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# THE HERMITE-HADAMARD INEQUALITY FOR $M_\varphi M_\psi$-$h$-CONVEX FUNCTIONS AND RELATED INTERPOLATIONS

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Abstract. In this paper we consider the Hermite-Hadamard inequality for $M_\varphi M_\psi$-$h$-convex functions. An $M_\varphi M_\psi$-$h$-convexity covers several particular types of generalized convexity such as a harmonic-$h$-convexity, a log-$h$-convexity, $(h,p)$-convexity, $M_pA$-$h$-convexity, $M_\varphi M_\psi$-convexity etc. The Hermite-Hadamard type inequalities with two and with $n$ nodes are given. Special attention is paid to a dyadic partition of an interval and related interpolations.

## 1. Introduction

In recent decades we have witnessed the emergence of various types of convexity. In addition to the classical convexity, we find the following variants of convexity in the literature: $s$-convexity, Godunova-Levin convexity, $P$-convexity, $h$-convexity, strong convexity, $m$-convexity, $MN$-convexity, $MT$ convexity, etc. For each type of convexity, one of the first results to be studied is the Hermite-Hadamard inequality. For the classical convexity, the Hermite-Hadamard inequality has the following statement.

*For an integrable convex function $f : [a,b] \to \mathbb{R}$, the following sequence of inequalities holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{2}[f(a)+f(b)]. \tag{1.1}$$

The natural question which arises in connection with this inequality is a question of its refinement. In recent literature, we find several articles on this topic. Here we have to mention article [9] where we find the following refinement.

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**Theorem A** *Assume that* $f : [a,b] \to \mathbb{R}$ *is a convex function on* $[a,b]$. *Then for all* $\lambda \in [0,1]$, *we have*

$$f\left(\frac{a+b}{2}\right) \le m(\lambda) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M(\lambda) \le \frac{1}{2}[f(a)+f(b)], \tag{1.2}$$

*where*

$$m(\lambda) := \lambda f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda) f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)$$

*and*

$$M(\lambda) := \frac{1}{2}\Big(f(\lambda b + (1-\lambda)a) + \lambda f(a) + (1-\lambda) f(b)\Big).$$

If $\lambda = \frac{1}{2}$, then points in the left refinement are $\frac{3a+b}{4}$ and $\frac{a+3b}{4}$, i.e

$$m\left(\frac{1}{2}\right) = \frac{1}{2} f\left(\frac{3a+b}{4}\right) + \frac{1}{2} f\left(\frac{a+3b}{4}\right)$$

in which we recognize the refinement which occurs in [15, p.37] and in articles about other type of convexity such as [2, 17].

Results from [9] were generalized in [7] for a more general class of functions. Namely, in [7], author obtained corresponding results for $h$-convex functions. Let us recall the definition of an $h$-convex function, [23].

DEFINITION 1.1. *Let* $h : J \to \mathbb{R}$ *be a non-negative function,* $\langle 0,1\rangle \subseteq J$. *A function* $f : I \to \mathbb{R}$ *is called* $h$*-convex if for any* $x, y$ *from the interval* $I$ *and any* $t \in \langle 0,1\rangle$ *the following holds*

$$f(tx + (1-t)y) \le h(t) f(x) + h(1-t) f(y).$$

This concept covers some classes such as a class of convex functions, a class of $s$-convex functions in the second sense ($h(t) = t^s, s \in \langle 0,1]$), a class of Godunova-Levin functions ($h(t) = \frac{1}{t}$), a class of P-convex functions ($h(t) = 1$). The Hermite-Hadamard inequality for an $h$-convex function was first given in [4] and [21] and has the following form:

**Theorem B** *If* $h$ *is an integrable function,* $h(\frac{1}{2}) \neq 0$, *then for an integrable* $h$*-convex function* $f : [a,b] \to \mathbb{R}$, *the following sequence of inequalities holds:*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le [f(a)+f(b)]\int_0^1 h(x)\,dx. \tag{1.3}$$

*If* $f$ *is* $h$*-concave, then the reversed signs of inequalities hold in* (1.3).

The following Hermite-Hadamard-type result for an $h$-convex function can be found in [7] as a consequence of Theorem 2 from [7] and the corresponding Remark in the same paper.

**Theorem C** *If $f$ is a non-negative, integrable, $h$-convex function on $[a,b]$ with $h \in L[0,1]$, $h(\frac{1}{2}) \neq 0$, then*

$$\begin{aligned}\delta_1 &\leq \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\leq \delta_2 \leq \Big[[h(1-\lambda)+\lambda]f(a)+[h(\lambda)+1-\lambda]f(b)\Big]\int_0^1 h(t)\,dt, \qquad (1.4)\end{aligned}$$

*where*

$$\delta_1 := \frac{1}{2h(\frac{1}{2})}\left\{(1-\lambda)f\left[\frac{(1-\lambda)a+(\lambda+1)b}{2}\right]+\lambda f\left[\frac{(2-\lambda)a+\lambda b}{2}\right]\right\}$$

$$\delta_2 := \Big[f\big((1-\lambda)a+\lambda b\big)+(1-\lambda)f(b)+\lambda f(a)\Big]\int_0^1 h(t)\,dt.$$

*Furthermore, if $\lambda \in \langle 0,1\rangle$ such that $h(\lambda) \neq 0$, then*

$$\frac{1}{2h(\frac{1}{2})}\min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}f\left(\frac{a+b}{2}\right) \leq \delta_1. \qquad (1.5)$$

A closer look into the proof of Theorem C gives that (1.4) is valid regardless of non-negativity of $f$. Non-negativity of $f$ in points $\frac{(1-\lambda)a+(\lambda+1)b}{2}$ and $\frac{(2-\lambda)a+\lambda b}{2}$ is necessary only in (1.5).

If $h(t) = t$, i.e. if $f$ is a convex function, then the result of Theorem C collapses to the refinement of Hermite-Hadamard inequality (1.2). It is a refinement which involves two nodes $\dfrac{(1-\lambda)a+(\lambda+1)b}{2}$ and $\dfrac{(2-\lambda)a+\lambda b}{2}$. In paper [8], a result including $n$ nodes was given. Here we give a version of that result for a real function of a real variable.

**Theorem D** *Let $f$ be an $h$-convex with $h \in L[0,1]$, $f \in L[a,b]$, $h(\frac{1}{2}) \neq 0$. Then for any partition*

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = 1, \qquad \textit{with } n \geq 1$$

*we have*

$$\begin{aligned}&\frac{1}{2h(\frac{1}{2})}\sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)f\left(\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)a+\frac{\lambda_j+\lambda_{j+1}}{2}b\right) \\ &\quad\leq \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\quad\leq \sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\times \\ &\quad\times\Big[f\big((1-\lambda_j)a+\lambda_j b\big)+f\big((1-\lambda_{j+1})a+\lambda_{j+1}b\big)\Big]\int_0^1 h(t)\,dt.\end{aligned}$$

In papers [7] and [8], a case of $h$-concavity was not considered, but from the proofs it is clear that if $f$ is $h$-concave, then inequalities in Theorems C and D hold with the reversed signs.

The topic of this paper is a counterpart of the Hermite-Hadamard inequality for a wider class of functions which covers $h$-convex functions.

Let $\varphi$ be a continuous, strictly monotone function defined on the interval $I$. By $M_\varphi$ we denote a quasi-arithmetic mean:

$$M_\varphi(x,y;t) := \varphi^{-1}(t\varphi(x)+(1-t)\varphi(y)), \quad x,y\in I, t\in[0,1].$$

It is obvious that the power mean $M_p$ corresponds to $\varphi(x)=x^p$ if $p\neq 0$ and to $\varphi(x)=\log x$ if $p=0$.

Definition 1.2. *Let $\varphi$ and $\psi$ be two continuous, strictly monotone functions defined on intervals $I$ and $K$ respectively. Let $h : J \to \mathbb{R}$ be a non-negative function, $\langle 0,1\rangle \subseteq J$ and let $f : I \to K$ such that $h(t)\psi(f(x)) + h(1-t)\psi(f(y)) \in \psi(K)$ for all $x,y\in I, t\in\langle 0,1\rangle$. We say that a function $f$ is $M_\varphi M_\psi$-$h$-convex if*

$$f(M_\varphi(x,y;t)) \leq \psi^{-1}\Big(h(t)\psi(f(x)) + h(1-t)\psi(f(y))\Big) \tag{1.6}$$

*for all $x,y\in I$ and all $t\in\langle 0,1\rangle$. If the sign of inequality is reversed in* (1.6)*, then $f$ is called $M_\varphi M_\psi$-$h$-concave.*

Some particular cases of $M_\varphi M_\psi$-$h$-convex functions have been recently investigated in last ten years. If $h(t)=t$, then $M_\varphi M_\psi$-$h$-convexity collapses to $M_\varphi M_\psi$-convexity which was described in [15]. Paper [1] consists several results about properties and the Jensen inequality for $M_\varphi M_\psi$-$h$-convex functions where $M_\varphi$, $M_\psi$ are an arithmetic mean ($A$), a geometric mean ($G$) or a harmonic mean ($H$). Furthermore, an $HA$-$h$-convexity or harmonic-$h$-convexity was described in [3] and [19]. An $HG$-$h$-convexity is investigated in [19] and an $AG$-$h$-convexity or log-$h$-convexity in [20]. An $AM_p$-$h$-convexity or $(h,p)$-convexity was described in [11] while some properties of $M_pA$-$h$-convex functions were given in [6]. Properties of $M_\varphi A$-$h$-convex functions were studied in [24].

In the second section, we prove the Hermite-Hadamard inequality for an $M_\varphi M_\psi$-$h$-convex function. The third section is devoted to different interpolation results related to the Hermite-Hadamard inequality. We end this paper with results related to a dyadic partition of interval $[a,b]$.

In this paper, if some inequality has a number $(n)$ then its reverse version, i.e. an inequality with another sign is denoted by (R$n$).

## 2. The Hermite-Hadamard inequality

The following result gives a connection between the theory of $h$-convexity and the theory of $M_\varphi M_\psi$-$h$-convexity. As we will see below, it is the powerful tool used in many proofs.

Proposition 2.1. *Let $\varphi$ and $\psi$ be strictly monotone continuous functions defined on intervals $I$ and $K$ respectively.*

*a) Let $\psi$ be an increasing function. A function $f : I \to \mathbb{R}$ is $M_\varphi M_\psi$-$h$-convex ($M_\varphi M_\psi$-$h$-concave) if and only if $\psi \circ f \circ \varphi^{-1}$ is $h$-convex ($h$-concave).*

*b) Let $\psi$ be an decreasing function. A function $f$ is $M_\varphi M_\psi$-$h$-convex ($M_\varphi M_\psi$-$h$-concave) if and only if $\psi \circ f \circ \varphi^{-1}$ is $h$-concave ($h$-convex).*

Proof. Let us suppose that $\psi$ is increasing. For any $u, v \in \mathrm{Im}(\varphi)$ there exist $x, y \in I$ such that $\varphi(x) = u, \varphi(y) = v$. If $f$ is $M_\varphi M_\psi$-$h$-convex and $\psi$ is increasing, then for any $t \in \langle 0, 1\rangle$

$$\psi(f(\varphi^{-1}(t\varphi(x) + (1-t)\varphi(y)))) \le h(t)\psi(f(x)) + h(1-t)\psi(f(y))$$

i.e.

$$(\psi \circ f \circ \varphi^{-1})(tu + (1-t)v) \le h(t)(\psi \circ f \circ \varphi^{-1})(u) + h(1-t)(\psi \circ f \circ \varphi^{-1})(v).$$

So, $\psi \circ f \circ \varphi^{-1}$ is $h$-convex. Other cases are proved in a similar way. ∎

Theorem 2.2 (The Hermite-Hadamard inequality for an $M_\varphi M_\psi$-$h$-convex function). *Let $h$ be a non-negative function defined on the interval $J$, $\langle 0,1\rangle \subseteq J$, $h(\frac{1}{2}) \neq 0$. Let $\varphi$ and $\psi$ be strictly monotone continuous functions defined on intervals $I$ and $K$ respectively such that $\varphi$ is differentiable on $[a,b] \subseteq I$.*

*a) If $\psi$ is increasing, then for an $M_\varphi M_\psi$-$h$-convex function $f : [a,b] \to \mathbb{R}$ the following holds*

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})}(\psi \circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big) &\le \frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\,\varphi'(x)\,dx \\ &\le \Big[\psi(f(a)) + \psi(f(b))\Big]\int_0^1 h(t)\,dt, \end{aligned} \tag{2.1}$$

*provided that all integrals exist.*

*If $f$ is $M_\varphi M_\psi$-$h$-concave, then* (R2.1) *holds.*

*b) If $\psi$ is decreasing, then for an $M_\varphi M_\psi$-$h$-convex function $f$* (R2.1) *holds. If $f$ is $M_\varphi M_\psi$-$h$-concave, then* (2.1) *holds.*

Proof. Let us suppose that $\psi$ is increasing and $f$ is $M_\varphi M_\psi$-$h$-convex. Then, by Proposition 2.1, a function $\psi \circ f \circ \varphi^{-1}$ is $h$-convex on $\varphi([a,b])$. If $\varphi$ is increasing, then $\varphi([a,b]) = [\varphi(a), \varphi(b)]$, while if $\varphi$ is decreasing, then $\varphi([a,b]) = [\varphi(b), \varphi(a)]$.

If $\varphi$ is increasing, then applying (1.3) for a function $\psi \circ f \circ \varphi^{-1}$, we get

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})}(\psi \circ f \circ \varphi^{-1})\Big(\frac{\varphi(a)+\varphi(b)}{2}\Big) &\le \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} (\psi \circ f \circ \varphi^{-1})(x)\,dx \\ &\le \Big[(\psi \circ f \circ \varphi^{-1})(\varphi(a)) + (\psi \circ f \circ \varphi^{-1})(\varphi(b))\Big]\int_0^1 h(t)\,dt.\end{aligned}$$

After substitution $\varphi^{-1}(x) = u$, the integral in the middle term becomes $\int_a^b \psi(f(x))\,\varphi'(x)\,dx$ and inequality (2.1) is proved.

If $\varphi$ is decreasing, then the middle term is $\frac{1}{\varphi(a)-\varphi(b)} \int_{\varphi(b)}^{\varphi(a)} (\psi \circ f \circ \varphi^{-1})(x)\,dx$ and after the same substitution we get $\frac{1}{\varphi(b)-\varphi(a)} \int_a^b \psi(f(x))\,\varphi'(x)\,dx$ and inequality (2.1) holds in this case.

All other cases are proved similarly. □

REMARK 2.3. Some particular cases of the above inequality are known. If $h(t) = t$, then the Hermite-Hadamard-type inequality for HG-convex, GG-convex, $M_pA$-convex, $AM_p$-convex, $M_\varphi A$-convex and $M_\varphi M_\psi$-convex functions can be found in [16], [13], [10], [5], [22] and [14] respectively.

The Hermite-Hadamard inequality for HA-$h$-convex, AG-$h$-convex, $AM_r$-$h$-convex functions are given in [19], [20], [11] respectively.

When $h$ has the form $h(t) = h_1(t^s)$ for the fixed $s \in \langle 0, 1]$, then results related to the Hermite-Hadamard inequality for $h$-convex functions are given in [18].

Note that Theorem 2.2 covers all the cases already mentioned. In the above-mentioned articles, the authors proved the Hermite-Hadamard type inequalities directly, *ab ovo*. But Proposition 2.1 allows us to prove such results much more elegantly using known results for $h$-convex functions.

## 3. Hermite-Hadamard type results with several nodes

In this section we direct our attention to Hermite-Hadamard-type results including two or more nodes. The section is finished with several results involving a dyadic partition of an interval. The following theorem is a generalization of Theorem C given in $M_\varphi M_\psi$-$h$-convexity settings. In fact, this is a Hermite-Hadamard-type result which on the left-hand side includes values of a function in two points:

$$\varphi^{-1}\left(\frac{(1-\lambda)\varphi(a) + (1+\lambda)\varphi(b)}{2}\right) \quad \text{and} \quad \varphi^{-1}\left(\frac{(2-\lambda)\varphi(a) + \lambda\varphi(b)}{2}\right)$$

and which, in particular case, leads to the refinement of the Hermite-Hadamard inequality for an $M_\varphi M_\psi$-convex function.

THEOREM 3.1. *Let $h$ be a non-negative function defined on the interval $J$, $\langle 0, 1\rangle \subseteq J$, $h(\frac{1}{2}) \neq 0$. Let $\varphi$ and $\psi$ be strictly monotone continuous functions defined on intervals $I$ and $K$ respectively such that $\varphi$ is differentiable on $[a, b] \subseteq I$. Let $f : I \to \mathbb{R}$.*

*(i) If $\psi$ is increasing, then for an $M_\varphi M_\psi$-$h$-convex function $f$ the following holds*

$$\Delta_1 \leq \frac{1}{\varphi(b)-\varphi(a)} \int_a^b \psi(f(x))\,\varphi'(x)\,dx$$

$$\leq \Delta_2 \leq \Big\{[h(1-\lambda)+\lambda]\psi(f(a)) + [h(\lambda)+1-\lambda]\psi(f(b))\Big\} \int_0^1 h(t)\,dt, \tag{3.1}$$

*where*

$$\Delta_1 := \frac{1}{2h(\frac{1}{2})} \Big\{(1-\lambda)(\psi\circ f)\Big(M_\varphi\Big(a,b;\frac{1-\lambda}{2}\Big)\Big) + \lambda(\psi\circ f)\left(M_\varphi\Big(a,b;\frac{2-\lambda}{2}\Big)\right)\Big\}$$

$$\Delta_2 := \Big[\psi(f(M_\varphi(a,b;1-\lambda))) + (1-\lambda)\psi(f(b)) + \lambda\psi(f(a))\Big] \int_0^1 h(t)\,dt,$$

*provided that all integrals exist.*

*Furthermore, if $h(\lambda), h(1-\lambda) \neq 0$ and $(\psi\circ f)\left(M_\varphi\Big(a,b;\frac{1-\lambda}{2}\Big)\right), (\psi\circ f)\left(M_\varphi\Big(a,b;\frac{2-\lambda}{2}\Big)\right) \geq 0$ for some $\lambda \in \langle 0,1\rangle$, then*

$$\frac{1}{2h(\frac{1}{2})} \min\left\{\frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)}\right\} (\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big) \leq \Delta_1. \tag{3.2}$$

*If $f$ is $M_\varphi M_\psi$-$h$-concave, then* (R3.1) *and* (R3.2) *(with change* min $\to$ max*) hold.*

*(ii) If $\psi$ is decreasing and $f$ is $M_\varphi M_\psi$-$h$-convex, then* (R3.1) *and* (R3.2) *(with change* min $\to$ max*) hold. If $\psi$ is decreasing and $f$ is $M_\varphi M_\psi$-$h$-concave, then* (3.1) *and* (3.2) *are valid.*

Proof. Let us prove the case when $\psi$ is increasing. Other cases are done in the similar manner. Denote $G := \psi\circ f$. Since $f$ is $M_\varphi M_\psi$-$h$-convex on $I$, then $G\circ\varphi^{-1}$ is $h$-convex on $\mathrm{Im}(\varphi)$ and applying Theorem C on function $G\circ\varphi^{-1}$, we get

$$\delta_1 = \frac{1}{2h(\frac{1}{2})} \left\{(1-\lambda)(G\circ\varphi^{-1})\left(\frac{(1-\lambda)\varphi(a)+(1+\lambda)\varphi(b)}{2}\right)\right.$$
$$\left. +\lambda(G\circ\varphi^{-1})\left(\frac{(2-\lambda)\varphi(a)+\lambda\varphi(b)}{2}\right)\right\}$$

$$\delta_2 = \Big[(G\circ\varphi^{-1})((1-\lambda)\varphi(a)+\lambda\varphi(b)) + (1-\lambda)G(b) + \lambda G(a)\Big] \int_0^1 h(t)\,dt.$$

The second term in (1.4) becomes $\frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\,\varphi'(x)\,dx$ and the fourth term in (1.4) becomes

$$\Big[\psi(f(M_\varphi(a,b;1-\lambda))) + (1-\lambda)\psi(f(b)) + \lambda\psi(f(a))\Big] \int_0^1 h(t)\,dt.$$

Since

$$(G\circ\varphi^{-1})\left(\frac{(1-\lambda)\varphi(a)+(1+\lambda)\varphi(b)}{2}\right)=(\psi\circ f)\left(M_\varphi\Big(a,b;\frac{1-\lambda}{2}\Big)\right)$$
$$(G\circ\varphi^{-1})\left(\frac{(2-\lambda)\varphi(a)+\lambda\varphi(b)}{2}\right)=(\psi\circ f)\left(M_\varphi\Big(a,b;\frac{2-\lambda}{2}\Big)\right)$$
$$(G\circ\varphi^{-1})\left((1-\lambda)\varphi(a)+\lambda\varphi(b)\right)=(\psi\circ f)(M_\varphi(a,b;1-\lambda))$$

we get (3.1).

Let us prove inequality (3.2). Let us rewrite $\delta_1$ on this way:

$$\begin{aligned}
2h\left(\frac{1}{2}\right)\delta_1 &= \frac{1-\lambda}{h(1-\lambda)}h(1-\lambda)(G\circ\varphi^{-1})\left(\frac{(1-\lambda)\varphi(a)+(1+\lambda)\varphi(b)}{2}\right)\\
&+\frac{\lambda}{h(\lambda)}h(\lambda)(G\circ\varphi^{-1})\left(\frac{(2-\lambda)\varphi(a)+\lambda\varphi(b)}{2}\right)\\
&\geq \min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}\times\\
&\times\left\{(h(1-\lambda)(G\circ\varphi^{-1})\left(\frac{(1-\lambda)\varphi(a)+(1+\lambda)\varphi(b)}{2}\right)\right.\\
&\left.+\,h(\lambda)(G\circ\varphi^{-1})\left(\frac{(2-\lambda)\varphi(a)+\lambda\varphi(b)}{2}\right)\right\}\\
&\geq \min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}\times\\
&\times(G\circ\varphi^{-1})\left[(1-\lambda)\frac{(1-\lambda)\varphi(a)+(\lambda+1)\varphi(b)}{2}+\lambda\frac{(2-\lambda)a+\lambda b}{2}\right]\\
&= \min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}(G\circ\varphi^{-1})\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\\
&= \min\left\{\frac{1-\lambda}{h(1-\lambda)},\frac{\lambda}{h(\lambda)}\right\}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big).
\end{aligned}$$

□

Corollary 3.2. *Let the assumptions of Theorem* 3.1 *hold.*

*(i) If* $\psi$ *is increasing, then for an* $M_\varphi M_\psi$*-*$h$*-convex function* $f:I\to\mathbb{R}$ *the following holds:*

$$\begin{aligned}
&\frac{1}{4h^2(\frac{1}{2})}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)\\
&\qquad\leq\frac{1}{4h(\frac{1}{2})}\left\{(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{4}\big)\Big)+(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{3}{4}\big)\Big)\right\}
\end{aligned}$$

$$\leq \frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\,\varphi'(x)\,dx$$

$$\leq \left\{(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)+\frac{\psi(f(a))+\psi(f(b))}{2}\right\}\int_0^1 h(t)\,dt$$

$$\leq \left[\frac{1}{2}+h\Big(\frac{1}{2}\Big)\right]\Big[\psi(f(a))+\psi(f(b))\Big]\int_0^1 h(t)\,dt, \tag{3.3}$$

*provided that all integrals exist.*

*If $f$ is $M_\varphi M_\psi$-$h$-concave, then* (R3.3) *holds.*

*(ii) If $\psi$ is decreasing and $f$ is $M_\varphi M_\psi$-$h$-convex, then* (R3.3) *holds. If $\psi$ is decreasing and $f$ is $M_\varphi M_\psi$-$h$-concave, then* (3.3) *is valid.*

Proof. Firstly we consider the case when $\psi$ is increasing and $f$ is $M_\varphi M_\psi$-$h$-convex. The second and the third inequalities in (3.3) are simple consequences of Theorem 3.1 for $\lambda = \frac{1}{2}$. Let us prove the first and the fourth inequalities.

For an $h$-convex function $F$ the following inequality holds:

$$F(A)+F(B) \;\geq\; \frac{1}{h(\frac{1}{2})}F\left(\frac{A+B}{2}\right). \tag{3.4}$$

Numbers $A := \frac{\varphi(a)+3\varphi(b)}{4}$ and $B := \frac{3\varphi(a)+\varphi(b)}{4}$ satisfy:

$$\frac{A+B}{2}=\frac{\varphi(a)+\varphi(b)}{2}$$

and applying (3.4) on function $F := \psi\circ f\circ\varphi^{-1}$, we get

$$(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{4}\big)\Big)+(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{3}{4}\big)\Big)\geq \frac{1}{h(\frac{1}{2})}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)$$

and the first inequality in (3.3) holds.

Let us prove the fourth inequality. From (3.4) we get

$$(\psi\circ f)\Big(M_\varphi(a,b;\frac{1}{2})\Big)\leq h\Big(\frac{1}{2}\Big)\Big[\psi(f(a))+\psi(f(b))\Big]$$

and hence

$$(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)+\frac{\psi(f(a))+\psi(f(b))}{2}\leq\left[\frac{1}{2}+h\Big(\frac{1}{2}\Big)\right]\Big[\psi(f(a))+\psi(f(b))\Big]$$

and the fourth inequality in (3.3) is valid. ∎

Corollary 3.3. *Let $h$ satisfies the assumptions of Theorem* 3.1*. Let $f$ be a positive GG-h-convex function on $[a,b] \subseteq [0,\infty\rangle$. Then*

$$\begin{aligned}
\left(f(\sqrt{ab})\right)^{\frac{1}{4h^2(\frac{1}{2})}} &\le \left[f(\sqrt[4]{a^3b})f(\sqrt[4]{ab^3})\right]^{\frac{1}{4h(\frac{1}{2})}} \\
&\le \exp\left(\frac{1}{\log b/a}\int_a^b \log f(x)\,\frac{dx}{x}\right) \\
&\le \left(f(\sqrt{ab})\sqrt{f(a)f(b)}\right)^H \le \left(\sqrt{f(a)f(b)}\right)^{H\left[\frac{1}{2}+h(\frac{1}{2})\right]}, \qquad (3.5)
\end{aligned}$$

*where $H = \int_0^1 h(t)\,dt$ and provided that all integrals exist.*

Proof. It is a consequence of Corollary 3.2 for $\psi = \varphi = \log$. □

Remark 3.4. Inequality (3.5) for $h(t) = t$ i.e. for $GG$-convex or multiplicatively convex function can be found in [15, p.62]. It is worth to mention that every polynomial with non-negative coefficients is $GG$-convex, every real analytic function $f(x) = \sum a_n x^n$ with $a_n \ge 0$ is $GG$-convex on $[0, R\rangle$ where $R$ is the radius of convergence. Also, the Gamma function is $GG$-convex.

Corollary 3.5. *Let $h$ satisfies the assumptions of Theorem* 3.1*. Let $f$ be a function on $[a,b] \subseteq [0,\infty\rangle$ and $\varphi(x) = x^p$, $p \neq 0$.*

*If $p > 0$ and $f$ is $M_\varphi A$-h-convex, then*

$$\begin{aligned}
&\frac{1}{4h^2(\frac{1}{2})} f\left(\left(\frac{a^p+b^p}{2}\right)^{1/p}\right) \\
&\le \frac{1}{4h(\frac{1}{2})}\left\{f\left(\left(\frac{a^p+3b^p}{4}\right)^{1/p}\right) + f\left(\left(\frac{3a^p+b^p}{4}\right)^{1/p}\right)\right\} \\
&\le \frac{p}{b^p-a^p}\int_a^b f(x)x^{p-1}\,dx \\
&\le \left\{f\left(\left(\frac{a^p+b^p}{2}\right)^{1/p}\right) + \frac{f(a)+f(b)}{2}\right\}\int_0^1 h(t)dt \\
&\le \left[\frac{1}{2} + h\left(\frac{1}{2}\right)\right][f(a)+f(b)]\int_0^1 h(t)dt, \qquad (3.6)
\end{aligned}$$

*provided that all integrals exist.*

*If $p < 0$ and $f$ is $M_\varphi A$-h-convex, then* (R3.6) *holds.*

Proof. It is a consequence of Corollary 3.2 for $\psi(x) = x$, $\varphi(x) = x^p$. □

Remark 3.6. If $h(t) = t$ and $p = 1$, then $4h^2(\frac{1}{2}) = 1$, $\frac{1}{2} + h(\frac{1}{2}) = 1$ and inequality (3.6) becomes the refinement of the Hermite-Hadamard inequality (1.1).

The following Hermite-Hadamard-type result involves more than two nodes.

THEOREM 3.7. *Let $h$ be a non-negative function defined on the interval $J, \langle 0,1\rangle \subseteq J$, $h(\frac{1}{2}) \neq 0$. Let $\varphi$ and $\psi$ be strictly monotone continuous functions defined on intervals $I$ and $K$ respectively such that $\varphi$ is differentiable on $[a,b] \subseteq I$.*

*(i) If $\psi$ is increasing, then for an $M_\varphi M_\psi$-$h$-convex function $f : I \to \mathbb{R}$ and for a partition*

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = 1, \qquad \textit{with } n \geq 1$$

*we have*

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2})} \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j)(\psi \circ f)\Big(M_\varphi\big(a,b; 1 - \frac{\lambda_j + \lambda_{j+1}}{2}\big)\Big) \\ &\quad\leq \frac{1}{\varphi(b) - \varphi(a)} \int_a^b \psi(f(x))\varphi'(x)\, dx \\ &\quad\leq \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j)\Big\{(\psi \circ f)\big(M_\varphi(a,b;1-\lambda_j)\big) \\ &\qquad + (\psi \circ f)\big(M_\varphi(a,b;1-\lambda_{j+1})\big)\Big\} \int_0^1 h(t)\, dt, \end{aligned} \tag{3.7}$$

*provided that all integrals exist.*

*If $f$ is $M_\varphi M_\psi$-$h$-concave, then* (R3.7) *holds.*

*(ii) If $\psi$ is decreasing and $f$ is $M_\varphi M_\psi$-$h$-convex, then* (R3.7) *holds. If $\psi$ is decreasing and $f$ is $M_\varphi M_\psi$-$h$-concave, then* (3.7) *is valid.*

PROOF. Let $\psi$ be increasing and $f$ be $M_\varphi M_\psi$-$h$-convex. Denote $G := \psi \circ f$. Then a function $\psi \circ f \circ \varphi^{-1}$ is $h$-convex on $\varphi([a,b])$ and applying Theorem D on function $G \circ \varphi^{-1}$, we get

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2})} \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) G\left(\varphi^{-1}\left(\left(1 - \frac{\lambda_j + \lambda_{j+1}}{2}\right)\varphi(a) + \frac{\lambda_j + \lambda_{j+1}}{2}\varphi(b)\right)\right) \\ &\quad\leq \frac{1}{\varphi(b) - \varphi(a)} \int_a^b G(x)\varphi'(x)\, dx \\ &\quad\leq \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j)\Big\{G\left(\varphi^{-1}\left((1-\lambda_j)\varphi(a) + \lambda_j\varphi(b)\right)\right) \\ &\qquad + G\left(\varphi^{-1}\left((1-\lambda_{j+1})\varphi(a) + \lambda_{j+1}\varphi(b)\right)\right)\Big\} \int_0^1 h(t)\, dt. \end{aligned} \tag{3.8}$$

Using the fact that $G\left(\varphi^{-1}\left(\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)\varphi(a)+\frac{\lambda_j+\lambda_{j+1}}{2}\varphi(b)\right)\right) = (\psi\circ f)\Big(M_\varphi\big(a,b;1-\frac{\lambda_j+\lambda_{j+1}}{2}\big)\Big)$ etc, we get (3.7). Other cases are done in a similar manner. ∎

If a partition is equidistant, then the series of inequalities in (3.7) can be extended. Namely, we have the following result.

THEOREM 3.8. *Let $h$ be a non-negative function defined on the interval $J$, $\langle 0,1\rangle\subseteq J$, $h(\frac{1}{2})\neq 0$. Let $\varphi$ and $\psi$ be strictly monotone continuous functions defined on intervals $I$ and $K$ respectively such that $\varphi$ is differentiable on $[a,b]\subseteq I$. Let $f: I\to\mathbb{R}$. Let $n\geq 2$.*

*(i) If $\psi$ is increasing, then for an $M_\varphi M_\psi$-$h$-convex function $f$ the following inequalities hold*

$$\begin{aligned}
&\frac{1}{4h^2(\frac{1}{2})}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)\\
&\qquad\leq l(n)\leq\frac{1}{\varphi(b)-\varphi(a)}\int_a^b\psi(f(x))\varphi'(x)\,dx\leq L(n)\\
&\qquad\leq\frac{1}{n}\Big[\psi(f(a))+\psi(f(b))\Big]\left\{1+2\sum_{j=1}^{n-1}h\left(\frac{j}{n}\right)\right\}\int_0^1 h(t)\,dt, \qquad (3.9)
\end{aligned}$$

*provided that all integrals exist and where*

$$l(n)=\frac{1}{2nh(\frac{1}{2})}\sum_{j=0}^{n-1}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{2n-2j-1}{2n}\big)\Big)$$

$$L(n)=\frac{2}{n}\int_0^1 h(t)\,dt\left\{\sum_{j=1}^{n-1}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{j}{n}\big)\Big)+\frac{\psi(f(a))+\psi(f(b))}{2}\right\}.$$

*If $f$ is $M_\varphi M_\psi$-$h$-concave, then* (R3.9) *holds.*

*(ii) If $\psi$ is decreasing and $f$ is $M_\varphi M_\psi$-$h$-convex, then* (R3.9) *holds. If $\psi$ is decreasing and $f$ is $M_\varphi M_\psi$-$h$-concave, then* (3.9) *is valid.*

PROOF. Let us suppose that $\psi$ is increasing and $f$ is $M_\varphi M_\psi$-$h$-convex. The second and the third inequalities in (3.9) are simply consequences of Theorem 3.7 when we apply it on points: $\lambda_j=\frac{j}{n}$. Let us prove the first inequality. Putting in (3.4) $F=\psi\circ f\circ\varphi^{-1}=G\circ\varphi^{-1}$ and

$$A=\frac{2n-2j-1}{2n}\varphi(a)+\frac{2j+1}{2n}\varphi(b),\quad B=\frac{2j+1}{2n}\varphi(a)+\frac{2n-2j-1}{2n}\varphi(b)$$

and since $A+B=\varphi(a)+\varphi(b)$, we get

$$\begin{aligned}G\Big(\varphi^{-1}\big(\frac{2n-2j-1}{2n}\varphi(a)+\frac{2j+1}{2n}\varphi(b)\big)\Big)&\\ +G\Big(\varphi^{-1}\big(\frac{2j+1}{2n}\varphi(a)+\frac{2n-2j-1}{2n}\varphi(b)\big)\Big)&\\ \geq \frac{1}{h\big(\frac{1}{2}\big)}G\Big(\varphi^{-1}\big(\frac{\varphi(a)+\varphi(b)}{2}\big)\Big),&\end{aligned}$$

i.e.

$$\begin{aligned}G\Big(M_\varphi\big(a,b;\frac{2n-2j-1}{2n}\big)\Big)+G\Big(M_\varphi\big(a,b;\frac{2j+1}{2n}\big)\Big)&\\ \geq \frac{1}{h\big(\frac{1}{2}\big)}G\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big).&\end{aligned}$$

Let us write the sum $\sum_{j=0}^{n-1}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{2n-2j-1}{2}\big)\Big)$ twice and add the addend indexed by $j$ from the first sum with the addend indexed by $(n-j-1)$ from the second sum. Then we get

$$\begin{aligned}&2\sum_{j=0}^{n-1}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{2n-2j-1}{2n}\big)\Big)\\ &\quad=\sum_{j=0}^{n-1}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{2n-2j-1}{2n}\big)\Big)+(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{2j+1}{2n}\big)\Big)\\ &\quad\geq\sum_{j=0}^{n-1}\frac{1}{h\big(\frac{1}{2}\big)}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)\\ &\quad=\frac{n}{h\big(\frac{1}{2}\big)}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big)\end{aligned}$$

and the first inequality in (3.9) follows.

In the proof of the fourth inequality in (3.9) we apply a definition of $M_\varphi M_\psi$-$h$-convexity on each addend in the sum and transform it:

$$\begin{aligned}&\psi(f(a))+\psi(f(b))+2\sum_{j=1}^{n-1}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{j}{n}\big)\Big)\\ &\qquad\leq\psi(f(a))+\psi(f(b))+2\sum_{j=1}^{n-1}\Big(h\Big(\frac{j}{n}\Big)\psi(f(a))+h\Big(\frac{n-j}{n}\Big)\psi(f(b))\Big)\\ &\qquad=\Big[\psi(f(a))+\psi(f(b))\Big]\left\{1+2\sum_{j=1}^{n-1}h\Big(\frac{j}{n}\Big)\right\}\end{aligned}$$

and from this estimate the fourth inequality in (3.9) follows. ∎

In the following theorem we consider a particular partition of interval $[0,1]$, so-called a dyadic partition. Let $m \geq 1$ be an integer and let

$$\lambda_j := \frac{j}{2^m}, \qquad j = 0, 1, 2, \dots, 2^m.$$

Note that Corollary 3.2 contains result of this type for $m = 1$. In literature, there are no similar results for $h$-convex functions. Therefore, we can not use Proposition 2.1 in the proof of the following theorem.

THEOREM 3.9. *Let $h$ be a non-negative function defined on the interval $J$, $\langle 0,1\rangle \subseteq J$, $h(\frac{1}{2}) \neq 0$. Let $\varphi$ and $\psi$ be strictly monotone continuous functions defined on intervals $I$ and $K$ respectively such that $\varphi$ is differentiable on $[a,b] \subseteq I$. Let $f : I \to \mathbb{R}$.*

*(i) If $\psi$ is increasing, then for an $M_\varphi M_\psi$-$h$-convex function $f$ and $m \in \mathbb{N}$ the following holds*

$$l(2^{m+1}) \geq \frac{1}{2h(\frac{1}{2})} l(2^m) \tag{3.10}$$

$$L(2^{m+1}) \leq \left(\frac{1}{2} + h\Big(\frac{1}{2}\Big)\right) L(2^m) \tag{3.11}$$

$$\begin{aligned} L(2^m) \leq\ & 8h^2\Big(\frac{1}{2}\Big) \int_0^1 h(t)dt \cdot l(2^m) + \frac{1}{2^m} \int_0^1 h(t)dt \Big\{ \psi(f(a)) + \psi(f(b)) \\ & - 2h\Big(\frac{1}{2}\Big)\psi\left(f\Big(M_\varphi(a,b,\frac{2^{m+1}-1}{2^{m+1}})\Big)\right) \\ & - 2h\Big(\frac{1}{2}\Big)\psi\left(f\Big(M_\varphi(a,b,\frac{1}{2^{m+1}})\Big)\right)\Big\}, \end{aligned} \tag{3.12}$$

*where $l(n)$ and $L(n)$ are defined as in Theorem* 3.8.

*If $f$ is $M_\varphi M_\psi$-$h$-concave, then* (R3.10), (R3.11) *and* (R3.12) *hold.*

*(ii) If $\psi$ is decreasing and $f$ is $M_\varphi M_\psi$-$h$-convex, then* (R3.10), (R3.11) *and* (R3.12) *hold. If $\psi$ is decreasing and $f$ is $M_\varphi M_\psi$-$h$-concave, then* (3.10), (3.11) *and* (3.12) *hold.*

PROOF. We prove the case when $\psi$ is increasing and $f$ is $M_\varphi M_\psi$-$h$-convex. We use notation: $F := \psi \circ f \circ \varphi^{-1}$, $A := \varphi(a)$ and $B := \varphi(B)$.

From Theorem 3.8 we get:

$$l(2^{m+1}) = \frac{1}{2^{m+2}h(\frac{1}{2})} \sum_{j=0}^{2^{m+1}-1} F\left(\frac{(2^{m+2}-2j-1)A + (2j+1)B}{2^{m+2}}\right).$$

Since

$$\begin{aligned} &\{0,1,2,\dots 2^{m+1}-1\} = \{0,2,4,\dots,2^{m+1}-2\} \cup \{1,3,5,\dots,2^{m+1}-1\} \\ &= \{2k : k = 0,1,\dots,2^m-1\} \cup \{2k+1 : k = 0,1,\dots,2^m-1\}, \end{aligned}$$

we obtain

$$l(2^{m+1}) = \frac{1}{2^{m+2}h(\frac{1}{2})}\left\{\sum_{k=0}^{2^m-1} F\left(\frac{(2^{m+2}-4k-1)A+(4k+1)B}{2^{m+2}}\right)\right.$$
$$\left.+\sum_{k=0}^{2^m-1} F\left(\frac{(2^{m+2}-4k-3)A+(4k+3)B}{2^{m+2}}\right)\right\}.$$

Since $F$ is $h$-convex, then $F(x)+F(y) \geq \frac{1}{h(\frac{1}{2})}F(\frac{x+y}{2})$. Putting in this inequality $x = \frac{(2^{m+2}-4k-1)A+(4k+1)B}{2^{m+2}}$ and $y = \frac{(2^{m+2}-4k-3)A+(4k+3)B}{2^{m+2}}$, we get that $l(2^{m+1})$ is bounded from below as follows

$$l(2^{m+1}) \geq \frac{1}{2^{m+2}h(\frac{1}{2})}\sum_{k=0}^{2^m-1}\frac{1}{h(\frac{1}{2})}F\left(\frac{(2^{m+1}-2k-1)A+(2k+1)B}{2^{m+1}}\right)$$
$$= \frac{1}{2h(\frac{1}{2})}l(2^m).$$

Hence (3.10) is proved.

Let us prove (3.11). Again, we split the sum in $L(2^{m+1})$ into two sums: one with odd indices and the second sum with even indices.

$$L(2^{m+1}) = \frac{1}{2^m}\int_0^1 h(t)dt\left\{\frac{F(A)+F(B)}{2}+\sum_{k=1}^{2^m-1} F\left(\frac{(2^{m+1}-2k)A+2kB}{2^{m+1}}\right)\right.$$
$$\left.+\sum_{k=0}^{2^m-1} F\left(\frac{(2^{m+1}-2k-1)A+(2k+1)B}{2^{m+1}}\right)\right\}$$
$$= \frac{1}{2^m}\int_0^1 h(t)dt\left\{\sum_{k=0}^{2^m-1} F\left(\frac{(2^{m+1}-2k-1)A+(2k+1)B}{2^{m+1}}\right)\right.$$
$$+\left[\frac{1}{2}\sum_{k=1}^{2^m-1} F\left(\frac{(2^{m+1}-2k)A+2kB}{2^{m+1}}\right)+\frac{F(A)}{2}\right]$$
$$\left.+\left[\frac{1}{2}\sum_{k=1}^{2^m-1} F\left(\frac{(2^{m+1}-2k)A+2kB}{2^{m+1}}\right)+\frac{F(B)}{2}\right]\right\}$$
$$= \frac{1}{2^m}\int_0^1 h(t)dt\left\{\sum_{k=0}^{2^m-1} F\left(\frac{[(2^m-k)A+kB]+[(2^m-k-1)A+(k+1)B]}{2\cdot 2^m}\right)\right.$$

$$+\frac{1}{2}\sum_{k=0}^{2^m-1} F\left(\frac{(2^{m+1}-2k)A+2kB}{2^{m+1}}\right)$$
$$+\frac{1}{2}\sum_{r=0}^{2^m-1} F\left(\frac{(2^m-r-1)A+(r+1)B}{2^m}\right)\Bigg\}$$
$$\leq \frac{1}{2^m}\int_0^1 h(t)dt\left\{\sum_{k=0}^{2^m-1} h\Big(\frac{1}{2}\Big)F\left(\frac{(2^m-k)A+kB}{2^m}\right)\right.$$
$$+\sum_{k=0}^{2^m-1} h\Big(\frac{1}{2}\Big)F\left(\frac{(2^m-k-1)A+(k+1)B}{2^m}\right)$$
$$\left.+\frac{1}{2}\sum_{k=0}^{2^m-1}F\left(\frac{(2^m-k)A+kB}{2^m}\right)+\frac{1}{2}\sum_{r=0}^{2^m-1}F\left(\frac{(2^m-r-1)A+(r+1)B}{2^m}\right)\right\}$$
$$=\frac{1}{2^m}\int_0^1 h(t)dt\left(\frac{1}{2}+h\Big(\frac{1}{2}\Big)\right)\times$$
$$\times\left\{\sum_{k=0}^{2^m-1}\left[F\left(\frac{(2^m-k)A+kB}{2^m}\right)+F\left(\frac{(2^m-k-1)A+(k+1)B}{2^m}\right)\right]\right\}$$
$$=\left(\frac{1}{2}+h\Big(\frac{1}{2}\Big)\right)L(2^m).$$

Let us prove (3.12). Note that for $k=1,2,\ldots,2^m-1$

$$\frac{(2^m-k)A+kB}{2^m}$$
$$=\frac{1}{2}\left(\frac{(2^{m+1}-2k+1)A+(2k-1)B}{2^{m+1}}+\frac{(2^{m+1}-2k-1)A+(2k+1)B}{2^{m+1}}\right).$$

Since $F$ is $h$-convex, we get

$$\sum_{k=1}^{2^m-1}F\Big(\frac{(2^m-k)A+kB}{2^m}\Big)\leq\sum_{k=1}^{2^m-1}h\Big(\frac{1}{2}\Big)\Big\{F\Big(\frac{(2^{m+1}-2k+1)A+(2k-1)B}{2^{m+1}}\Big)$$
$$+F\left(\frac{(2^{m+1}-2k-1)A+(2k+1)B}{2^{m+1}}\right)\Bigg\}$$
$$=h\Big(\frac{1}{2}\Big)\left[2\sum_{j=0}^{2^m-1}F\left(\frac{(2^{m+1}-2j-1)A+(2j+1)B}{2^{m+1}}\right)\right.$$
$$\left.-F\left(\frac{(2^{m+1}-1)A+B}{2^{m+1}}\right)-F\left(\frac{A+(2^{m+1}-1)B}{2^{m+1}}\right)\right].$$

Adding on the both sides $\frac{F(A)+F(B)}{2}$ and using notations for $l$ and $L$, we get

$$\frac{2^{m-1}}{\int_0^1 h(t)dt} L(2^m) \leq 2^{m+2}h^2\Big(\frac{1}{2}\Big)\cdot l(2^m) + \frac{F(A)+F(B)}{2}$$
$$- h^2\Big(\frac{1}{2}\Big)F\left(\frac{(2^{m+1}-1)A+B}{2^{m+1}}\right) - h^2\Big(\frac{1}{2}\Big)F\left(\frac{A+(2^{m+1}-1)B}{2^{m+1}}\right)$$

and (3.12) is proved. ∎

If $h(\frac{1}{2}) \leq \frac{1}{2}$, then the previous Theorem gives a sequence of interpolations of the Hermite-Hadamard inequality.

Corollary 3.10. *Suppose that the assumptions of Theorem* 3.9 *hold. Let* $h(\frac{1}{2}) \leq \frac{1}{2}$.

*If $\psi$ is increasing and $f$ is an $M_\varphi M_\psi$-$h$-convex integrable function such that $\psi\circ f\circ\varphi^{-1}$ is non-negative, then the following holds*

$$\frac{1}{4h^2(\frac{1}{2})}(\psi\circ f)\Big(M_\varphi\big(a,b;\frac{1}{2}\big)\Big) \leq l(2) \leq l(2^2) \leq \ldots \leq l(2^m) \leq \ldots$$
$$\leq \frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\varphi'(x)\,dx$$
$$\leq \ldots \leq L(2^m) \leq \ldots \leq L(2^2) \leq L(2)$$
$$\leq \left[\frac{1}{2}+h\Big(\frac{1}{2}\Big)\right]\Big[\psi(f(a))+\psi(f(b))\Big]\int_0^1 h(t)\,dt. \tag{3.13}$$

*Additionally, if $\int_0^1 h(t)\,dt \leq \frac{1}{2}$ and if $\psi\circ f\circ\varphi^{-1}$ is bounded on $\varphi([a,b])$, then*

$$\lim_{m\to\infty}(L(2^m)-l(2^m)) = 0 \tag{3.14}$$

*and*

$$\lim_{m\to\infty} l(2^m) = \frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\varphi'(x)\,dx = \lim_{m\to\infty} L(2^m). \tag{3.15}$$

Proof. If $h(\frac{1}{2}) \leq \frac{1}{2}$, then $\frac{1}{2h(\frac{1}{2})} \geq 1$ and $\frac{1}{2}+h\Big(\frac{1}{2}\Big) \leq 1$ and from (3.10) and (3.11) we have that for any $m \geq 1$

$$l(2^{m+1}) \geq l(2^m) \quad \text{and} \quad L(2^{m+1}) \leq L(2^m).$$

Hence, applying Theorem 3.8, Corollary 3.2 and above inequalities, we get (3.13).

If $h(\frac{1}{2}) \leq \frac{1}{2}$ and $\int_0^1 h(t)\,dt \leq \frac{1}{2}$, then $8h^2\Big(\frac{1}{2}\Big)\int_0^1 h(t)dt \leq 1$ and (3.14) follows from (3.12). The sequence $(l(2^m))_m$ is a non-decreasing sequence, bounded from above with $\frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\varphi'(x)\,dx$, so, it is convergent. Similarly, $(L(2^m))_m$ is convergent and from (3.14) and from inequality

$$l(2^m) \leq \frac{1}{\varphi(b)-\varphi(a)}\int_a^b \psi(f(x))\varphi'(x)\,dx \leq L(2^m)$$

we get (3.15). □

Under assumptions of Corollary 3.10 we conclude that the larger $m$ makes $l(2^m)$ and $L(2^m)$ closer to the integral mean of $\psi \circ f \circ \varphi^{-1}$. The behavior of convex functions involving dyadic partition is studied in [12]. Here we extend those results to a more general function class.

**Conclusion.** In this paper, we study Hermite-Hadamard-type inequalities for $M_\varphi M_\psi$-$h$-convex functions. Until now we have found similar results only for particular subclasses of the class of $M_\varphi M_\psi$-$h$-convex functions. The connection between $h$-convex function and $M_\varphi M_\psi$-$h$-convex function which is described in Proposition 2.1 has a crucial role in the proofs and the use of it makes proofs more elegant. It would be interesting to see how this method impacts the study of other properties of $M_\varphi M_\psi$-$h$-convex functions.

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## Hermite-Hadamardova nejednakost za $M_\varphi M_\psi$-$h$-konveksne funkcije i odgovarajuće interpolacije

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Sažetak. U članku se promatra Hermite-Hadamardova nejednakost za $M_\varphi M_\psi$-$h$-konveksne funkcije. Kao što je poznato, $M_{\varphi M_\psi}$-$h$-konveksnost generalizira nekoliko klasa funkcija kao što su harmonijski-$h$-konveksne funkcije, logaritamski $h$-konveksne, $(h,p)$-konveksne, $M_pA$-$h$-konveksne, $M_\varphi M_\psi$ konveksne funkcije i druge. Dokazane su nejednakosti Hermite-Hadamardovog tipa koje uključuju dva i više čvorova, a posebna je pažnja posvećena dijadskoj particiji intervala i profinjenju nejednakosti koja se javlja u tom slučaju.

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