recall some useful definitions and results related to our topic.

In literature, the famous Hermite-Hadamard dual inequality obtained by the contribution of Hermite and Hadamard as:

THEOREM 1.1 ([10]). *If $\xi : I \to \mathbb{R}$ is a convex function, then*

$$\xi\left(\frac{\varpi_1 + \varpi_2}{2}\right) \le \frac{1}{\varpi_2 - \varpi_1} \int_{\varpi_1}^{\varpi_2} \xi(\zeta) d\zeta \le \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2}. \tag{1.1}$$

REMARK 1.2. It is to be observed that Hadamard's inequality may be regarded as the refinement of Jensen's inequality.

The well-known inequality has been given an illustrious attention in recent years (see [1]–[5], [11]–[15] and [27]).

Here, we recall the definition of p-convex functions, extracted from [12]:

DEFINITION 1.3. *If $p \in \mathbb{R} - \{0\}$ and $\xi : I \subset (0, \infty) \to \mathbb{R}$ is a p-convex, then*

$$\xi\left([ux^p + (1-u)\zeta^p]^{\frac{1}{p}}\right) \le uf(x) + (1-u)\xi(\zeta),$$

$\forall\ x, y \in I$ and $u \in [0, 1]$.

REMARK 1.4. If we take $p = 1$ and $p = -1$ in Definition 1.3, we attain the ordinary convex function [6] and harmonically convex function [11], respectively.

Recently, in [2], we have introduced the definition of $s-p$ convex functions in the mixed kind (or $(r, s)-p$ convex functions) by generalizing the definitions of $s-p$ convex functions of the 1^{st} and 2^{nd} kinds in such a way that we can easily obtained both the definitions by imposing certain restrictions on r and s.

Definition 1.5. *If* $(r,s)\in[0,1]^2$, $p\in\mathbb{R}-\{0\}$ *and* $\xi: I\subset(0,\infty)\to[0,\infty)$ *is a* $(r,s)-p$ *convex function, then*

$$\xi\left(\left[ux^p+(1-u)\zeta^p\right]^{\frac{1}{p}}\right)\leq u^{rs}\xi(x)+(1-u^r)^s\xi(\zeta), \tag{1.2}$$

$\forall\ x,y\in I$ *and* $u\in[0,1]$.

Remark 1.6. Following well known results will be obtained by taking different combinations for the values of r, s and p.

1. If we take $r=0$ with $s\neq 0$ in (1.2), we attain the refinement of quasi$-p$ convex functions [2].
2. If we take $s=0$ in (1.2), we attain $P-p$ convex functions [3].
3. If we take $r=1$ in (1.2), we attain $s-p$ convex functions of the 1^{st} kind [2].
4. If we take $s=1$ in (1.2), we attain $s-p$ convex functions of the 2^{nd} kind [2].
5. If we take $p=1$ in (1.2), we attain (r,s) convex functions [14].
6. If we take $p=-1$ in (1.2), we attain harmonically (r,s) convex functions [4].
7. If we take $r=s=1$ in (1.2), we attain Definition 1.3.
8. If we take $r=0$ and $p=1$ with $s\neq 0$ in (1.2), we attain refinement of quasi convex functions [4].
9. If we take $r=0$ and $p=-1$ with $s\neq 0$ in (1.2), we attain refinement of harmonically quasi convex functions [4].
10. If we take $p=r=1$ in (1.2), we attain s-convex functions of the 2^{nd} kind [14].
11. If we take $r=1$ and $p=-1$ in (1.2), we attain harmonically s-convex functions of the 2^{nd} kind [4].
12. If we take $p=1$ and $s=0$ in (1.2), we attain P-convex functions [8].
13. If we take $p=-1$ and $s=0$ in (1.2), we attain harmonically P-convex functions [8].
14. If we take $p=s=1$ in (1.2), we attain s-convex functions of the 1^{st} kind [14].
15. If we take $s=1$ and $p=-1$ in (1.2), we attain harmonically s-convex functions of the 1^{st} kind [4].
16. If we take $p=r=s=1$ in (1.2), we attain ordinary convex functions [6].
17. If we take $r=s=1$ and $p=-1$ in (1.2), we attain harmonically convex functions [11].

Well-known Classical Hölder's integral inequality in its general integral form is stated as follows [19]:

THEOREM 1.7. *If* $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $g \in L_q$, *then* $\xi g \in L_1$ *and*

$$\int |\xi(\zeta)g(\zeta)|d\zeta \le \|\xi\|_p \|g\|_q, \tag{1.3}$$

where $\xi \in L_p$ *if* $\|\xi\|_p = \left(\int |\xi(\zeta)|^p d\zeta\right)^{\frac{1}{p}} < \infty$.

REMARK 1.8. Note that if we put $p = q = 2$, the above inequality becomes Cauchy-Schwarz inequality. Also, if we put $q = 1$ and let $p \to \infty$, then we attain,

$$\int |\xi(\zeta)g(\zeta)|d\zeta \le ||\xi||_\infty ||g||_1,$$

where $||\xi||_\infty$ stands for the essential supremum of $|\xi|$, i.e.,

$$||\xi||_\infty = ess \sup_{\forall y} |\xi(\zeta)|.$$

PROPOSITION 1.9. *If* ξ *and* g *are real valued functions defined on* I *with* $|\xi|$ *and* $|\xi||g|^q$ *are integrable on* I, *then for* $q \ge 1$ *we have:*

$$\int_{\varpi_1}^{\varpi_2} |\xi(\zeta)||g(\zeta)|d\zeta \le \left(\int_{\varpi_1}^{\varpi_2} |\xi(\zeta)|d\zeta\right)^{1-\frac{1}{q}} \left(\int_{\varpi_1}^{\varpi_2} |\xi(\zeta)||g(\zeta)|^q d\zeta\right)^{\frac{1}{q}}. \tag{1.4}$$

The above inequality is known in literature as Power mean integral inequality (see [24]).

In [2], we proposed some results related to left bound estimation of Hermite-Hadamard type inequalities for the class of differentiable $(r, s) - p$ convex functions which we recall here:

THEOREM 1.10. *Let* $\xi : I \subset (0, +\infty) \to \mathbb{R}$ *be a first differentiable function on* I° *s.t.* $\xi' \in L[\varpi_1, \varpi_2]$, *where* $\varpi_1, \varpi_2 \in I^\circ$ *and* $\varpi_1 < \varpi_2$. *If* $|\xi'|$ *is* $(r, s) - p$ *convex function on* I *for fixed* $r, s \in [0, 1]$ *on* I *for* $p \in \mathbb{R} - \{0\}$, *then following inequality holds:*

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p \xi\left(\left[\frac{\varpi_1{}^p + \varpi_2{}^p}{2}\right]^{\frac{1}{p}}\right)\right| \le M_p^2 [A(p)|\xi'(\varpi_1)| + B(p)|\xi'(\varpi_2)|],$$

where

$$A(p) = \left[\int_0^{1/2} \frac{u^{rs+1}}{[u\varpi_1{}^p + (1-u)\varpi_2{}^p]^{1-\frac{1}{p}}} du + \int_{1/2}^{1} \frac{u^{rs} - u^{rs+1}}{[u\varpi_1{}^p + (1-u)\varpi_2{}^p]^{1-\frac{1}{p}}} du\right]$$

and

$$B(p)=\left[\int_0^{1/2}\frac{u(1-u^r)^s}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du+\int_{1/2}^{1}\frac{(1-u)(1-u^r)^s}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du\right].$$

THEOREM 1.11. *Let $\xi: I\subset(0,+\infty)\to\mathbb{R}$ be a first differentiable function on I° s.t. $\xi'\in L[\varpi_1,\varpi_2]$, where $\varpi_1,\varpi_2\in I^\circ$ and $\varpi_1<\varpi_2$. If $|\xi'|^q$ with $q\geq 1$ is a $(r,s)-p$ convex function on I for fixed $r,s\in[0,1]$ and $p\in\mathbb{R}-\{0\}$, then following inequality holds:*

$$\begin{aligned}&\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^{1-p}}d\zeta-M_p\xi\left(\left[\frac{\varpi_1{}^p+\varpi_2{}^p}{2}\right]^{\frac{1}{p}}\right)\right|\\&\leq M_p^2\left\{(C(p))^{1-\frac{1}{q}}\left[(D(p))|\xi'(\varpi_1)|^q+E(p)|\xi'(\varpi_2)|^q\right]^{\frac{1}{q}}\right.\\&\left.+(F(p))^{1-\frac{1}{q}}\left[(G(p))|\xi'(\varpi_1)|^q+H(p)|\xi'(\varpi_2)|^q\right]^{\frac{1}{q}}\right\},\end{aligned}$$

where

$$C(p)=\int_0^{1/2}\frac{u}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du,\quad D(p)=\int_0^{1/2}\frac{u^{rs+1}}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du,$$

$$E(p)=\int_0^{1/2}\frac{u(1-u^r)^s}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du,\quad F(p)=\int_{1/2}^{1}\frac{(1-u)}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du,$$

$$G(p)=\int_{1/2}^{1}\frac{u^{rs}(1-u)}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du,\quad H(p)=\int_{1/2}^{1}\frac{(1-u)(1-u^r)^s}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du.$$

THEOREM 1.12. *Let $\xi: I\subset(0,+\infty)\to\mathbb{R}$ be a first differentiable function on I° s.t. $\xi'\in L[\varpi_1,\varpi_2]$, where $\varpi_1,\varpi_2\in I^\circ$ and $\varpi_1<\varpi_2$. If $|\xi'|^{q_2}$ with $q_2\geq 1$ and $\frac{1}{q_1}+\frac{1}{q_2}=1$ is a $(r,s)-p$ convex function on I for fixed $r,s\in[0,1]$ and $p\in\mathbb{R}-\{0\}$, then following inequality holds for $r\neq 0$:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^{1-p}}d\zeta-M_p\xi\left(\left[\frac{\varpi_1{}^p+\varpi_2{}^p}{2}\right]^{\frac{1}{p}}\right)\right|$$

$$\leq M_p^2 \left[J(p) \left(\frac{|\xi'(\varpi_1)|^{q_2}}{2^{rs+1}(rs+1)} + \frac{\beta_{1/2^r}\left(\frac{1}{r}, s+1\right)|\xi'(\varpi_2)|^{q_2}}{r} \right)^{\frac{1}{q_2}} \right.$$
$$\left. +K(p) \left(\frac{(2^{rs+1}-1)|\xi'(\varpi_1)|^{q_2}}{2^{rs+1}(rs+1)} + \frac{\beta_{1-1/2^r}\left(s+1, \frac{1}{r}\right)|\xi'(\varpi_2)|^{q_2}}{r} \right)^{\frac{1}{q_2}} \right],$$

where

$$J(p) = \left(\int_0^{1/2} \left(\frac{u}{[u\varpi_1{}^p + (1-u)\varpi_2{}^p]^{1-\frac{1}{p}}} \right)^{q_1} du \right)^{\frac{1}{q_1}}$$

and

$$K(p) = \left(\int_{1/2}^{1} \left(\frac{1-u}{[u\varpi_1{}^p + (1-u)\varpi_2{}^p]^{1-\frac{1}{p}}} \right)^{q_1} du \right)^{\frac{1}{q_1}}.$$

Now we state the following identity which will be used later to derive our main results of this article.

Lemma 1.13 ([21]). *If $\xi : I \subset (0,\infty) \to \mathbb{R}$ is a first differentiable function on I° with $\xi' \in L[\varpi_1, \varpi_2]$, $\varpi_1, \varpi_2 \in I^\circ$ with $\varpi_1 < \varpi_2$ and $p \in \mathbb{R} - \{0\}$, then following identity holds:*

$$\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p \xi \left(\left[\frac{\varpi_1{}^p + \varpi_2{}^p}{2} \right]^{\frac{1}{p}} \right)$$
$$= M_p^2 \int_{\varpi_1}^{\varpi_2} \frac{K(u)}{[u\varpi_1{}^p + (1-u)\varpi_2{}^p]^{1-\frac{1}{p}}} \xi' \left([u\varpi_1{}^p + (1-u)\varpi_2{}^p]^{\frac{1}{p}} \right) du,$$

where

$$K(u) = \begin{cases} u, & u \in [0, \frac{1}{2}), \\ u-1, & u \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

2. Left bound estimations of Hermite-Hadamard inequality for $(\alpha, \eta, \gamma, \delta) - p$ convex functions

As we mentioned earlier the main objective of this article is to introduce another new class of convex functions and then use this class to estimate the left bound of Hermite-Hadamard inequality using three distinct techniques including Power mean and Hölder's integral inequalities.

Firstly, we are going to introduce another new class of p-convex functions, which we call as $(\alpha,\eta,\gamma,\delta)-p$ convex functions. This new class also contain five new and various well reputed classes of convex functions.

Definition 2.1. *If* $(\alpha,\eta,\gamma,\delta)\in[0,1]^4$, $p\in\mathbb{R}-\{0\}$ *and* $\xi: I\subset(0,\infty)\to[0,\infty)$ *is a* $(\alpha,\eta,\gamma,\delta)-p$ *convex function, then*

$$\xi\left([ux^p+(1-u)\zeta^p]^{\frac{1}{p}}\right)\le u^{\alpha\gamma}\xi(x)+(1-u^{\eta})^{\delta}\xi(\zeta),$$

$\forall\ x,y\in I$ *and* $u\in[0,1]$.

Remark 2.2. Following new and well-known interesting results would be obtained by substituting different of values of α, η, γ, δ and p:

1. If we take $p=1$ in Definition 2.1, we attain $(\alpha,\eta,\gamma,\delta)-$convex functions [9].
2. If we take $p=-1$ in Definition 2.1, we attain harmonically $(\alpha,\eta,\gamma,\delta)-$convex functions, i.e.,
$$\xi\left(\frac{x\zeta}{(1-u)x+u\zeta}\right)\le u^{\alpha\gamma}\xi(x)+(1-u^{\eta})^{\delta}\xi(\zeta).$$
3. If we take $\gamma=\delta=1$ in Definition 2.1, we attain $(\alpha,\eta)-p$ convex functions of the 1^{st} kind, i.e.,
$$\xi\left([ux^p+(1-u)\zeta^p]^{\frac{1}{p}}\right)\le u^{\alpha}\xi(x)+(1-u^{\eta})\xi(\zeta).$$
4. If we take $\gamma=\delta=1$ and $p=1$ in Definition 2.1, we attain $(\alpha,\eta)-$convex functions of the 1^{st} kind [9].
5. If we take $\gamma=\delta=1$ and $p=-1$ in Definition 2.1, we attain harmonically $(\alpha,\eta)-$convex functions of the 1^{st} kind, i.e.,
$$\xi\left(\frac{x\zeta}{(1-u)x+u\zeta}\right)\le u^{\alpha}\xi(x)+(1-u^{\eta})\xi(\zeta).$$
6. If we take $\alpha=\eta=1$ in Definition 2.1, we attain $(\alpha,\eta)-p$ convex functions of the 2^{nd} kind, i.e.,
$$\xi\left([ux^p+(1-u)\zeta^p]^{\frac{1}{p}}\right)\le u^{\gamma}\xi(x)+(1-u)^{\delta}\xi(\zeta).$$
7. If we take $\alpha=\eta=1$ and $p=1$ in Definition 2.1, we attain $(\alpha,\eta)-$convex functions of the 2^{nd} kind [9].
8. If we take $\alpha=\eta=1$ and $p=-1$ in Definition 2.1, we attain harmonically $(\alpha,\eta)-$convex functions of the 2^{nd} kind, i.e.,
$$\xi\left(\frac{x\zeta}{(1-u)x+u\zeta}\right)\le u^{\gamma}\xi(x)+(1-u)^{\delta}\xi(\zeta).$$
9. If we take $\alpha=\eta=r$ and $\gamma=\delta=s$ in Definition (2.1), we attain $(r,s)-p$ convex functions [2].

10. If we take $\alpha = \eta = r$, $\gamma = \delta = s$ and $p = 1$ in Definition (2.1), we attain $(r, s)-$convex functions [4].
11. If we take $\alpha = \eta = r$, $\gamma = \delta = s$ and $p = -1$ in Definition (2.1), we attain harmonically $(r, s)-$convex functions [4].
12. If we take $\alpha = \eta = s$ and $\gamma = \delta = 1$ in Definition 2.1, we attain $s - p$ convex functions of the 1^{st} kind [2].
13. If we take $\alpha = \eta = s$ and $\gamma = \delta = 1$ and $p = 1$ in Definition 2.1, we attain s-convex functions of the 1^{st} kind [21].
14. If we take $\alpha = \eta = s$ and $\gamma = \delta = 1$ and $p = -1$ in Definition 2.1, we attain harmonically s-convex functions of the 1^{st} kind [21].
15. If we take $\alpha = \eta = 1$ and $\gamma = \delta = s$ in Definition 2.1, we attain $s - p$ convex functions of the 2^{nd} kind [2].
16. If we take $\alpha = \eta = 1$ and $\gamma = \delta = s$ and $p = 1$ in Definition 2.1, we attain s-convex functions of the 2^{nd} kind [2].
17. If we take $\alpha = \eta = 1$ and $\gamma = \delta = s$ and $p = -1$ in Definition 2.1, we attain harmonically s-convex functions of the 2^{nd} kind [21].
18. If we take $\alpha = \eta = 0$ and $\delta \neq 0$ in Definition 2.1, we attain refinement of quasi$-p$ convex functions [3].
19. If we take $\alpha = \eta = 0$, $\delta \neq 0$ and $p = 1$ in Definition 2.1, we attain refinement of quasi convex functions [4].
20. If we take $\alpha = \eta = 0$, $\delta \neq 0$ and $p = -1$ in Definition 2.1, we attain refinement of harmonically quasi convex functions [4].
21. If we take $\gamma = \delta = 0$ in Definition 2.1, we attain $P - p$ convex functions [3].
22. If we take $\gamma = \delta = 0$ and $p = 1$ in Definition 2.1, we attain P-convex functions [3].
23. If we take $\gamma = \delta = 0$ and $p = -1$ in Definition 2.1, we attain harmonically P-convex functions [3].
24. If we take $\alpha = \eta = \gamma = \delta = 1$ in Definition 2.1, we attain p-convex functions [3].
25. If we take $\alpha = \eta = \gamma = \delta = 1$ and $p = 1$ in Definition 2.1, we attain ordinary convex functions [6].
26. If we take $\alpha = \eta = \gamma = \delta = 1$ and $p = -1$ in Definition 2.1, we attain harmonically convex functions [4].

Now, we are going to reveal three results in the form of theorem related to the sum, product and composition of $(\alpha, \eta, \gamma, \delta) - p$ convex function.

Let ξ and g are said to be similarly ordered functions on I. Then the following inequality holds

$$(\xi(\zeta) - \xi(x))(g(\zeta) - g(x)) \geq 0 \tag{2.1}$$

for all $\zeta, x \in I$ ([26]).

THEOREM 2.3. *Let ξ and g be a similarly ordered functions on I. If ξ and g are $(\alpha,\eta,\gamma,\delta)-p$ convex functions and $u^{\alpha\gamma}+(1-u^{\eta})^{\delta}\leq c$ for all $(\alpha,\eta,\gamma,\delta)\in(0,1)$, where $u=\max\{u_1,u_2\}$ and c is a fixed positive number, then the product ξg is a $(\alpha,\eta,\gamma,\delta)-cp$ convex function.*

PROOF. From (2.1), we have

$$\xi(\zeta)g(\zeta)+\xi(x)g(x)\geq\xi(\zeta)g(x)+\xi(x)g(\zeta).$$

Since ξ and g are $(\alpha,\eta,\gamma,\delta)-p$ convex functions we obtain

$$\begin{aligned}
& \xi g\left([ux^p+(1-u)\zeta^p]^{\frac{1}{p}}\right)\\
= & \xi\left([u_1x^p+(1-u_1)\zeta^p]^{\frac{1}{p}}\right)g\left([u_2x^p+(1-u_2)\zeta^p]^{\frac{1}{p}}\right)\\
\leq & \left(u_1^{\alpha\gamma}\xi(x)+(1-u_1^{\eta})^{\delta}\xi(\zeta)\right)\left(u_2^{\alpha\gamma}g(x)+(1-u_2^{\eta})^{\delta}g(\zeta)\right)\\
\leq & \left(u^{\alpha\gamma}\right)^2\xi(x)g(x)+u^{\alpha\gamma}(1-u^{\eta})^{\delta}\xi(x)g(\zeta)+u^{\alpha\gamma}(1-u^{\eta})^{\delta}\xi(\zeta)g(x)\\
+ & \left((1-u^{\eta})^{\delta}\right)^2\xi(\zeta)g(\zeta)\\
\leq & \left(u^{\alpha\gamma}\right)^2\xi(x)g(x)+u^{\alpha\gamma}(1-u^{\eta})^{\delta}\xi(x)g(x)+u^{\alpha\gamma}(1-u^{\eta})^{\delta}\xi(\zeta)g(\zeta)\\
+ & \left((1-u^{\eta})^{\delta}\right)^2\xi(\zeta)g(\zeta)\\
= & \left(u^{\alpha\gamma}+(1-u^{\eta})^{\delta}\right)\left(u^{\alpha\gamma}\xi g(x)+(1-u^{\eta})^{\delta}\xi g(\zeta)\right)\\
\leq & cu^{\alpha\gamma}\xi g(x)+c(1-u^{\eta})^{\delta}\xi g(\zeta).
\end{aligned}$$

□

REMARK 2.4. It is to be noted that if $c=1$, then the product of simply ordered $(\alpha,\eta,\gamma,\delta)-p$ convex functions is a $(\alpha,\eta,\gamma,\delta)-p$ convex function.

THEOREM 2.5. *If ξ and g are $(\alpha,\eta,\gamma,\delta)-p$ convex functions for all $(\alpha,\eta,\gamma,\delta)\in(0,1)^4$ and $u=\max\{u_1,u_2\}$, then the addition $\xi+g$ is $(\alpha,\eta,\gamma,\delta)-p$ convex function.*

PROOF. Since ξ and g are $(\alpha,\eta,\gamma,\delta)-p$ convex functions, so we have

$$\xi\left([u_1x^p+(1-u_1)\zeta^p]^{\frac{1}{p}}\right)\leq\left(u_1^{\alpha\gamma}\xi(x)+(1-u_1^{\eta})^{\delta}\xi(\zeta)\right)$$

and

$$g\left([u_2x^p+(1-u_2)\zeta^p]^{\frac{1}{p}}\right)\leq\left(u_2^{\alpha\gamma}g(x)+(1-u_2^{\eta})^{\delta}g(\zeta)\right).$$

By adding the above two results and applying the definition of maximum value we get

$$\begin{aligned}
&\xi\left([u_1x^p+(1-u_1)\zeta^p]^{\frac{1}{p}}\right)+g\left([u_2x^p+(1-u_2)\zeta^p]^{\frac{1}{p}}\right)\\
&\quad\leq\left(u^{\alpha\gamma}\left(\xi(x)+g(x)\right)+(1-u^{\eta})^{\delta}\left(\xi(\zeta)+g(\zeta)\right)\right).
\end{aligned}$$

□

THEOREM 2.6. *Let g be a $(\alpha,\eta,\gamma,\delta)-p$ convex function for all $(\alpha,\eta,\gamma,\delta)\in(0,1)^4$. If ξ is linear, then $\xi\circ g$ is $(\alpha,\eta,\gamma,\delta)-p$ convex function.*

PROOF. As g is $(\alpha,\eta,\gamma,\delta)-p$ convex function, we have

$$\xi\circ g\left([ux^p+(1-u)\zeta^p]^{\frac{1}{p}}\right)\le\xi\left(u^{\alpha\gamma}g(x)+(1-u^\eta)^\delta g(\zeta)\right).$$

The result will be accomplished by taking ξ to be linear.

$$\begin{aligned}&\xi\left(u^{\alpha\gamma}g(x)+(1-u^\eta)^\delta g(\zeta)\right)\\&=u^{\alpha\gamma}\xi\left(g(x)\right)+(1-u^\eta)^\delta\xi\left(g(\zeta)\right)\\&=u^{\alpha\gamma}\xi\circ g(x)+(1-u^\eta)^\delta\xi\circ g(\zeta)\end{aligned}$$

□

Now we are going to state and prove three distinct generalized results related to Hermite-Hadamard type inequality for $(\alpha,\eta,\gamma,\delta)-p$ convex functions using Definition 2.1, Theorem 1.7, Proposition 1.9 and Lemma 1.13.

THEOREM 2.7. *Assume that $\xi'\in L[\varpi_1,\varpi_2]$ with $\varpi_1,\varpi_2\in I^\circ$ and $\varpi_2>\varpi_1$. If $\xi:I\subset(0,\infty)\to\mathbb{R}$ is a differentiable function on I° s. t. $|\xi'|$ is $(\alpha,\eta,\gamma,\delta)-p$ convex on I for fixed $(\alpha,\eta,\gamma,\delta)\in[0,1]^4$ on I and $p\in\mathbb{R}-\{0\}$, then below stated result holds:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^{1-p}}d\zeta-M_p\xi\left(\left[\frac{{\varpi_1}^p+{\varpi_2}^p}{2}\right]^{\frac{1}{p}}\right)\right|\le M_p^2[Z_1(p)|\xi'(\varpi_1)|+Z_2(p)|\xi'(\varpi_2)|],$$

where

$$Z_1(p)=\int_0^{1/2}\frac{\dagger^{\alpha\gamma+1}}{[\dagger{\varpi_1}^p+(1-\dagger){\varpi_2}^p]^{1-\frac{1}{p}}}d\dagger+\int_{1/2}^{1}\frac{\dagger^{\alpha\gamma}-\dagger^{\alpha\gamma+1}}{[\dagger{\varpi_1}^p+(1-\dagger){\varpi_2}^p]^{1-\frac{1}{p}}}d\dagger$$

and

$$Z_2(p)=\int_0^{1/2}\frac{\dagger(1-\dagger^\eta)^\delta}{[\dagger{\varpi_1}^p+(1-\dagger){\varpi_2}^p]^{1-\frac{1}{p}}}d\dagger+\int_{1/2}^{1}\frac{(1-\dagger)(1-\dagger^\eta)^\delta}{[\dagger{\varpi_1}^p+(1-\dagger){\varpi_2}^p]^{1-\frac{1}{p}}}d\dagger.$$

PROOF. By using Lemma 1.13 and the definition of absolute value, we have,

$$\begin{aligned}&\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^{1-p}}d\zeta-M_p\xi\left(\left[\frac{{\varpi_1}^p+{\varpi_2}^p}{2}\right]^{\frac{1}{p}}\right)\right|\\&\le M_p^2\left[\int_0^{1/2}\frac{\dagger}{[\dagger{\varpi_1}^p+(1-\dagger){\varpi_2}^p]^{1-\frac{1}{p}}}\left|\xi'\left([\dagger{\varpi_1}^p+(1-\dagger){\varpi_2}^p]^{\frac{1}{p}}\right)\right|d\dagger\right.\end{aligned}$$

$$(2.2)\qquad + \int_{1/2}^{1} \frac{1-\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right| d\dagger \Bigg].$$

As we have $|\xi'|$ is a $(\alpha, \eta, \gamma, \delta) - p$-convex function, so we can take,

$$\left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right| \leq \dagger^{\alpha\gamma}|\xi'(\varpi_1)| + (1-\dagger^{\eta})^{\delta}|\xi'(\varpi_2)|.$$

Utilizing the above two results, (2.2) becomes

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p \xi\left(\left[\frac{\varpi_1{}^p + \varpi_2{}^p}{2}\right]^{\frac{1}{p}}\right)\right| \leq M_p^2$$

$$\Bigg[\left\{\int_0^{1/2} \frac{\dagger^{\alpha\gamma+1}}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger + \int_{1/2}^{1} \frac{\dagger^{\alpha\gamma} - \dagger^{\alpha\gamma+1}}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger\right\} |\xi'(\varpi_1)|$$

$$+\left\{\int_0^{1/2} \frac{\dagger(1-\dagger^{\eta})^{\delta}}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger + \int_{1/2}^{1} \frac{(1-\dagger)(1-\dagger^{\eta})^{\delta}}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger\right\} |\xi'(\varpi_2)|\Bigg],$$

which completes the proof. □

REMARK 2.8. By varying different values of $\alpha, \eta, \gamma, \delta$ and p in Theorem 2.7, one can capture the following well-known inequalities:

1. If we replace $\alpha = \eta = r$ and $\gamma = \delta = s$, then we acquire the result of Hermite-Hadamard type inequality for $(r, s) - p$ convex functions (see Theorem 4 of [2]).
2. If we replace $p = 1$, $\alpha = \eta = r$ and $\gamma = \delta = s$, then we acquire the result of Hermite-Hadamard type inequality for $(r, s)-$convex functions (see third result of Corollary 1 of [2]).
3. If we replace $\alpha = \eta = r$ and $\gamma = \delta = 1$, then we acquire the result of Hermite-Hadamard type inequality for $s - p$ convex functions of the 1st kind (see first result of Corollary 1 of [2]).
4. If we replace $\alpha = \eta = r$ and $\gamma = \delta = p = 1$, then we acquire the result of Hermite-Hadamard type inequality for s-convex functions of the 1st kind (see fourth result of Corollary 1 of [2]).
5. If we replace $\alpha = \eta = 1$ and $\gamma = \delta = s$, then we acquire the result of Hermite-Hadamard type inequality for $s - p$ convex functions of the 2nd kind (see second result of Corollary 1 of [2]).
6. If we replace $\alpha = \eta = p = 1$ and $\gamma = \delta = s$, then we acquire the result of Hermite-Hadamard type inequality for s-convex functions of the 2nd kind (see fifth result of Corollary 1 of [2]).

7. If we replace $\alpha=\eta=\gamma=\delta=1$, then we acquire the result of Hermite-Hadamard type inequality for p-convex functions (see Theorem 3.3 of [21]).
8. If we replace $p=\alpha=\eta=\gamma=\delta=1$, then we acquire the result of Hermite-Hadamard type inequality for ordinary convex functions (see Theorem 2.2 of [16]).
9. If we replace $p=-1$ and $\alpha=\eta=\gamma=\delta=1$, then we acquire the result of Hermite-Hadamard type inequality for harmonically convex functions (see third result of Corollary 1 of [15]).

Corollary 2.9. *Under the assumptions of Theorem* 2.7, *one can achieve the following results:*

1. *If we replace* $p=1$, *then we acquire the following Hermite-Hadamard type inequality for* $(\alpha,\eta,\gamma,\delta)-$*convex functions:*

$$\left|\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta-M_1\xi\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|\leq M_1^2\left[\frac{(2^{\alpha\gamma+1}-1)}{2^{\alpha\gamma+1}(\alpha\gamma+1)(\alpha\gamma+2)}|\xi'(\varpi_1)|+\right.$$
$$\left.\left(\beta_{1/2^\eta}\left(\frac{2}{\eta},\delta+1\right)+\beta_{1-1/2^\eta}\left(\delta+1,\frac{1}{\eta}\right)-\beta_{1-1/2^\eta}\left(\delta+1,\frac{2}{\eta}\right)\right)\frac{|\xi'(\varpi_2)|}{\eta}\right].$$

2. *If we replace* $p=-1$, *then we acquire the following Hermite-Hadamard type inequality for harmonically* $(\alpha,\eta,\gamma,\delta)-$*convex functions:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^2}d\zeta-M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right|\leq(\varpi_1\varpi_2M_{-1})^2\times$$
$$\left[\left\{\int_0^{1/2}\frac{u^{\alpha\gamma+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du+\int_{1/2}^{1}\frac{u^{\alpha\gamma}-u^{\alpha\gamma+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du\right\}|\xi'(\varpi_1)|\right.$$
$$\left.+\left\{\int_0^{1/2}\frac{u(1-u^\eta)^\delta}{[u\varpi_2+(1-u)\varpi_1]^2}du+\int_{1/2}^{1}\frac{(1-u)(1-u^\eta)^\delta}{[u\varpi_2+(1-u)\varpi_1]^2}du\right\}|\xi'(\varpi_2)|\right].$$

3. *If we replace* $\gamma=\delta=1$, *then we acquire the following Hermite-Hadamard type inequality for* $(\alpha,\eta)-p$ *convex functions of the* 1^{st} *kind:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^{1-p}}d\zeta-M_p\xi\left(\left[\frac{{\varpi_1}^p+{\varpi_2}^p}{2}\right]^{\frac{1}{p}}\right)\right|\leq M_p^2\times$$

$$\left[\left\{\int_0^{1/2}\frac{u^{\alpha+1}}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du+\int_{1/2}^{1}\frac{u^{\alpha}-u^{\alpha+1}}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du\right\}|\xi'(\varpi_1)|\right.$$

$$\left.+\left\{\int_0^{1/2}\frac{u-u^{\eta+1}}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du+\int_{1/2}^{1}\frac{1-u-u^{\eta}+u^{\eta+1}}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du\right\}|\xi'(\varpi_2)|\right].$$

4. *If we replace $\gamma=\delta=p=1$, then we acquire the following Hermite-Hadamard type inequality for $(\alpha,\eta)-$convex functions of the 1^{st} kind:*

$$\left|\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta-M_1\xi\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|\le M_1^2\times$$

$$\left[\frac{(2^{\alpha+1}-1)}{2^{\alpha+1}(\alpha+1)(\alpha+2)}|\xi'(\varpi_1)|+\left(\frac{1}{4}-\frac{(2^{\eta+1}-1)}{2^{\eta+1}(\eta+1)(\eta+2)}\right)|\xi'(\varpi_2)|\right].$$

5. *If we replace $\gamma=\delta=1$ and $p=-1$, then we acquire the following Hermite-Hadamard type inequality for harmonically $(\alpha,\eta)-$convex functions of the 1^{st} kind:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{y^2}d\zeta-M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right|\le(\varpi_1\varpi_2M_{-1})^2\times$$

$$\left[\left\{\int_0^{1/2}\frac{u^{\alpha+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du+\int_{1/2}^{1}\frac{u^{\alpha}-u^{\alpha+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du\right\}|\xi'(\varpi_1)|\right.$$

$$\left.+\left\{\int_0^{1/2}\frac{u-u^{\eta+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du+\int_{1/2}^{1}\frac{1-u-u^{\eta}+u^{\eta+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du\right\}|\xi'(\varpi_2)|\right].$$

6. *If we replace $\alpha=\eta=1$, then we acquire the following Hermite-Hadamard type inequality for $(\alpha,\eta)-p$ convex functions of the 2^{nd} kind:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^{1-p}}d\zeta-M_p\xi\left(\left[\frac{\varpi_1{}^p+\varpi_2{}^p}{2}\right]^{\frac{1}{p}}\right)\right|\le M_p^2\times$$

$$\left[\left\{\int_0^{1/2}\frac{u^{\gamma+1}}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du+\int_{1/2}^{1}\frac{u^{\gamma}-u^{\gamma+1}}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du\right\}|\xi'(\varpi_1)|\right.$$

$$\left.+\left\{\int_0^{1/2}\frac{u(1-u)^{\delta}}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du+\int_{1/2}^{1}\frac{(1-u)^{\delta+1}}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du\right\}|\xi'(\varpi_2)|\right].$$

7. *If we replace $\alpha = \eta = p = 1$, then we acquire the following Hermite-Hadamard type inequality for $(\alpha, \eta)-$convex functions of the 2^{nd} kind:*

$$\left| \int_{\varpi_1}^{\varpi_2} \xi(\zeta) d\zeta - M_1 \xi\left(\frac{\varpi_1 + \varpi_2}{2}\right) \right|$$

$$\leq M_1^2 \left[\frac{(2^{\gamma+1} - 1)}{2^{\gamma+1}(\gamma+1)(\gamma+2)} |\xi'(\varpi_1)| + \frac{(2^{\delta+1} - 1)}{2^{\delta+1}(\delta+1)(\delta+2)} |\xi'(\varpi_2)| \right].$$

8. *If we replace $\alpha = \eta = 1$ and $p = -1$, then we acquire the following Hermite-Hadamard type inequality for harmonically $(\alpha, \eta)-$convex functions of the 2^{nd} kind:*

$$\left| \int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^2} d\zeta - M_{-1} \xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right) \right| \leq (\varpi_1\varpi_2 M_{-1})^2 \times$$

$$\left[\left\{ \int_0^{1/2} \frac{u^{\gamma+1}}{[u\varpi_2 + (1-u)\varpi_1]^2} du + \int_{1/2}^1 \frac{u^\gamma - u^{\gamma+1}}{[u\varpi_2 + (1-u)\varpi_1]^2} du \right\} |\xi'(\varpi_1)| \right.$$

$$\left. + \left\{ \int_0^{1/2} \frac{u(1-u)^\delta}{[u\varpi_2 + (1-u)\varpi_1]^2} du + \int_{1/2}^1 \frac{(1-u)^{\delta+1}}{[u\varpi_2 + (1-u)\varpi_1]^2} du \right\} |\xi'(\varpi_2)| \right].$$

9. *If we replace $p = -1$, $\alpha = \eta = r$ and $\gamma = \delta = s$, then we acquire the following Hermite-Hadamard type inequality for harmonically $(r, s)-$convex functions:*

$$\left| \int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^2} d\zeta - M_{-1} \xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right) \right| \leq (\varpi_1\varpi_2 M_{-1})^2 \times$$

$$\left[\left\{ \int_0^{1/2} \frac{u^{rs+1}}{[u\varpi_2 + (1-u)\varpi_1]^2} du + \int_{1/2}^1 \frac{u^{rs} - u^{rs+1}}{[u\varpi_2 + (1-u)\varpi_1]^2} du \right\} |\xi'(\varpi_1)| \right.$$

$$\left. + \left\{ \int_0^{1/2} \frac{u(1-u^r)^s}{[u\varpi_2 + (1-u)\varpi_1]^2} du + \int_{1/2}^1 \frac{(1-u)(1-u^r)^s}{[u\varpi_2 + (1-u)\varpi_1]^2} du \right\} |\xi'(\varpi_2)| \right].$$

10. *If we replace $p = -1$, $\alpha = \eta = r$ and $\gamma = \delta = 1$, then we acquire the following Hermite-Hadamard type inequality for harmonically s-convex*

functions of the 1^{st} *kind:*

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^2} d\zeta - M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right| \leq (\varpi_1\varpi_2 M_{-1})^2 \times$$

$$\left[\left\{\int_0^{1/2} \frac{u^{r+1}}{[u\varpi_2+(1-u)\varpi_1]^2} du + \int_{1/2}^1 \frac{u^r - u^{r+1}}{[u\varpi_2+(1-u)\varpi_1]^2} du\right\} |\xi'(\varpi_1)|\right.$$

$$\left. + \left\{\int_0^{1/2} \frac{u-u^{r+1}}{[u\varpi_2+(1-u)\varpi_1]^2} du + \int_{1/2}^1 \frac{1-u-u^r+u^{r+1}}{[u\varpi_2+(1-u)\varpi_1]^2} du\right\} |\xi'(\varpi_2)|\right].$$

11. *If we replace* $p=-1$, $\alpha=\eta=1$ *and* $\gamma=\delta=s$, *then we acquire the following Hermite-Hadamard type inequality for harmonically* s*-convex functions of the* 2^{nd} *kind:*

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^2} d\zeta - M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right| \leq (\varpi_1\varpi_2 M_{-1})^2 \times$$

$$\left[\left\{\int_0^{1/2} \frac{u^{s+1}}{[u\varpi_2+(1-u)\varpi_1]^2} du + \int_{1/2}^1 \frac{u^s - u^{s+1}}{[u\varpi_2+(1-u)\varpi_1]^2} du\right\} |\xi'(\varpi_1)|\right.$$

$$\left. + \left\{\int_0^{1/2} \frac{u(1-u)^s}{[u\varpi_2+(1-u)\varpi_1]^2} du + \int_{1/2}^1 \frac{(1-u)^{s+1}}{[u\varpi_2+(1-u)\varpi_1]^2} du\right\} |\xi'(\varpi_2)|\right].$$

12. *If we replace* $\alpha=\eta=0$ *with* $\delta\neq 0$, *then we acquire the following refinement of Hermite-Hadamard type inequality for quasi*$-p$ *convex functions:*

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p\xi\left(\left[\frac{{\varpi_1}^p+{\varpi_2}^p}{2}\right]^{\frac{1}{p}}\right)\right|$$

$$\leq \frac{|\xi'(\varpi_1)|}{p+1}\left[{\varpi_1}^{p+1} - 2\left(\frac{{\varpi_1}^p+{\varpi_2}^p}{2}\right)^{\frac{1}{p}+1} + {\varpi_2}^{p+1}\right].$$

13. *If we replace* $p=1$, $\alpha=\eta=0$ *with* $\delta\neq 0$, *then we acquire the following refinement of Hermite-Hadamard type inequality for quasi convex functions:*

$$\left|\int_{\varpi_1}^{\varpi_2} \xi(\zeta) d\zeta - M_1\xi\left(\frac{\varpi_1+\varpi_2}{2}\right)\right| \leq M_1^2 \frac{|\xi'(\varpi_1)|}{4}.$$

14. *If we replace $\gamma = \delta = 0$, then we acquire the following Hermite-Hadamard type inequality for $P - p$ convex functions:*

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p \xi\left(\left[\frac{\varpi_1{}^p + \varpi_2{}^p}{2}\right]^{\frac{1}{p}}\right)\right|$$
$$\leq \left[\varpi_1{}^{p+1} - 2\left(\frac{\varpi_1{}^p + \varpi_2{}^p}{2}\right)^{\frac{1}{p}+1} + \varpi_2{}^{p+1}\right]\left(\frac{|\xi'(\varpi_1)| + |\xi'(\varpi_2)|}{p+1}\right).$$

15. *If we replace $\gamma = \delta = 0$ and $p = 1$, then we acquire the following Hermite-Hadamard type inequality for P-convex functions:*

$$\left|\int_{\varpi_1}^{\varpi_2} \xi(\zeta) d\zeta - M_1 \xi\left(\frac{\varpi_1 + \varpi_2}{2}\right)\right| \leq M_1^2 \left[\frac{|\xi'(\varpi_1)| + |\xi'(\varpi_2)|}{4}\right].$$

THEOREM 2.10. *Assume that $\xi' \in L[\varpi_1, \varpi_2]$ with $\varpi_1, \varpi_2 \in I^\circ$ and $\varpi_2 > \varpi_1$. If $\xi : I \subset (0, \infty) \to \mathbb{R}$ is a differentiable function on I° s. t. $|\xi'|^q$ for $q \geq 1$ is a $(\alpha, \eta, \gamma, \delta) - p$-convex on I for $(\alpha, \eta, \gamma, \delta) \in [0, 1]^4$ and $p \in \mathbb{R} - \{0\}$, then below stated result holds:*

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p \xi\left(\left[\frac{\varpi_1{}^p + \varpi_2{}^p}{2}\right]^{\frac{1}{p}}\right)\right|$$
$$\leq M_p^2 \Big[(Z_3(p))^{1-\frac{1}{q}} \left\{(Z_4(p))|\xi'(\varpi_1)|^q + Z_5(p)|\xi'(\varpi_2)|^q\right\}^{\frac{1}{q}}$$
$$+ (Z_6(p))^{1-\frac{1}{q}} \left\{(Z_7(p))|\xi'(\varpi_1)|^q + Z_8(p)|\xi'(\varpi_2)|^q\right\}^{\frac{1}{q}}\Big],$$

where

$$Z_3(p) = \int_0^{1/2} \frac{\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger, \quad Z_4(p) = \int_0^{1/2} \frac{\dagger^{\alpha\gamma+1}}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger,$$
$$Z_5(p) = \int_0^{1/2} \frac{\dagger(1-\dagger^\eta)^\delta}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger, \quad Z_6(p) = \int_{1/2}^{1} \frac{(1-\dagger)}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger,$$
$$Z_7(p) = \int_{1/2}^{1} \frac{\dagger^{\alpha\gamma}(1-\dagger)}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger, \quad Z_8(p) = \int_{1/2}^{1} \frac{(1-\dagger)(1-\dagger^\eta)^\delta}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger.$$

Proof. By using Lemma 1.13 and the definition of absolute value, we have,

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p \xi\left(\left[\frac{\varpi_1{}^p + \varpi_2{}^p}{2}\right]^{\frac{1}{p}}\right)\right|$$

$$\leq M_p^2 \left[\int_0^{1/2} \frac{\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right| d\dagger\right.$$

$$\left. + \int_{1/2}^{1} \frac{1-\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right| d\dagger\right]. \tag{2.3}$$

Applying (1.4) to

$$\int_0^{1/2} \frac{\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right| d\dagger$$

and

$$\int_{1/2}^{1} \frac{1-\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right| d\dagger$$

implies

$$\int_0^{1/2} \frac{\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right| d\dagger$$

$$\leq \left(\int_0^{1/2} \frac{\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger\right)^{1-\frac{1}{q}}$$

$$\left(\int_0^{1/2} \frac{\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right|^q d\dagger\right)^{\frac{1}{q}}$$

and

$$\int_{1/2}^{1} \frac{1-\dagger}{[\dagger\varpi_1{}^p+(1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p+(1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right| d\dagger$$

$$\leq \left(\int_{1/2}^{1} \frac{1-\dagger}{[\dagger\varpi_1{}^p+(1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger\right)^{1-\frac{1}{q}}$$

$$\left(\int_{1/2}^{1} \frac{1-\dagger}{[\dagger\varpi_1{}^p+(1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p+(1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right|^q d\dagger\right)^{\frac{1}{q}}.$$

As we have $|\xi'|^q$ is a $(\alpha,\eta,\gamma,\delta)-p$-convex function, so we can take,

$$\left|\xi'\left([\dagger\varpi_1{}^p+(1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right|^q \leq \dagger^{\alpha\gamma}|\xi'(\varpi_1)|^q+(1-\dagger^{\eta})^{\delta}|\xi'(\varpi_2)|^q.$$

Utilizing the above two results, (2.3) becomes

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p\xi\left(\left[\frac{\varpi_1{}^p+\varpi_2{}^p}{2}\right]^{\frac{1}{p}}\right)\right|$$

$$\leq M_p^2\left[\left(\int_{0}^{1/2} \frac{\dagger}{[\dagger\varpi_1{}^p+(1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger\right)^{1-\frac{1}{q}} \times\right.$$

$$\left(|\xi'(\varpi_1)|^q\int_{0}^{1/2} \frac{\dagger^{\alpha\gamma+1}}{[\dagger\varpi_1{}^p+(1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger + |\xi'(\varpi_2)|^q\int_{0}^{1/2} \frac{\dagger(1-\dagger^{\eta})^{\delta}}{[\dagger\varpi_1{}^p+(1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger\right)^{\frac{1}{q}}$$

$$+\left(\int_{1/2}^{1} \frac{1-\dagger}{[\dagger\varpi_1{}^p+(1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger\right)^{1-\frac{1}{q}}$$

$$\left.\left(|\xi'(\varpi_1)|^q\int_{1/2}^{1} \frac{\dagger^{\alpha\gamma}-\dagger^{\alpha\gamma+1}}{[\dagger\varpi_1{}^p+(1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger + |\xi'(\varpi_2)|^q\int_{1/2}^{1} \frac{(1-\dagger)(1-\dagger^{\eta})^{\delta}}{[\dagger\varpi_1{}^p+(1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} d\dagger\right)^{\frac{1}{q}}\right]$$

which completes the proof. □

REMARK 2.11. By varying different values of $\alpha,\eta,\gamma,\delta$ and p in Theorem 2.10, one can captured the following well-known inequalities:

1. If we replace $\alpha=\eta=r$ and $\gamma=\delta=s$, then we acquire the result of Hermite-Hadamard type inequality for $(r,s)-p$ convex functions (see Theorem 5 of [2]).

2. If we replace $p=1$, $\alpha=\eta=r$ and $\gamma=\delta=s$, then we acquire the result of Hermite-Hadamard type inequality for $(r,s)-$convex functions (see third result of Corollary 2 of [2]).
3. If we replace $\alpha=\eta=r$ and $\gamma=\delta=1$, then we acquire the result of Hermite-Hadamard type inequality for $s-p$ convex functions of the 1^{st} kind (see first result of Corollary 2 of [2]).
4. If we replace $\alpha=\eta=r$ and $\gamma=\delta=p=1$, then we acquire the result of Hermite-Hadamard type inequality for s-convex functions of the 1^{st} kind (see fourth result of Corollary 2 of [2]).
5. If we replace $\alpha=\eta=1$ and $\gamma=\delta=s$, then we acquire the result of Hermite-Hadamard type inequality for $s-p$ convex functions of the 2^{nd} kind (see second result of Corollary 2 of [2]).
6. If we replace $\alpha=\eta=p=1$ and $\gamma=\delta=s$, then we acquire the result of Hermite-Hadamard type inequality for s-convex functions of the 2^{nd} kind (see fifth result of Corollary 2 of [2]).
7. If we replace $\alpha=\eta=\gamma=\delta=1$, then we acquire the result of Hermite-Hadamard type inequality for p-convex functions (see second result of Corollary 2 of [15]).
8. If we replace $p=\alpha=\eta=\gamma=\delta=1$, then we acquire the result of Hermite-Hadamard type inequality for ordinary convex functions (see first result of Corollary 2 of [15]).
9. If we replace $p=-1$ and $\alpha=\eta=\gamma=\delta=1$, then we acquire the result of Hermite-Hadamard type inequality for harmonically convex functions (see fifth result of Corollary 2 of [15]).

COROLLARY 2.12. *Under the assumptions of Theorem* 2.10, *one can achieve the following results:*

1. *If we replace $p=1$, then we acquire the following Hermite-Hadamard type inequality for $(\alpha,\eta,\gamma,\delta)-$convex functions:*

$$\left|\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta-M_1\xi\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|\leq M_1^2\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left[\left\{\frac{|\xi'(\varpi_1)|^q}{2^{\alpha\gamma+2}(\alpha\gamma+2)}\right.\right.$$
$$\left.+\frac{|\xi'(\varpi_2)|^q}{\eta}\beta_{1/2^\eta}\left(\frac{2}{\eta},\delta+1\right)\right\}^{\frac{1}{q}}+\left\{\frac{(2^{\alpha\gamma+2}-\alpha\gamma-3)|\xi'(\varpi_1)|^q}{2^{\alpha\gamma+2}(\alpha\gamma+1)(\alpha\gamma+2)}\right.$$
$$\left.\left.+\frac{|\xi'(\varpi_2)|^q}{\eta}\left(\beta_{1-1/2^\eta}\left(\delta+1,\frac{1}{\eta}\right)-\beta_{1-1/2^\eta}\left(\delta+1,\frac{2}{\eta}\right)\right)\right\}^{\frac{1}{q}}\right].$$

2. *If we replace $p=-1$, then we acquire the following Hermite-Hadamard type inequality for harmonically $(\alpha,\eta,\gamma,\delta)-$convex functions:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^2}d\zeta-M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right|$$

$$\leq(\varpi_1\varpi_2M_{-1})^2\left[\left(\int_0^{1/2}\frac{u}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{1-\frac{1}{q}}\times\right.$$

$$\left(|\xi'(\varpi_1)|^q\int_0^{1/2}\frac{u^{\alpha\gamma+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du+|\xi'(\varpi_2)|^q\int_0^{1/2}\frac{u(1-u^\eta)^\delta}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{\frac{1}{q}}$$

$$+\left(\int_{1/2}^{1}\frac{1-u}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{1-\frac{1}{q}}\times$$

$$\left.\left(|\xi'(\varpi_1)|^q\int_{1/2}^{1}\frac{u^{\alpha\gamma}-u^{\alpha\gamma+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du+|\xi'(\varpi_2)|^q\int_{1/2}^{1}\frac{(1-u)(1-u^\eta)^\delta}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{\frac{1}{q}}\right].$$

3. *If we replace $\gamma=\delta=1$, then we acquire the following Hermite-Hadamard type inequality for $(\alpha,\eta)-p$ convex functions of the 1^{st} kind:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^{1-p}}d\zeta-M_p\xi\left(\left[\frac{\varpi_1{}^p+\varpi_2{}^p}{2}\right]^{\frac{1}{p}}\right)\right|$$

$$\leq M_p^2\left[\left(\int_0^{1/2}\frac{u}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(|\xi'(\varpi_1)|^q\int_0^{1/2}\frac{u^{\alpha+1}}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du+|\xi'(\varpi_2)|^q\int_0^{1/2}\frac{u^\eta-u^{\eta+1}}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du\right)^{\frac{1}{q}}$$

$$+\left(\int_{1/2}^{1}\frac{1-u}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du\right)^{1-\frac{1}{q}}\times$$

$$\left.\left(|\xi'(\varpi_1)|^q\int_{1/2}^{1}\frac{u^\alpha-u^{\alpha+1}}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du+|\xi'(\varpi_2)|^q\int_{1/2}^{1}\frac{1-u-u^\eta+u^{\eta+1}}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}du\right)^{\frac{1}{q}}\right].$$

4. *If we replace $\gamma = \delta = p = 1$, then we acquire the following Hermite-Hadamard type inequality for $(\alpha, \eta)-$convex functions of the 1^{st} kind:*

$$\left|\int_{\varpi_1}^{\varpi_2} \xi(\zeta)d\zeta - M_1 \xi\left(\frac{\varpi_1+\varpi_2}{2}\right)\right| \le M_1^2 \left(\frac{1}{8}\right)^{1-\frac{1}{q}} \times$$
$$\left[\left(\frac{|\xi'(\varpi_1)|^q}{2^{\alpha+2}(\alpha+2)} + \left\{\frac{1}{8} - \frac{1}{2^{\eta+2}(\eta+2)}\right\}|\xi'(\varpi_2)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(2^{\alpha+2}-\alpha-3)}{2^{\alpha+2}(\alpha+1)(\alpha+2)}|\xi'(\varpi_1)|^q + \left\{\frac{1}{8} - \frac{(2^{\eta+2}-\eta-3)}{2^{\eta+2}(\eta+1)(\eta+2)}\right\}|\xi'(\varpi_2)|^q\right)^{\frac{1}{q}}\right].$$

5. *If we replace $\gamma = \delta = 1$ and $p = -1$, then we acquire the following Hermite-Hadamard type inequality for harmonically $(\alpha, \eta)-$convex functions of the 1^{st} kind:*

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{y^2} d\zeta - M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right|$$
$$\le (\varpi_1\varpi_2 M_{-1})^2 \left[\left(\int_0^{1/2} \frac{u}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(|\xi'(\varpi_1)|^q \int_0^{1/2} \frac{u^{\alpha+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du + |\xi'(\varpi_2)|^q \int_0^{1/2} \frac{u-u^{\eta+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{\frac{1}{q}}$$
$$+\left(\int_{1/2}^{1} \frac{1-u}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{1-\frac{1}{q}} \times$$
$$\left.\left(|\xi'(\varpi_1)|^q \int_{1/2}^{1} \frac{u^{\alpha}-u^{\alpha+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du + |\xi'(\varpi_2)|^q \int_{1/2}^{1} \frac{1-u-u^{\eta}+u^{\eta+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{\frac{1}{q}}\right].$$

6. *If we replace $\alpha = \eta = 1$, then we acquire the following Hermite-Hadamard type inequality for $(\alpha, \eta) - p$ convex functions of the 2^{nd}*

kind:

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p \xi\left(\left[\frac{\varpi_1{}^p + \varpi_2{}^p}{2}\right]^{\frac{1}{p}}\right)\right|$$

$$\leq M_p^2 \left[\left(\int_0^{1/2} \frac{u}{\left[u\varpi_1{}^p + (1-u)\varpi_2{}^p\right]^{1-\frac{1}{p}}} du\right)^{1-\frac{1}{q}}\right.$$

$$\times \left(|\xi'(\varpi_1)|^q \int_0^{1/2} \frac{u^{\gamma+1}}{\left[u\varpi_1{}^p + (1-u)\varpi_2{}^p\right]^{1-\frac{1}{p}}} du + |\xi'(\varpi_2)|^q \int_0^{1/2} \frac{u(1-u)^{\delta}}{\left[u\varpi_1{}^p + (1-u)\varpi_2{}^p\right]^{1-\frac{1}{p}}} du\right)^{\frac{1}{q}}$$

$$+ \left(\int_{1/2}^{1} \frac{1-u}{\left[u\varpi_1{}^p + (1-u)\varpi_2{}^p\right]^{1-\frac{1}{p}}} du\right)^{1-\frac{1}{q}} \times$$

$$\left.\left(|\xi'(\varpi_1)|^q \int_{1/2}^{1} \frac{u^{\gamma} - u^{\gamma+1}}{\left[u\varpi_1{}^p + (1-u)\varpi_2{}^p\right]^{1-\frac{1}{p}}} du + |\xi'(\varpi_2)|^q \int_{1/2}^{1} \frac{(1-u)^{\delta+1}}{\left[u\varpi_1{}^p + (1-u)\varpi_2{}^p\right]^{1-\frac{1}{p}}} du\right)^{\frac{1}{q}}\right].$$

7. *If we replace $\alpha = \eta = p = 1$, then we acquire the following Hermite-Hadamard type inequality for $(\alpha, \eta)-$convex functions of the 2^{nd} kind:*

$$\left|\int_{\varpi_1}^{\varpi_2} \xi(\zeta) d\zeta - M_1 \xi\left(\frac{\varpi_1 + \varpi_2}{2}\right)\right|$$

$$\leq M_1^2 \left(\frac{1}{8}\right)^{1-\frac{1}{q}} \left[\left(\frac{|\xi'(\varpi_1)|^q}{2^{\gamma+2}(\gamma+2)} + |\xi'(\varpi_2)|^q \frac{2^{\delta+2} - \delta - 3}{2^{\delta+2}(\delta+1)(\delta+2)}\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\frac{(2^{\gamma+2} - \gamma - 3)|\xi'(\varpi_1)|^q}{2^{\gamma+2}(\gamma+1)(\gamma+2)} + \frac{|\xi'(\varpi_2)|^q}{2^{\delta+2}(\delta+2)}\right)^{\frac{1}{q}}\right].$$

8. *If we replace $\alpha = \eta = 1$ and $p = -1$, then we acquire the following Hermite-Hadamard type inequality for harmonically $(\alpha, \eta)-$convex functions of the 2^{nd} kind:*

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^2} d\zeta - M_{-1} \xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right)\right|$$

$$\leq (\varpi_1\varpi_2 M_{-1})^2 \left[\left(\int_0^{1/2} \frac{u}{[u\varpi_2 + (1-u)\varpi_1]^2} du \right)^{1-\frac{1}{q}} \times \right.$$

$$\left(|\xi'(\varpi_1)|^q \int_0^{1/2} \frac{u^{\gamma+1}}{[u\varpi_2 + (1-u)\varpi_1]^2} du + |\xi'(\varpi_2)|^q \int_0^{1/2} \frac{u(1-u)^\delta}{[u\varpi_2 + (1-u)\varpi_1]^2} du \right)^{\frac{1}{q}}$$

$$+ \left(\int_{1/2}^{1} \frac{1-u}{[u\varpi_2 + (1-u)\varpi_1]^2} du \right)^{1-\frac{1}{q}} \times$$

$$\left. \left(|\xi'(\varpi_1)|^q \int_{1/2}^{1} \frac{u^\gamma - u^{\gamma+1}}{[u\varpi_2 + (1-u)\varpi_1]^2} du + |\xi'(\varpi_2)|^q \int_{1/2}^{1} \frac{(1-u)^{\delta+1}}{[u\varpi_2 + (1-u)\varpi_1]^2} du \right)^{\frac{1}{q}} \right].$$

9. *If we replace $p = -1$, $\alpha = \eta = r$ and $\gamma = \delta = s$, then we acquire the following Hermite-Hadamard type inequality for harmonically $(r, s)-$convex functions:*

$$\left| \int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^2} d\zeta - M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right) \right|$$

$$\leq (\varpi_1\varpi_2 M_{-1})^2 \left[\left(\int_0^{1/2} \frac{u}{[u\varpi_2 + (1-u)\varpi_1]^2} du \right)^{1-\frac{1}{q}} \times \right.$$

$$\left(|\xi'(\varpi_1)|^q \int_0^{1/2} \frac{u^{rs+1}}{[u\varpi_2 + (1-u)\varpi_1]^2} du + |\xi'(\varpi_2)|^q \int_0^{1/2} \frac{u(1-u^r)^s}{[u\varpi_2 + (1-u)\varpi_1]^2} du \right)^{\frac{1}{q}}$$

$$+ \left(\int_{1/2}^{1} \frac{1-u}{[u\varpi_2 + (1-u)\varpi_1]^2} du \right)^{1-\frac{1}{q}} \times$$

$$\left. \left(|\xi'(\varpi_1)|^q \int_{1/2}^{1} \frac{u^{rs} - t^{rs+1}}{[u\varpi_2 + (1-u)\varpi_1]^2} du + |\xi'(\varpi_2)|^q \int_{1/2}^{1} \frac{(1-u)(1-u^r)^s}{[u\varpi_2 + (1-u)\varpi_1]^2} du \right)^{\frac{1}{q}} \right].$$

10. *If we replace $p = -1$, $\alpha = \eta = r$ and $\gamma = \delta = 1$, then we acquire the following Hermite-Hadamard type inequality for harmonically s-convex*

functions of the 1^{st} *kind:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^2}d\zeta - M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right|$$

$$\leq(\varpi_1\varpi_2M_{-1})^2\left[\left(\int_0^{1/2}\frac{u}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(|\xi'(\varpi_1)|^q\int_0^{1/2}\frac{u^{r+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du+|\xi'(\varpi_2)|^q\int_0^{1/2}\frac{u-u^{r+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{\frac{1}{q}}$$

$$+\left(\int_{1/2}^{1}\frac{1-u}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{1-\frac{1}{q}}\times$$

$$\left.\left(|\xi'(\varpi_1)|^q\int_{1/2}^{1}\frac{u^r-u^{r+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du+|\xi'(\varpi_2)|^q\int_{1/2}^{1}\frac{1-u-u^r+u^{r+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{\frac{1}{q}}\right].$$

11. *If we replace* $p=-1$, $\alpha=\eta=1$ *and* $\gamma=\delta=s$, *then we acquire the following Hermite-Hadamard type inequality for harmonically* s*-convex functions of the* 2^{nd} *kind:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^2}d\zeta - M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right|$$

$$\leq(\varpi_1\varpi_2M_{-1})^2\left[\left(\int_0^{1/2}\frac{u}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(|\xi'(\varpi_1)|^q\int_0^{1/2}\frac{u^{s+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du+|\xi'(\varpi_2)|^q\int_0^{1/2}\frac{u(1-u)^s}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{\frac{1}{q}}$$

$$+\left(\int_{1/2}^{1}\frac{1-u}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{1-\frac{1}{q}}\times$$

$$\left.\left(|\xi'(\varpi_1)|^q\int_{1/2}^{1}\frac{u^s-u^{s+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du+|\xi'(\varpi_2)|^q\int_{1/2}^{1}\frac{(1-u)^{s+1}}{[u\varpi_2+(1-u)\varpi_1]^2}du\right)^{\frac{1}{q}}\right].$$

12. *If we replace $\alpha = \eta = 0$ with $\delta \neq 0$, then we acquire the following refinement of Hermite-Hadamard type inequality for quasi−p convex functions:*

$$\left| \int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p \xi \left(\left[\frac{\varpi_1{}^p + \varpi_2{}^p}{2} \right]^{\frac{1}{p}} \right) \right| \leq M_p^2 |\xi'(\varpi_1)| \times$$

$$\left[\int_0^{1/2} \frac{u}{[u\varpi_1{}^p + (1-u)\varpi_2{}^p]^{1-\frac{1}{p}}} du + \int_{1/2}^1 \frac{1-u}{[u\varpi_1{}^p + (1-u)\varpi_2{}^p]^{1-\frac{1}{p}}} du \right].$$

13. *If we replace $p = 1$, $\alpha = \eta = 0$ with $\delta \neq 0$, then we acquire the following refinement of Hermite-Hadamard type inequality for quasi convex functions:*

$$\left| \int_{\varpi_1}^{\varpi_2} \xi(\zeta) d\zeta - M_1 \xi \left(\frac{\varpi_1 + \varpi_2}{2} \right) \right| \leq M_1^2 \frac{|\xi'(\varpi_1)|}{4}.$$

14. *If we replace $p = -1$, $\alpha = \eta = 0$ with $\delta \neq 0$, then we acquire the following refinement of Hermite-Hadamard type inequality for harmonically quasi convex functions:*

$$\left| \int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^2} d\zeta - M_{-1} \xi \left(\frac{2\varpi_1 \varpi_2}{\varpi_1 + \varpi_2} \right) \right| \leq (\varpi_1 \varpi_2 M_{-1})^2 |\xi'(\varpi_1)| \times$$

$$\left[\int_0^{1/2} \frac{u}{[u\varpi_2 + (1-u)\varpi_1]^2} du + \int_{1/2}^1 \frac{1-u}{[u\varpi_2 + (1-u)\varpi_1]^2} du \right].$$

15. *If we replace $\gamma = \delta = 0$, then we acquire the following Hermite-Hadamard type inequality for $P - p$ convex functions:*

$$\left| \int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p \xi \left(\left[\frac{\varpi_1{}^p + \varpi_2{}^p}{2} \right]^{\frac{1}{p}} \right) \right| \leq M_p^2 \left(|\xi'(\varpi_1)|^q + |\xi'(\varpi_2)|^q \right)^{\frac{1}{q}}$$

$$\times \left[\int_0^{1/2} \frac{u}{[u\varpi_1{}^p + (1-u)\varpi_2{}^p]^{1-\frac{1}{p}}} du + \int_{1/2}^1 \frac{1-u}{[u\varpi_1{}^p + (1-u)\varpi_2{}^p]^{1-\frac{1}{p}}} du \right].$$

16. *If we replace $\gamma = \delta = 0$ and $p = 1$, then we acquire the following Hermite-Hadamard type inequality for P-convex functions:*

$$\left| \int_{\varpi_1}^{\varpi_2} \xi(\zeta) d\zeta - M_1 \xi \left(\frac{\varpi_1 + \varpi_2}{2} \right) \right| \leq \left(\frac{M_1}{2} \right)^2 \left(|\xi'(\varpi_1)|^q + |\xi'(\varpi_2)|^q \right)^{\frac{1}{q}}.$$

17. If we replace $\gamma = \delta = 0$ and $p = -1$, then we acquire the following Hermite-Hadamard type inequality for harmonically P-convex functions:

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^2} d\zeta - M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right| \leq (\varpi_1\varpi_2 M_{-1})^2 \left(|\xi'(\varpi_1)|^q + |\xi'(\varpi_2)|^q\right)^{\frac{1}{q}}$$

$$\times \left[\int_0^{1/2} \frac{u}{[bu+(1-u)\varpi_1]^2} du + \int_{1/2}^{1} \frac{1-u}{[bu+(1-u)\varpi_1]^2} du\right].$$

THEOREM 2.13. *Assume that $\xi' \in L[\varpi_1, \varpi_2]$, with $\varpi_1, \varpi_2 \in I^\circ$ and $\varpi_2 > \varpi_1$. If $\xi : I \subset (0, \infty) \to \mathbb{R}$ is a differentiable function on I° s. t. $|\xi'|^{q_2}$ is a $(\alpha, \eta, \gamma, \delta) - p$-convex on I for $(\alpha, \eta, \gamma, \delta) \in [0, 1]^4$ and $p \in \mathbb{R} - \{0\}$ with $\eta \neq 0$, then below stated result holds for $\frac{1}{q_1} + \frac{1}{q_2} = 1$:*

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p\xi\left(\left[\frac{{\varpi_1}^p + {\varpi_2}^p}{2}\right]^{\frac{1}{p}}\right)\right|$$

$$\leq M_p^2 \left\{ (Z_9(p)) \left[\frac{|\xi'(\varpi_1)|^{q_2}}{2^{\alpha\gamma+1}(\alpha\gamma+1)} + \frac{\beta_{1/2^\eta}\left(\frac{1}{\eta}, \delta+1\right)|\xi'(\varpi_2)|^{q_2}}{\eta}\right]^{\frac{1}{q_2}}\right.$$

$$\left. + (Z_{10}(p)) \left[\frac{(2^{\alpha\gamma+1}-1)|\xi'(\varpi_1)|^{q_2}}{2^{\alpha\gamma+1}(\alpha\gamma+1)} + \frac{\beta_{1-1/2^\eta}\left(\delta+1, \frac{1}{\eta}\right)|\xi'(\varpi_2)|^{q_2}}{\eta}\right]^{\frac{1}{q_2}}\right\}.$$

where

$$Z_9(p) = \left(\int_0^{1/2} \left(\frac{\dagger}{[\dagger{\varpi_1}^p + (1-\dagger){\varpi_2}^p]^{1-\frac{1}{p}}}\right)^{q_1} d\dagger\right)^{\frac{1}{q_1}}$$

and

$$Z_{10}(p) = \left(\int_{1/2}^{1} \left(\frac{1-\dagger}{[\dagger{\varpi_1}^p + (1-\dagger){\varpi_2}^p]^{1-\frac{1}{p}}}\right)^{q_1} d\dagger\right)^{\frac{1}{q_1}}.$$

Proof. By using Lemma 1.13 and the definition of absolute value, we attain,

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p \xi\left(\left[\frac{\varpi_1{}^p + \varpi_2{}^p}{2}\right]^{\frac{1}{p}}\right)\right|$$

$$\leq M_p^2 \left[\int_0^{1/2} \frac{\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right| d\dagger\right.$$

$$\left. + \int_{1/2}^{1} \frac{1-\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right| d\dagger\right]. \tag{2.4}$$

Applying (1.3) to

$$\int_0^{1/2} \frac{\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right| d\dagger$$

and

$$\int_{1/2}^{1} \frac{1-\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right| d\dagger$$

implies

$$\int_0^{1/2} \frac{\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right| d\dagger$$

$$\leq \left(\int_0^{1/2} \left(\frac{\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}}\right)^{q_1} d\dagger\right)^{\frac{1}{q_1}} \left(\int_0^{1/2} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right|^{q_2} d\dagger\right)^{\frac{1}{q_2}}$$

and

$$\int_{1/2}^{1} \frac{1-\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right| d\dagger$$

$$\leq \left(\int_{1/2}^{1} \left(\frac{1-\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}}\right)^{q_1} d\dagger\right)^{\frac{1}{q_1}} \left(\int_{1/2}^{1} \left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right|^{q_2} d\dagger\right)^{\frac{1}{q_2}}.$$

As we have $|\xi'|^q$ is a $(\alpha, \eta, \gamma, \delta) - p$-convex function, so we can take,

$$\left|\xi'\left([\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{\frac{1}{p}}\right)\right|^{q_2} \leq \dagger^{\alpha\gamma}|\xi'(\varpi_1)|^{q_2} + (1-\dagger^{\eta})^{\delta}|\xi'(\varpi_2)|^{q_2}.$$

Utilizing the above three results, (2.4) becomes

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p \xi\left(\left[\frac{\varpi_1{}^p + \varpi_2{}^p}{2}\right]^{\frac{1}{p}}\right)\right|$$

$$\leq M_p^2 \left[\left(\int_0^{1/2} \left(\frac{\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}}\right)^{q_1} d\dagger\right)^{\frac{1}{q_1}} \times\right.$$

$$\left(\int_0^{1/2} \left(\dagger^{\alpha\gamma}|\xi'(\varpi_1)|^{q_2} + (1-\dagger^{\eta})^{\delta}|\xi'(\varpi_2)|^{q_2}\right) d\dagger\right)^{\frac{1}{q_2}}$$

$$+ \left(\int_{1/2}^{1} \left(\frac{1-\dagger}{[\dagger\varpi_1{}^p + (1-\dagger)\varpi_2{}^p]^{1-\frac{1}{p}}}\right)^{q_1} d\dagger\right)^{\frac{1}{q_1}} \times$$

$$\left.\left(\int_{1/2}^{1} \left(\dagger^{\alpha\gamma}|\xi'(\varpi_1)|^{q_2} + (1-\dagger^{\eta})^{\delta}|\xi'(\varpi_2)|^{q_2}\right) d\dagger\right)^{\frac{1}{q_2}}\right].$$

After using the following facts, the result of Theorem 2.13 is accomplished:

$$\int_0^{1/2} \dagger^{\alpha\gamma} d\dagger = \frac{1}{2^{\alpha\gamma+1}(\alpha\gamma+1)},$$

$$\int_0^{1/2} (1-\dagger^{\eta})^{\delta} d\dagger = \frac{\beta_{1/2^{\eta}}\left(\frac{1}{\eta}, \delta+1\right)}{\eta},$$

$$\int_{1/2}^{1} \dagger^{\alpha\gamma} d\dagger = \frac{2^{\alpha\gamma+1}-1}{2^{\alpha\gamma+1}(\alpha\gamma+1)}$$

and

$$\int_{1/2}^{1} (1-\dagger^{\eta})^{\delta} d\dagger = \frac{\beta_{1-1/2^{\eta}}\left(\delta+1, \frac{1}{\eta}\right)}{\eta}.$$

□

Remark 2.14. By varying different values of $\alpha, \eta, \gamma, \delta$ and p in Theorem 2.13, one can captured the following well-known inequalities:

1. If we replace $\alpha = \eta = r$ and $\gamma = \delta = s$, then we acquire the result of Hermite-Hadamard type inequality for $(r, s) - p$ convex functions (see Theorem 6 of [2]).
2. If we replace $p = 1$, $\alpha = \eta = r$ and $\gamma = \delta = s$, then we acquire the result of Hermite-Hadamard type inequality for $(r, s)-$convex functions (see third result of Corollary 3 of [2]).
3. If we replace $\alpha = \eta = r$ and $\gamma = \delta = 1$, then we acquire the result of Hermite-Hadamard type inequality for $s - p$ convex functions of the 1^{st} kind (see first result of Corollary 3 of [2]).
4. If we replace $\alpha = \eta = r$ and $\gamma = \delta = p = 1$, then we acquire the result of Hermite-Hadamard type inequality for s-convex functions of the 1^{st} kind (see fourth result of Corollary 3 of [2]).
5. If we replace $\alpha = \eta = 1$ and $\gamma = \delta = s$, then we acquire the result of Hermite-Hadamard type inequality for $s - p$ convex functions of the 2^{nd} kind (see second result of Corollary 3 of [2]).
6. If we replace $\alpha = \eta = p = 1$ and $\gamma = \delta = s$, then we acquire the result of Hermite-Hadamard type inequality for s-convex functions of the 2^{nd} kind (see fifth result of Corollary 3 of [2]).
7. If we replace $\alpha = \eta = \gamma = \delta = 1$, then we acquire Hermite-Hadamard type inequality for p-convex functions (see first result of Corollary 3 of [15]).
8. If we replace $p = \alpha = \eta = \gamma = \delta = 1$, then we acquire Hermite-Hadamard type inequality for ordinary convex functions (see Theorem 2.3 of [16]).
9. If we replace $p = -1$ and $\alpha = \eta = \gamma = \delta = 1$, then we acquire Hermite-Hadamard type inequality for harmonically convex functions (see fourth result of Corollary 3 of [15]).

Corollary 2.15. *Under the assumptions of Theorem* 2.13, *one can achieve the following results:*

1. *If we replace* $p = 1$, *then we acquire the following Hermite-Hadamard type inequality for* $(\alpha, \eta, \gamma, \delta)-$*convex functions:*

$$\left| \int_{\varpi_1}^{\varpi_2} \xi(\zeta) d\zeta - M_1 \xi\left(\frac{\varpi_1 + \varpi_2}{2}\right) \right| \leq M_1^2 \left(\frac{1}{2^{q_1+1}(q_1+1)}\right)^{\frac{1}{q_1}} \times$$

$$\left[\left(\frac{|\xi'(\varpi_1)|^{q_2}}{2^{\alpha\gamma+1}(\alpha\gamma+1)}+\frac{\beta_{1/2^\eta}\left(\frac{1}{\eta},\delta+1\right)|\xi'(\varpi_2)|^{q_2}}{\eta}\right)^{\frac{1}{q_2}}\right.$$

$$\left.+\left(\frac{(2^{\alpha\gamma+1}-1)|\xi'(\varpi_1)|^{q_2}}{2^{\alpha\gamma+1}(\alpha\gamma+1)}+\frac{\beta_{1-1/2^\eta}\left(\delta+1,\frac{1}{\eta}\right)|\xi'(\varpi_2)|^{q_2}}{\eta}\right)^{\frac{1}{q_2}}\right].$$

2. *If we replace $p=-1$, then we acquire the following Hermite-Hadamard type inequality for harmonically $(\alpha,\eta,\gamma,\delta)-$convex functions:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^2}d\zeta-M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right|$$

$$\leq(\varpi_1\varpi_2M_{-1})^2\left[\left(\int_0^{1/2}\left(\frac{u}{[u\varpi_2+(1-u)\varpi_1]^2}\right)^{q_1}du\right)^{\frac{1}{q_1}}\times\right.$$

$$\left(\frac{|\xi'(\varpi_1)|^{q_2}}{2^{\alpha\gamma+1}(\alpha\gamma+1)}+\frac{\beta_{1/2^\eta}\left(\frac{1}{\eta},\delta+1\right)|\xi'(\varpi_2)|^{q_2}}{\eta}\right)^{\frac{1}{q_2}}$$

$$+\left(\int_{1/2}^{1}\left(\frac{1-u}{[u\varpi_2+(1-u)\varpi_1]^2}\right)^{q_1}du\right)^{\frac{1}{q_1}}\times$$

$$\left.\left(\frac{(2^{\alpha\gamma+1}-1)|\xi'(\varpi_1)|^{q_2}}{2^{\alpha\gamma+1}(\alpha\gamma+1)}+\frac{\beta_{1-1/2^\eta}\left(\delta+1,\frac{1}{\eta}\right)|\xi'(\varpi_2)|^{q_2}}{\eta}\right)^{\frac{1}{q_2}}\right].$$

3. *If we replace $\gamma=\delta=1$, then we acquire the following Hermite-Hadamard type inequality for $(\alpha,\eta)-p$ convex functions of the 1^{st} kind:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^{1-p}}d\zeta-M_p\xi\left(\left[\frac{{\varpi_1}^p+{\varpi_2}^p}{2}\right]^{\frac{1}{p}}\right)\right|$$

$$\leq M_p^2\left[\left(\int_0^{1/2}\left(\frac{u}{[u{\varpi_1}^p+(1-u){\varpi_2}^p]^{1-\frac{1}{p}}}\right)^{q_1}du\right)^{\frac{1}{q_1}}\times\right.$$

$$\left(\frac{|\xi'(\varpi_1)|^{q_2}}{2^{\alpha+1}(\alpha+1)}+\left(\frac{1}{2}-\frac{1}{2^{\eta+1}(\eta+1)}\right)|\xi'(\varpi_2)|^{q_2}\right)^{\frac{1}{q_2}}+$$
$$\left(\int_{1/2}^{1}\left(\frac{1-u}{[u\varpi_1{}^p+(1-u)\varpi_2{}^p]^{1-\frac{1}{p}}}\right)^{q_1}du\right)^{\frac{1}{q_1}}\times$$
$$\left(\frac{(2^{\alpha+1}-1)|\xi'(\varpi_1)|^{q_2}}{2^{\alpha+1}(\alpha+1)}+\left(\frac{1}{2}-\frac{(2^{\eta+1}-1)}{2^{\eta+1}(\eta+1)}\right)|\xi'(\varpi_2)|^{q_2}\right)^{\frac{1}{q_2}}\Bigg].$$

4. *If we replace $\gamma=\delta=p=1$, then we acquire the following Hermite-Hadamard type inequality for $(\alpha,\eta)-$convex functions of the 1^{st} kind:*

$$\left|\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta-M_1\xi\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|\le M_1^2\left(\frac{1}{2^{q_1+1}(q_1+1)}\right)^{\frac{1}{q_1}}\times$$
$$\left[\left(\frac{|\xi'(\varpi_1)|^{q_2}}{2^{\alpha+1}(\alpha+1)}+\left(\frac{1}{2}-\frac{1}{2^{\eta+1}(\eta+1)}\right)|\xi'(\varpi_2)|^{q_2}\right)^{\frac{1}{q_2}}+\right.$$
$$\left.\left(\frac{(2^{\alpha+1}-1)|\xi'(\varpi_1)|^{q_2}}{2^{\alpha+1}(\alpha+1)}+\left(\frac{1}{2}-\frac{(2^{\eta+1}-1)}{2^{\eta+1}(\eta+1)}\right)|\xi'(\varpi_2)|^{q_2}\right)^{\frac{1}{q_2}}\right].$$

5. *If we replace $\gamma=\delta=1$ and $p=-1$, then we acquire the following Hermite-Hadamard type inequality for harmonically $(\alpha,\eta)-$convex functions of the 1^{st} kind:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{y^2}d\zeta-M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right|$$
$$\le(\varpi_1\varpi_2M_{-1})^2\left[\left(\int_0^{1/2}\left(\frac{u}{[u\varpi_2+(1-u)\varpi_1]^2}\right)^{q_1}du\right)^{\frac{1}{q_1}}\times\right.$$
$$\left(\frac{|\xi'(\varpi_1)|^{q_2}}{2^{\alpha+1}(\alpha+1)}+\left(\frac{1}{2}-\frac{1}{2^{\eta+1}(\eta+1)}\right)|\xi'(\varpi_2)|^{q_2}\right)^{\frac{1}{q_2}}+$$
$$\left(\int_{1/2}^{1}\left(\frac{1-u}{[u\varpi_2+(1-u)\varpi_1]^2}\right)^{q_1}du\right)^{\frac{1}{q_1}}\times$$
$$\left.\left(\frac{(2^{\alpha+1}-1)|\xi'(\varpi_1)|^{q_2}}{2^{\alpha+1}(\alpha+1)}+\left(\frac{1}{2}-\frac{(2^{\eta+1}-1)}{2^{\eta+1}(\eta+1)}\right)|\xi'(\varpi_2)|^{q_2}\right)^{\frac{1}{q_2}}\right].$$

6. *If we replace* $\alpha = \eta = 1$, *then we acquire the following Hermite-Hadamard type inequality for* $(\alpha, \eta) - p$ *convex functions of the* 2^{nd} *kind:*

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p \xi\left(\left[\frac{{\varpi_1}^p + {\varpi_2}^p}{2}\right]^{\frac{1}{p}}\right)\right|$$
$$\leq M_p^2 \left[\left(\int_0^{1/2} \left(\frac{u}{[u{\varpi_1}^p + (1-u){\varpi_2}^p]^{1-\frac{1}{p}}}\right)^{q_1} du\right)^{\frac{1}{q_1}} \times\right.$$
$$\left(\frac{|\xi'(\varpi_1)|^{q_2}}{2^{\gamma+1}(\gamma+1)} + \frac{(2^{\delta+1}-1)|\xi'(\varpi_2)|^{q_2}}{2^{\delta+1}(\delta+1)}\right)^{\frac{1}{q_2}} +$$
$$\left(\int_{1/2}^{1} \left(\frac{1-u}{[u{\varpi_1}^p + (1-u){\varpi_2}^p]^{1-\frac{1}{p}}}\right)^{q_1} du\right)^{\frac{1}{q_1}} \times$$
$$\left.\left(\frac{(2^{\gamma+1}-1)|\xi'(\varpi_1)|^{q_2}}{2^{\gamma+1}(\gamma+1)} + \frac{|\xi'(\varpi_2)|^{q_2}}{2^{\delta+1}(\delta+1)}\right)^{\frac{1}{q_2}}\right].$$

7. *If we replace* $\alpha = \eta = p = 1$, *then we acquire the following Hermite-Hadamard type inequality for* $(\alpha, \eta)-$*convex functions of the* 2^{nd} *kind:*

$$\left|\int_{\varpi_1}^{\varpi_2} \xi(\zeta) d\zeta - M_1 \xi\left(\frac{\varpi_1 + \varpi_2}{2}\right)\right| \leq M_1^2 \left(\frac{1}{2^{q_1+1}(q_1+1)}\right)^{\frac{1}{q_1}} \times$$
$$\left[\left(\frac{|\xi'(\varpi_1)|^{q}}{2^{\gamma+1}(\gamma+1)} + \frac{(2^{\delta+1}-1)|\xi'(\varpi_2)|^{q}}{2^{\delta+1}(\delta+1)}\right)^{\frac{1}{q}} +\right.$$
$$\left.\left(\frac{(2^{\gamma+1}-1)|\xi'(\varpi_1)|^{q}}{2^{\gamma+1}(\gamma+1)} + \frac{|\xi'(\varpi_2)|^{q}}{2^{\delta+1}(\delta+1)}\right)^{\frac{1}{q}}\right].$$

8. *If we replace* $\alpha = \eta = 1$ *and* $p = -1$, *then we acquire the following Hermite-Hadamard type inequality for harmonically* $(\alpha, \eta)-$*convex functions of the* 2^{nd} *kind:*

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^2} d\zeta - M_{-1} \xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right)\right| \leq (\varpi_1\varpi_2 M_{-1})^2 \times$$

$$\left[\left(\int_0^{1/2}\left(\frac{u}{[u\varpi_2+(1-u)\varpi_1]^2}\right)^{q_1}du\right)^{\frac{1}{q_1}}\times\right.$$
$$\left(\frac{|\xi'(\varpi_1)|^{q_2}}{2^{\gamma+1}(\gamma+1)}+\frac{(2^{\delta+1}-1)|\xi'(\varpi_2)|^{q_2}}{2^{\delta+1}(\delta+1)}\right)^{\frac{1}{q_2}}+$$
$$\left(\int_{1/2}^{1}\left(\frac{1-u}{[u\varpi_2+(1-u)\varpi_1]^2}\right)^{q_1}du\right)^{\frac{1}{q_1}}\times$$
$$\left.\left(\frac{(2^{\gamma+1}-1)|\xi'(\varpi_1)|^{q_2}}{2^{\gamma+1}(\gamma+1)}+\frac{|\xi'(\varpi_2)|^{q_2}}{2^{\delta+1}(\delta+1)}\right)^{\frac{1}{q_2}}\right].$$

9. *If we replace $p=-1$, $\alpha=\eta=r$ and $\gamma=\delta=s$, then we acquire the following Hermite-Hadamard type inequality for harmonically $(r,s)-$convex functions:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^2}d\zeta-M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right|$$
$$\leq(\varpi_1\varpi_2M_{-1})^2\left[\left(\int_0^{1/2}\left(\frac{u}{[u\varpi_2+(1-u)\varpi_1]^2}\right)^{q_1}du\right)^{\frac{1}{q_1}}\times\right.$$
$$\left(\frac{|\xi'(\varpi_1)|^{q_2}}{2^{rs+1}(rs+1)}+\frac{\beta_{1/2^r}\left(\frac{1}{r},s+1\right)|\xi'(\varpi_2)|^{q_2}}{r}\right)^{\frac{1}{q_2}}$$
$$+\left(\int_{1/2}^{1}\left(\frac{1-u}{[u\varpi_2+(1-u)\varpi_1]^2}\right)^{q_1}du\right)^{\frac{1}{q_1}}\times$$
$$\left.\left(\frac{(2^{rs+1}-1)|\xi'(\varpi_1)|^{q_2}}{2^{rs+1}(rs+1)}+\frac{\beta_{1-1/2^r}\left(s+1,\frac{1}{r}\right)|\xi'(\varpi_2)|^{q_2}}{r}\right)^{\frac{1}{q_2}}\right].$$

10. *If we replace $p=-1$, $\alpha=\eta=r$ and $\gamma=\delta=1$, then we acquire the following Hermite-Hadamard type inequality for harmonically s-convex*

functions of the 1^{st} *kind:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{y^2}d\zeta - M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right|$$

$$\leq (\varpi_1\varpi_2 M_{-1})^2\left[\left(\int_0^{1/2}\left(\frac{u}{[u\varpi_2+(1-u)\varpi_1]^2}\right)^{q_1}du\right)^{\frac{1}{q_1}}\times\right.$$

$$\left(\frac{|\xi'(\varpi_1)|^{q_2}}{2^{r+1}(r+1)}+\left(\frac{1}{2}-\frac{1}{2^{r+1}(r+1)}\right)|\xi'(\varpi_2)|^{q_2}\right)^{\frac{1}{q_2}}+$$

$$\left(\int_{1/2}^{1}\left(\frac{1-u}{[u\varpi_2+(1-u)\varpi_1]^2}\right)^{q_1}du\right)^{\frac{1}{q_1}}\times$$

$$\left.\left(\frac{(2^{r+1}-1)|\xi'(\varpi_1)|^{q_2}}{2^{r+1}(r+1)}+\left(\frac{1}{2}-\frac{(2^{r+1}-1)}{2^{r+1}(r+1)}\right)|\xi'(\varpi_2)|^{q_2}\right)^{\frac{1}{q_2}}\right].$$

11. *If we replace* $p=-1$, $\alpha=\eta=1$ *and* $\gamma=\delta=s$, *then we acquire the following Hermite-Hadamard type inequality for harmonically* s*-convex functions of the* 2^{nd} *kind:*

$$\left|\int_{\varpi_1}^{\varpi_2}\frac{\xi(\zeta)}{\zeta^2}d\zeta - M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right| \leq (\varpi_1\varpi_2 M_{-1})^2\times$$

$$\left[\left(\int_0^{1/2}\left(\frac{u}{[u\varpi_2+(1-u)\varpi_1]^2}\right)^{q_1}du\right)^{\frac{1}{q_1}}\times\right.$$

$$\left(\frac{|\xi'(\varpi_1)|^{q_2}+(2^{s+1}-1)|\xi'(\varpi_2)|^{q_2}}{2^{s+1}(s+1)}\right)^{\frac{1}{q_2}}+$$

$$\left(\int_{1/2}^{1}\left(\frac{1-u}{[u\varpi_2+(1-u)\varpi_1]^2}\right)^{q_1}du\right)^{\frac{1}{q_1}}\times$$

$$\left.\left(\frac{(2^{s+1}-1)|\xi'(\varpi_1)|^{q_2}+|\xi'(\varpi_2)|^{q_2}}{2^{s+1}(s+1)}\right)^{\frac{1}{q_2}}\right].$$

12. *If we replace $\gamma = \delta = 0$, then we acquire the following Hermite-Hadamard type inequality for $P-p$ convex functions:*

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^{1-p}} d\zeta - M_p \xi\left(\left[\frac{{\varpi_1}^p + {\varpi_2}^p}{2}\right]^{\frac{1}{p}}\right)\right|$$
$$\leq M_p^2 \left(\frac{|\xi'(\varpi_1)|^{q_2} + |\xi'(\varpi_2)|^{q_2}}{2}\right)^{\frac{1}{q_2}} \times$$
$$\left[\left(\int_0^{1/2}\left(\frac{u}{[u{\varpi_1}^p + (1-u){\varpi_2}^p]^{1-\frac{1}{p}}}\right)^{q_1} du\right)^{\frac{1}{q_1}} + \right.$$
$$\left.\left(\int_{1/2}^{1}\left(\frac{1-u}{[u{\varpi_1}^p + (1-u){\varpi_2}^p]^{1-\frac{1}{p}}}\right)^{q_1} du\right)^{\frac{1}{q_1}}\right].$$

13. *If we replace $\gamma = \delta = 0$ and $p = 1$, then we acquire the following Hermite-Hadamard type inequality for P-convex functions:*

$$\left|\int_{\varpi_1}^{\varpi_2} \xi(\zeta) d\zeta - M_1 \xi\left(\frac{\varpi_1 + \varpi_2}{2}\right)\right|$$
$$\leq 2M_1^2 \left(\frac{1}{2^{q_1+1}(q_1+1)}\right)^{\frac{1}{q_1}} \left(\frac{|\xi'(\varpi_1)|^{q_2} + |\xi'(\varpi_2)|^{q_2}}{2}\right)^{\frac{1}{q_2}}$$

14. *If we replace $\gamma = \delta = 0$ and $p = -1$, then we acquire the following Hermite-Hadamard type inequality for harmonically P-convex functions:*

$$\left|\int_{\varpi_1}^{\varpi_2} \frac{\xi(\zeta)}{\zeta^2} d\zeta - M_{-1}\xi\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)\right|$$
$$\leq (\varpi_1\varpi_2 M_{-1})^2 \left(\frac{|\xi'(\varpi_1)|^{q_2}+|\xi'(\varpi_2)|^{q_2}}{2}\right)^{\frac{1}{q_2}} \times$$
$$\left[\left(\int_0^{1/2}\left(\frac{u}{[u\varpi_2+(1-u)\varpi_1]^2}\right)^{q_1} du\right)^{\frac{1}{q_1}}\right.$$
$$\left.+\left(\int_{1/2}^{1}\left(\frac{1-u}{[u\varpi_2+(1-u)\varpi_1]^2}\right)^{q_1} du\right)^{\frac{1}{q_1}}\right].$$

3. Application to Numerical Integration: Midpoint Formula

Let d be the division of the interval I such that $d : \varpi_1 = \zeta_0 < \zeta_1 < \cdots < \zeta_{k-1} < \zeta_k = \varpi_2$, ξ is integrable on $[\varpi_1, \varpi_2]$ and consider the Quadrature formula:

$$J = \int_{\varpi_1}^{\varpi_2} \xi(\zeta) d\zeta = M(\xi, d) + R(\xi, d),$$

where

$$M(\xi, d) = \sum_{k=0}^{n-1} \xi\left(\frac{\zeta_k+\zeta_{k+1}}{2}\right)(\zeta_{k+1}-\zeta_k)$$

is the midpoint formula and $R(\xi, d)$ denotes the associated approximation error of the interval I.

Now, we are going to drive some estimations for midpoint formula:

Theorem 3.1. *Assume that $\xi' \in L[\varpi_1, \varpi_2]$ with $\varpi_1, \varpi_2 \in I^\circ$ and $\varpi_2 > \varpi_1$. If $\xi : I \subset (0, \infty) \to \mathbb{R}$ is a differentiable function on I° s. t. $|\xi'|$ is $(\alpha, \eta, \gamma, \delta)-$convex on I for fixed $(\alpha, \eta, \gamma, \delta) \in [0, 1]^4$ on I, then below stated result holds with $\eta \neq 0$:*

$$|R(\xi, d)| \leq \sum_{k=0}^{n-1} (\zeta_{k+1}-\zeta_k)^2 \left[\frac{(2^{\alpha\gamma+1}-1)}{2^{\alpha\gamma+1}(\alpha\gamma+1)(\alpha\gamma+2)}|\xi'(\zeta_k)| + \right.$$

(3.1)
$$\left.\left(\beta_{1/2^\eta}\left(\frac{2}{\eta}, \delta+1\right) + \beta_{1-1/2^\eta}\left(\delta+1, \frac{1}{\eta}\right) - \beta_{1-1/2^\eta}\left(\delta+1, \frac{2}{\eta}\right)\right)\frac{|\xi'(\zeta_{k+1})|}{\eta}\right].$$

PROOF. For $p = 1$, we can write the inequality of Theorem 2.7 for $[\zeta_k, \zeta_{k+1}]$ $(k = 0, 1, \ldots, k-1)$ of the division d,

$$\left|\int_{\zeta_k}^{\zeta_{k+1}} \xi(\zeta)d\zeta - (\zeta_{k+1}-\zeta_k)\xi\left(\frac{\zeta_k+\zeta_{k+1}}{2}\right)\right|$$

$$\leq (\zeta_{k+1}-\zeta_k)^2\left[\frac{(2^{\alpha\gamma+1}-1)}{2^{\alpha\gamma+1}(\alpha\gamma+1)(\alpha\gamma+2)}|\xi'(\zeta_k)|+\right.$$

$$\left.\left(\beta_{1/2^\eta}\left(\frac{2}{\eta},\delta+1\right)+\beta_{1-1/2^\eta}\left(\delta+1,\frac{1}{\eta}\right)-\beta_{1-1/2^\eta}\left(\delta+1,\frac{2}{\eta}\right)\right)\frac{|\xi'(\zeta_{k+1})|}{\eta}\right].$$

Hence,

$$\left|\int_{\varpi_1}^{\varpi_2} \xi(\zeta)d\zeta - M(\xi,d)\right|$$

$$=\left|\sum_{k=0}^{n-1}\left[\int_{\zeta_k}^{\zeta_{k+1}} \xi(\zeta)d\zeta - (\zeta_{k+1}-\zeta_k)\xi\left(\frac{\zeta_k+\zeta_{k+1}}{2}\right)\right]\right|$$

$$\leq \sum_{k=0}^{n-1}\left|\left[\int_{\zeta_k}^{\zeta_{k+1}} \xi(\zeta)d\zeta - (\zeta_{k+1}-\zeta_k)\xi\left(\frac{\zeta_k+\zeta_{k+1}}{2}\right)\right]\right|$$

$$\leq \sum_{k=0}^{n-1}(\zeta_{k+1}-\zeta_k)^2\left[\frac{(2^{\alpha\gamma+1}-1)}{2^{\alpha\gamma+1}(\alpha\gamma+1)(\alpha\gamma+2)}|\xi'(\zeta_k)|+\right.$$

$$\left.\left(\beta_{1/2^\eta}\left(\frac{2}{\eta},\delta+1\right)+\beta_{1-1/2^\eta}\left(\delta+1,\frac{1}{\eta}\right)-\beta_{1-1/2^\eta}\left(\delta+1,\frac{2}{\eta}\right)\right)\frac{|\xi'(\zeta_{k+1})|}{\eta}\right].$$

□

THEOREM 3.2. *Assume that $\xi' \in L[\varpi_1, \varpi_2]$ with $\varpi_1, \varpi_2 \in I^\circ$ and $\varpi_2 > \varpi_1$. If $\xi : I \subset (0,\infty) \to \mathbb{R}$ is a differentiable function on I° s. t. $|\xi'|^q$ for $q \geq 1$ is a $(\alpha,\eta,\gamma,\delta)-$convex on I for $(\alpha,\eta,\gamma,\delta) \in [0,1]^4$, then below stated result holds with $\eta \neq 0$:*

$$|R(\xi,d)| \leq \frac{1}{(8)^{1-\frac{1}{q}}}\sum_{k=0}^{n-1}(\zeta_{k+1}-\zeta_k)^2\left[\left\{\frac{|\xi'(\zeta_k)|^q}{2^{\alpha\gamma+2}(\alpha\gamma+2)}\right.\right.$$

$$\left.+\frac{|\xi'(\zeta_{k+1})|^q}{\eta}\beta_{1/2^\eta}\left(\frac{2}{\eta},\delta+1\right)\right\}^{\frac{1}{q}}+\left\{\frac{(2^{\alpha\gamma+2}-\alpha\gamma-3)|\xi'(\zeta_k)|^q}{2^{\alpha\gamma+2}(\alpha\gamma+1)(\alpha\gamma+2)}\right.$$

$$\left.\left.+\frac{|\xi'(\zeta_{k+1})|^q}{\eta}\left(\beta_{1-1/2^\eta}\left(\delta+1,\frac{1}{\eta}\right)-\beta_{1-1/2^\eta}\left(\delta+1,\frac{2}{\eta}\right)\right)\right\}^{\frac{1}{q}}\right]. \tag{3.2}$$

PROOF. The proof follows the same technique of the previous theorem, applying to the inequality of Theorem 2.10 on $[\zeta_k, \zeta_{k+1}]$. ∎

THEOREM 3.3. *Assume that* $\xi' \in L[\varpi_1, \varpi_2]$, *with* $\varpi_1, \varpi_2 \in I^\circ$ *and* $\varpi_2 > \varpi_1$. *If* $\xi : I \subset (0, \infty) \to \mathbb{R}$ *is a differentiable function on* I° *s. t.* $|\xi'|^{q_2}$ *is a* $(\alpha, \eta, \gamma, \delta)-$*convex on* I *for* $(\alpha, \eta, \gamma, \delta) \in [0, 1]^4$ *with* $\eta \neq 0$, *then below stated result holds for* $\frac{1}{q_1} + \frac{1}{q_2} = 1$*:*

$$
\begin{aligned}
|R(\xi, d)| \leq & \left(\frac{1}{2^{q_1+1}(q_1+1)}\right)^{\frac{1}{q_1}} \sum_{k=0}^{n-1} (\zeta_{k+1} - \zeta_k)^2 \\
& \times \left[\left(\frac{|\xi'(\zeta_k)|^{q_2}}{2^{\alpha\gamma+1}(\alpha\gamma+1)} + \frac{\beta_{1/2^\eta}\left(\frac{1}{\eta}, \delta+1\right) |\xi'(\zeta_{k+1})|^{q_2}}{\eta} \right)^{\frac{1}{q_2}} \right. \\
& \left. + \left(\frac{(2^{\alpha\gamma+1}-1)|\xi'(\zeta_k)|^{q_2}}{2^{\alpha\gamma+1}(\alpha\gamma+1)} + \frac{\beta_{1-1/2^\eta}\left(\delta+1, \frac{1}{\eta}\right) |\xi'(\zeta_{k+1})|^{q_2}}{\eta} \right)^{\frac{1}{q_2}} \right]. \qquad (3.3)
\end{aligned}
$$

PROOF. The proof will be followed with the same technique of Theorem 3.1, applying to the inequality of Theorem 2.13 on $[\zeta_k, \zeta_{k+1}]$. ∎

4. Conclusion and Remarks

4.1. *Conclusion.* Hermite-Hadamard dual inequality is one of the most celebrated inequalities. We can find its various generalizations and variants in literature. We have given its generalization by introducing new generalized notion of $(\alpha, \eta, \gamma, \delta) - p$ convex functions. This new class of functions contains many important classes including class of $(\alpha, \eta)-$convex of the 1^{st} and 2^{nd} kinds, s-convex of the 1^{st}, 2^{nd} and mixed kinds (and hence contains class of ordinary convex functions). It also contains class of $P-p$ convex functions, P-convex functions, quasi$-p$ convex functions and class of quasi convex functions. In Section 2, we have also stated three distinct results related to estimation of bound of difference of left and middle term of Hermite-Hadamard dual inequality in the absolute sense. Here, we used various techniques including Power mean and Hölder's integral inequalities. These results capture various results stated in the articles [2, 15, 16] and [21]. In the third section, we established a relationship of our obtained results with midpoint formula.

4.2. *Results Summary of Section 2.* Now, we are going to summarize the already existed results of Section 2 in tabular form (see next page).

In the above table HHTI, CF, Har. and $-$ stand for Hermite-Hadamard type inequality, Convex function, harmonically and For any value, respectively.

TABLE 1. Result Summary of Section 2

No.	α	η	γ	δ	p	**Results**	**Found in**
1	–	–	–	–	1	HHTI for $(\alpha, \eta, \gamma, \delta)-$CF	This article
2	–	–	–	–	-1	HHTI for Har. $(\alpha, \eta, \gamma, \delta)-$CF	This article
3	–	–	1	1	–	HHTI for $(\alpha, \eta) - p-$CF of the 1st kind	This article
4	–	–	1	1	1	HHTI for $(\alpha, \eta)-$CF of the 1st kind	This article
5	–	–	1	1	-1	HHTI for Har. $(\alpha, \eta)-$CF of the 1st kind	This article
6	1	1	–	–	–	HHTI for $(\alpha, \eta) - p-$CF of the 2nd kind	This article
7	1	1	–	–	1	HHTI for $(\alpha, \eta)-$CF of the 2nd kind	This article
8	1	1	–	–	-1	HHTI for Har. $(\alpha, \eta)-$CF of the 2nd kind	This article
9	r	r	s	s	–	HHTI for $(r, s) - p-$CF	[2]
10	r	r	s	s	1	HHTI for $(r, s)-$CF	[2]
11	r	r	s	s	-1	HHTI for Har. $(r, s)-$CF	This article
12	r	r	1	1	–	HHTI for $s - p-$CF of the 1st kind	[2]
13	r	r	1	1	1	HHTI for $s-$CF of the 1st kind	[2]
14	r	r	1	1	-1	HHTI for Har. $s-$CF of the 1st kind	This article
15	1	1	s	s	–	HHTI for $s - p-$CF of the 2nd kind	[2]
16	1	1	s	s	1	HHTI for $s-$CF of the 2nd kind	[2]
17	1	1	s	s	-1	HHTI for Har. $s-$CF of the 2nd kind	This article
18	0	0	$\neq 0$	–	–	HHTI for quasi$-p-$CF	This article
19	0	0	$\neq 0$	–	1	HHTI for quasi CF	This article
20	–	–	0	0	–	HHTI for $P - p-$CF	This article
21	–	–	0	0	1	HHTI for $P-$CF	This article
22	1	1	1	1	–	HHTI for $p-$CF	[21]
23	1	1	1	1	1	HHTI for ordinary CF	[16]
24	1	1	1	1	-1	HHTI for Har. CF	[15]

Now we are going to give some remarks and future ideas for researchers.

4.3. *Remarks and Future Ideas.*

1. All the inequalities given in this article can be stated in reverse direction for concave function using simple relation ξ is concave if and only if $-\xi$ is convex.
2. One can extend this work to time scale domain or Quantum Calculus.
3. One can try to attain this work for Fuzzy theory.
4. One can try to work for finding refined bounds of all results.
5. One may also work on Fejér inequality by introducing weights in Hermite-Hadamard dual inequality.
6. One can extend this work to Fractional Calculus.
7. One can try to state all results in discrete case.
8. One can also extend all results in multi-dimensions.

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Generalizirane nejednakosti Hermite-Hadamardovog tipa za $(\alpha, \eta, \gamma, \delta) - p$ konveksne funkcije

Muhamed Bilal, Silvestru S. Dragomir i Asif R. Khan

Sažetak. U ovom članku želimo prikazati još jednu generaliziranu klasu konveksnih funkcija koje nazivamo $(\alpha, \eta, \gamma, \delta) - p$ konveksne funkcije. Ova nova klasa sadrži još dvije nove klase, naime, $(\alpha, \eta) - p$ konveksne funkcije prve i druge vrste. Nadalje, također generaliziramo neke rezultate koji se odnose na čuvenu nejednakost Hermite-Hadamardovog tipa za gore spomenutu klasu funkcija s različitim tehnikama. Stoga će različiti postojeći i novi rezultati biti obuhvaćeni kao poseban slučaj naših dobivenih rezultata. Štoviše, prikazana je i primjena na formulu srednje točke.

Muhammad Bilal
Department of Mathematics
University of Karachi
University Road, Karachi-75270, Pakistan
E-mail: mbilalfawad@gmail.com

Silvestru Sever Dragomir
Mathematics
College of Engineering and Science
Victoria University
PO Box 14428, Melbourne City, MC 8001, Australia

DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences
School of Computer Science and Applied Mathematics
University of the Witwatersrand
Private Bag 3, Johannesburg 2050, South Africa
E-mail: sever.dragomir@vu.edu.au

Asif Raza Khan
Department of Mathematics
University of Karachi
University Road, Karachi-75270, Pakistan
E-mail: asifrk@uok.edu.pk

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