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Vol. 29 = 564 (2025): 231-241
DOI: https://doi.org/10.21857/ydkx2cvdg9

ASYMPTOTIC BEHAVIOUR OF THE QUASI-ARITHMETIC MEANS

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Abstract. In this paper we study the asymptotic behaviour of the quasi-arithmetic means M_φ, for large values of its arguments. We extend and simplify known results form the literature. Asymptotic expansions of these means are derived under very weak assumptions on a given function φ. The coefficients in the asymptotic expansions are defined by recursive formulas, and the general algorithms for their calculation are then demonstrated on some interesting examples of means.

1. Introduction

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be an n-tuple of positive real numbers, and $\mathbf{e} = (1, 1, \ldots, 1)$. For a positive strictly monotone function φ, quasi-arithmetic or φ-mean is defined by

$$M_\varphi(\mathbf{a}) = \varphi^{-1}\left(\sum_{k=1}^{n} q_k \varphi(a_k)\right), \tag{1.1}$$

where weights $q_1, q_2, \ldots, q_n$ are non-negative real numbers with sum equal to 1.

The most important example of φ-mean is the power mean, obtained for $\varphi = x^r$, which we will denote by M_r. In particular, this class of means covers quadratic ($Q = M_2$), arithmetic ($A = M_1$), geometric ($G = M_0$) and harmonic ($H = M_{-1}$) mean.

The asymptotic behaviour of the quasi-arithmetic mean M_φ was studied by Boas and Brenner. They proved the following result.

Theorem A ([3]). *Suppose φ satisfies the following conditions*

(*i*) *φ is positive and strictly monotonic.*
(*ii*) *If φ increases to ∞ as $x \to \infty$, then $\varphi'(x)/\varphi(x) = o(1)$.*
(*iii*) *$\varphi'(x+y)/\varphi'(x) \to 1$ as $x \to \infty$, uniformly on finite positive y interval.*

2020 *Mathematics Subject Classification.* 41A60, 26E60.

Key words and phrases. Asymptotic expansion, quasi-arithmetic mean, digamma function.

(*iv*) *Inverse function* φ^{-1} *has the same properties as* φ.
Moreover, if $\varphi(x)$ *decreases to 0 as* $x \to \infty$*, then* φ *has the same properties as before, but* $\psi \equiv \varphi^{-1}$ *(which decreases to 0) should satisfy* $\psi'(x+y)/\psi'(x) \to 1$ *and that*

$$\psi'(x) \circ [\varphi(x)(1+\varepsilon)] = \psi' \circ [\varphi(x)](1+\eta).$$

Under this assumptions, it holds

$$M_\varphi(x\mathbf{e}+\mathbf{a}) - x \to M_1(\mathbf{a}).$$

The conditions of this theorem were constructed to cover the case of power means, where $\varphi = x^r$ and $r \neq 0$. Therefore, Theorem A improves the result of Hoehn and Niven [17] on power means, but it does not cover such a simple example as the geometric mean, for which one should take $\varphi(x) = \log x$. Namely, the inverse function does not satisfy assumptions of Theorem A.

In this paper we will discuss the existence of a complete asymptotic expansion of the quasi-arithmetic mean, in the form

$$M_\varphi(x\mathbf{e}+\mathbf{a}) \sim x \sum_{k=0}^{\infty} d_k x^{-k}. \tag{1.2}$$

A detailed analysis of bivariate means through their asymptotic expansions was given in a series of papers [4–6, 8–10, 12–15]. This approach was generalized to n-variable means in [11], where, in contrast to the expansion in terms of Bell polynomials by Abel and Ivan ([1]), the asymptotic expansion of the power mean M_r was obtained using a simple recursive algorithm.

Let

$$m_k := q_1 a_1^k + q_2 a_2^k + \ldots + q_n a_n^k, \quad k \in \mathbb{N}_0.$$

Theorem B ([11]). *General power mean has the following asymptotic expansion*

$$M_r(x\mathbf{e}+\mathbf{a}) = x \cdot \sum_{k=0}^{\infty} c_k(r) x^{-k},$$

where $c_0(r) = 1$ *and*

$$c_k(r) = \frac{1}{k} \sum_{j=1}^{k} \left[j\left(1+\frac{1}{r}\right) - k \right] \binom{r}{j} m_j c_{k-j}(r), \quad k \in \mathbb{N}. \tag{1.3}$$

Then the asymptotic expansions of the quadratic, arithmetic, geometric and harmonic mean are

$$\begin{aligned}
Q(x\mathbf{e}+\mathbf{a}) &= x + A(\mathbf{a}) + \tfrac{1}{2}(-A(\mathbf{a})^2 + Q(\mathbf{a})^2)x^{-1} + \mathcal{O}(x^{-2}),\\
A(x\mathbf{e}+\mathbf{a}) &= x + A(\mathbf{a}),\\
G(x\mathbf{e}+\mathbf{a}) &= x + A(\mathbf{a}) + \tfrac{1}{2}(A(\mathbf{a})^2 - Q(\mathbf{a})^2)x^{-1} + \mathcal{O}(x^{-2}),\\
H(x\mathbf{e}+\mathbf{a}) &= x + A(\mathbf{a}) - Q(\mathbf{a})^2 x^{-1} + \mathcal{O}(x^{-2}),
\end{aligned}$$

and in general

$$M_r(x\mathbf{e}+\mathbf{a}) = x + A(\mathbf{a}) - \tfrac{1}{2}(r-1)(A(\mathbf{a})^2 - Q(\mathbf{a})^2)x^{-1} + \mathcal{O}(x^{-2}).$$

The results of this paper are as follows. First, we will extend and simplify the theorem of Boas and Brenner, so that it also covers the geometric mean. In the main part, we will prove that the general quasi-arithmetic mean has a complete asymptotic expansion, if this is the case with the function φ. Consequently, we will also provide an efficient algorithm for computing the coefficients in the asymptotic expansion of the quasi-arithmetic mean. General results will be presented on some well-known examples of means and also applied to a function whose inverse is not known by an explicit formula.

2. Asymptotic behaviour

In this section, we give an improvement of Boas and Brenner's Theorem A.

Theorem 2.1. *Let φ be continuous, strictly monotone function such that*

1. $\varphi'(x+L)/\varphi'(x) \to 1$ as $x\to\infty$, for any fixed $L>0$,

2. φ' is strictly monotone on some interval $\langle b,\infty\rangle$.

Then

$$M_\varphi(x\mathbf{e}+\mathbf{a}) - x \to m_1, \qquad \textit{as } x\to\infty.$$

Proof. We shall prove some auxiliary results first. For a fixed L, let ξ be defined by

$$\varphi(x+L) = \varphi(x) + L\varphi'(x)(1+\xi). \tag{2.1}$$

By the mean value theorem we have

$$\varphi(x+L) = \varphi(x) + L\varphi'(x+\gamma), \quad 0<\gamma<L.$$

Combining these two equalities, we obtain

$$1+\xi = \frac{\varphi'(x+\gamma)}{\varphi'(x)}$$

which lies between $\frac{\varphi'(x)}{\varphi'(x)}$ and $\frac{\varphi'(x+L)}{\varphi'(x)}$ proving $\xi\to 0$ as $x\to\infty$.

Denote

$$J = \sum_{k=1}^{n} q_k\varphi(x+a_k).$$

Now we have

$$\begin{aligned} J &= \sum_{k=1}^{n} q_k\left[\varphi(x) + a_k\varphi'(x)(1+\xi_k)\right], \quad \xi_k\to 0 \\ &= \varphi(x) + \varphi'(x)\left[m_1 + \sum_{k=1}^{n} q_k a_k \xi_k\right] \\ &= \varphi(x) + \varphi'(x)m_1(1+\varepsilon), \quad \varepsilon\to 0. \end{aligned}$$

On the other side, by the mean value theorem we can define η to be such that

$$J = \varphi(x + m_1(1+\eta)) = \varphi(x) + m_1\varphi'(x)(1+\varepsilon).$$

The mean M_φ now equals

$$M_\varphi(x\mathbf{e}+\mathbf{a}) = \varphi^{-1}\left(\sum_{k=1}^{n} q_k\varphi(x+a_k)\right) = \varphi^{-1}(J) = x + m_1(1+\eta).$$

We want to prove that $\eta \to 0$ as $x \to \infty$.

Using the mean value theorem again and then the inverse function rule we obtain

$$\begin{aligned} x + m_1(1+\eta) &= \varphi^{-1}\left(\varphi(x) + m_1\varphi'(x)(1+\varepsilon)\right) \\ &= \varphi^{-1}(\varphi(x)) + m_1\varphi'(x)(1+\varepsilon)[\varphi^{-1}]'(\varphi(x) + m_1\varphi'(x)(1+\varepsilon)\vartheta) \\ &= x + m_1(1+\varepsilon)\frac{\varphi'(x)}{\varphi'(\varphi^{-1}(\varphi(x)+m_1\varphi'(x)(1+\varepsilon)\vartheta))} \end{aligned} \tag{2.2}$$

where $|\vartheta| < 1$.

We claim that for x large enough and some $L > m_1$ following bounds hold

$$x - L \le \varphi^{-1}(\varphi(x) + m_1\varphi'(x)(1+\varepsilon)\vartheta) \le x + L. \tag{2.3}$$

Since φ is monotone, (2.3) is equivalent with one set of the inequalities

$$\varphi(x-L) \underset{(\ge)}{\le} \varphi(x) + m_1\varphi'(x)(1+\varepsilon)\vartheta \underset{(\ge)}{\le} \varphi(x+L). \tag{2.4}$$

With similar reasoning as in (2.1), there exist ξ_+ and ξ_- such that

$$\varphi(x \pm L) = \varphi(x) \pm L\varphi'(x)(1+\xi_\pm), \quad \xi_\pm \to 0 \quad \text{as } x \to \infty. \tag{2.5}$$

By combining (2.4) with (2.5) we obtain

$$-L(1+\xi_-) \le m_1(1+\varepsilon)\vartheta \le L(1+\xi_+)$$

which is true for x large enough.

Returning to (2.2) with inequalities (2.3) and using conditions (1) and (2) we see that $\eta \to 0$ and therefore theorem is proved. ∎

REMARK 2.2. A similar result in a more general setting was obtained in [16] where uniform convergence in (1) was required.

3. Complete asymptotic expansion of Quasi-arithmetic mean

For a given function φ in the definition of the quasi-arithmetic mean (1.1), our aim is to find the asymptotic expansion (1.2). This problem is equivalent with solving the equation

$$\varphi\left(x\sum_{k=0}^{\infty} d_k x^{-k}\right) = \sum_{i=1}^{n} q_i\varphi(x+a_i) \tag{3.1}$$

in terms of coefficients (d_k). A general problem of this type, i.e. solving the equation $B(A(x)) = C(x)$ for known functions B and C in terms of asymptotic series, was studied in [14]. It was shown that such equations can be solved in the form of recursive relations, for a wide class of functions φ. It is sufficient to require that φ possesses an asymptotic expansion, but the algorithm depends on the assumed form of that asymptotic representation. More precisely, depending on whether it contains a logarithm or not, we will observe two cases.

3.1. *Ordinary case.* Let φ have the asymptotic expansion of the form

$$\varphi(x) \sim x^u \sum_{k=0}^{\infty} b_k x^{-k}, \tag{3.2}$$

where $b_0 \neq 0$. Since the quasi-arithmetic mean M_φ is invariant on non-trivial affine transformation of function φ, we may assume that $b_0 = 1$.

Regarding the expansion on the right side of (3.1), with rearranging of sums we obtain

$$\begin{aligned} C(x) &= \sum_{i=1}^{n} q_i \varphi(x + a_i) \sim \sum_{i=1}^{n} q_i \sum_{j=0}^{\infty} b_j (x + a_i)^{u-j} = \sum_{i=1}^{n} q_i \sum_{j=0}^{\infty} b_j x^{u-j} \left(1 + \frac{a_i}{x}\right)^{u-j} \\ &= x^u \sum_{i=1}^{n} q_i \sum_{j=0}^{\infty} b_j \sum_{k=0}^{\infty} \binom{u-j}{k} a_i^k x^{-(j+k)} = x^u \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{u-j}{k} b_j m_k x^{-(j+k)} \\ &= x^u \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} \binom{u-j}{k-j} b_j m_{k-j} \right] x^{-k}. \end{aligned}$$

It can easily be seen that all assumptions of Theorem 2.3 from [14] are satisfied which provides recursive formula for d_k.

Theorem 3.1. *Let the mean M_φ be defined by* (1.1) *and let φ have the asymptotic expansion* (3.2) *with $u \neq 0$. Then the coefficients d_k in the asymptotic expansion* (1.2) *of the mean M_φ are given by the expression*

$$\begin{aligned} d_0 &= 1, \\ d_k &= -\frac{1}{u} \left[\sum_{j=1}^{k} b_j P_{k-j}(u-j) + \frac{1}{k} \sum_{j=1}^{k-1} [j(1+u) - k] d_j P_{k-j}(u) \right. \\ &\qquad \left. - \sum_{j=0}^{k} \binom{u-j}{k-j} b_j m_{k-j} \right], \quad k \in \mathbb{N}, \end{aligned}$$

where coefficients $P_k(u)$ are defined by formula (3.3) *in Lemma 3.2.*

The coefficients in the asymptotic representation of the r-th power of the asymptotic series whose coefficients are given by the sequence $\mathbf{d} = (d_k)_{k \in \mathbb{N}_0}$

can be obtained by the recursive formula. The following lemma about such functional transformation was used to obtain the main result.

LEMMA 3.2 ([8]). *Let $d_0 \neq 0$ and $D(x)$ be a function with asymptotic expansion (as $x \to \infty$)*

$$D(x) \sim \sum_{k=0}^{\infty} d_k x^{-k}.$$

Then for all real numbers r it holds

$$[D(x)]^r \sim \sum_{k=0}^{\infty} P_k(r) x^{-k},$$

where

$$\begin{aligned} &P_0(r) = d_0^r, \\ (3.3) \qquad &P_k(r) = \frac{1}{kd_0} \sum_{j=1}^{k} [j(1+r)-k] a_j P_{k-j}(r), \quad k \in \mathbb{N}. \end{aligned}$$

By the Theorem 3.1 we obtain first few coefficients d_k:

$$\begin{aligned} d_0 &= 1, \\ d_1 &= m_1, \\ d_2 &= \tfrac{1}{2}(u-1)(m_2 - m_1^2), \\ d_3 &= -\frac{u-1}{2u}(m_2 - m_1^2)b_1 + \tfrac{1}{6}(u-1)\big((2u-1)m_1^3 - 3(u-1)m_1 m_2 + (u-2)m_3\big), \\ d_4 &= \frac{(u-1)^2}{2u^2}(m_2 - m_1^2)b_1^2 - \frac{u-2}{2}(m_2 - m_1^2)b_2 \\ &\quad - \frac{u-1}{3u}\big((2u-1)m_1^3 - 3(u-1)m_1 m_2 + (u-2)m_3\big)b_1 \\ &\quad - \tfrac{1}{24}(u-1)\big((2u-1)(3u-1)m_1^4 - 6(u-1)(2u-1)m_1^2 m_2 \\ &\quad + 4(u-1)(u-2)m_1 m_3 + 3(u-1)^2 m_2^2 - (u-2)(u-3)m_4\big). \end{aligned}$$

EXAMPLE 3.3. One of the special cases of this mean is the power mean obtained for $f(x) = x^r$. In this case u is equal to r, $b_0 = 1$ and $b_k = 0$ for all $k \geq 1$. Expression in the Theorem 3.1 reduces to

$$(3.4) \qquad d_0 = 1, \qquad d_k = -\frac{1}{k} \sum_{j=1}^{k-1} \left[j\left(1 + \frac{1}{r}\right) - k \right] d_j P_{k-j}(r) + \binom{r}{k} m_k.$$

We obtained a recursive formula for the coefficients in the asymptotic expansion of the weighted power mean. Although it seems different from (1.3), it leads to the same coefficients.

EXAMPLE 3.4. Let $\varphi(x) = x + e^{-x}$. Then the corresponding asymptotic expansion equals $\varphi(x) \sim x$. Therefore, $u = 1$, $b_0 = 1$ and $b_k = 0$ for $k \geq 1$.

We obtain a formula similar to (3.4) with $r = 1$ which by direct computation leads to

$$M_\varphi(x\mathbf{e} + \mathbf{a}) \sim x + m_1.$$

This is not the coincidence since φ has the same asymptotic expansion as the identity function.

EXAMPLE 3.5. Let $\varphi(x) = x^2 - x$. Then using Theorem 3.1 we have

$$\begin{aligned} M_\varphi(x\mathbf{e} + \mathbf{a}) \sim x + m_1 &+ \tfrac{1}{2}\left(m_2 - m_1^2\right)x^{-1} - \tfrac{1}{4}\left(m_2 - m_1^2\right)(2m_1 - 1)\,x^{-2} \\ &+ \tfrac{1}{8}\left(m_2 - m_1^2\right)\left(m_1(5m_1 - 4) - m_2 + 1\right)x^{-3} \\ &- \tfrac{1}{16}\left(m_2 - m_1^2\right)(2m_1 - 1)\left(m_1(7m_1 - 4) - 3m_2 + 1\right)x^{-4} + \cdots \end{aligned}$$

Based on the fact the mean of identical numbers is equal to their common value, i.e. $x+a = M_\varphi(x\mathbf{e}+a\mathbf{e})$, and recursion from the Theorem 3.1, we obtain the result analogous to Theorem 2.2. from [11].

THEOREM 3.6. *The coefficient* d_k *(*$k \geq 2$*), can be considered as polynomial in the variables* $(b_0, \dots, b_{k-2})$ *whose coefficients have the following form:*

$$\sum_{\substack{\alpha_1,\alpha_2,\dots,\alpha_j \geq 0 \\ \alpha_1 + 2\alpha_2 + \cdots + j\alpha_j = j}} q_{\alpha_1,\dots,\alpha_j}(u)\, m_1^{\alpha_1} \cdots m_j^{\alpha_j},$$

where

$$\sum_{\substack{\alpha_1,\alpha_2,\dots,\alpha_j \geq 0 \\ \alpha_1 + 2\alpha_2 + \cdots + j\alpha_j = j}} q_{\alpha_1,\dots,\alpha_j}(u) = 0, \quad 2 \leq j \leq k.$$

The aforementioned Theorem 2.2. from [11] was later used in [7] for computing the expectations of coefficients in the asymptotic expansion of the large data power means.

3.2. *Logarithmic case.* Consider the geometric mean $G(\mathbf{a}) = \Pi_{i=1}^n a_i^{q_i}$. It is the limit case of the r-th power mean as $r \to 0$ but also the quasi-arithmetic mean for $\varphi(x) = \log x$. We will now cover this case as well. First, we recall the following result about the logarithm of an asymptotic series.

LEMMA 3.7 ([8]). *Let* $d_0 \neq 0$ *and*

$$D(x) \sim \sum_{k=0}^{\infty} d_k x^{-k}$$

be a given asymptotic expansion. Then its logarithm has the asymptotic expansion of the following form

$$\ln(D(x)) \sim \sum_{k=1}^{\infty} L_k x^{-k}$$

where

$$L_k = \frac{d_k}{d_0} - \frac{1}{kd_0}\sum_{j=1}^{k-1} jL_j d_{k-j}, \qquad k \geq 1. \tag{3.5}$$

THEOREM 3.8. *Let φ have the asymptotic expansion*

$$\varphi(x) \sim \log x + x^{-1}\sum_{k=0}^{\infty} b_k x^{-k}. \tag{3.6}$$

Then the quasi-arithmetic mean M_φ defined by (1.1) has the asymptotic expansion (1.2) in which $d_0 = 1$ and for $k \geq 1$ the coefficients d_k are calculated by following recursive formula:

$$d_k = \sum_{j=0}^{k-1} b_{k-1-j}\left[\binom{j-k}{j} m_j - P_j(j-k)\right] + \frac{(-1)^{k+1}}{k} m_k + \frac{1}{k}\sum_{j=1}^{k-1} jL_j d_{k-j},$$

where P_k and L_k are given by (3.3) and (3.5) respectively.

PROOF. On the left hand side of (3.1) we have

$$\begin{aligned}
\varphi\left(x\sum_{k=0}^{\infty} d_k x^{-k}\right) &= \log\left(x\sum_{k=0}^{\infty} d_k x^{-k}\right) + \sum_{k=0}^{\infty} b_k \left(x\sum_{j=0}^{\infty} d_j x^{-j}\right)^{-1-k} \\
&= \log x + \log\left(\sum_{k=0}^{\infty} d_k x^{-k}\right) + \sum_{k=0}^{\infty} b_k x^{-1-k}\left(\sum_{j=0}^{\infty} d_j x^{-j}\right)^{-1-k} \\
&= \log x + \sum_{k=1}^{\infty} L_k x^{-k} + \sum_{k=0}^{\infty} b_k x^{-1-k}\sum_{j=0}^{\infty} P_j(-1-k)x^{-j} \\
&= \log x + \sum_{k=1}^{\infty} L_k x^{-k} + \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} b_{k-1-j}P_j(-k+j)x^{-k}.
\end{aligned}$$

On the other side

$$\begin{aligned}
\sum_{i=1}^{n} q_i f(x+a_i) &= \sum_{i=1}^{n} q_i \left[\log(x+a_i) + \sum_{j=0}^{\infty} b_j (x+a_i)^{-1-j}\right] \\
&= \sum_{i=1}^{n} q_i \left[\log x + \log\left(1+\frac{a_i}{x}\right) + \sum_{j=0}^{\infty} b_j x^{-1-j}\left(1+\frac{a_i}{x}\right)^{-1-j}\right] \\
&= \log x + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} m_k x^{-k} + \sum_{j=0}^{\infty} b_j x^{-1-j}\sum_{k=0}^{\infty}\binom{-1-j}{k} m_k x^{-k}
\end{aligned}$$

$$= \log x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} m_k x^{-k} + \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} b_{k-1-j} \binom{-k+j}{j} m_j x^{-k}.$$

By equating the coefficients by x^k on both sides, we obtain

$$L_k + \sum_{j=0}^{k-1} b_{k-1-j} P_j(-k+j) = \frac{(-1)^{k+1}}{k} m_k + \sum_{j=0}^{k-1} b_{k-1-j} \binom{-k+j}{j} m_j x^{-k}.$$

Coefficient d_k appears only in

$$L_k = d_k - \frac{1}{k} \sum_{j=1}^{k-1} j L_j d_{k-j}.$$

Thus we have proved the theorem. □

REMARK 3.9. If $\varphi(x) \sim a \log x + x^{-1} \sum_{k=0}^{\infty} b_k x^{-k}$, $a \neq 0$, then coefficients in the asymptotic expansion of M_φ are obtained form Theorem 3.8 with substitution $b_k \mapsto \frac{b_k}{a}$.

According to the Theorem 3.8 the first few coefficients d_k are:

$$\begin{aligned}
d_0 &= 1, \\
d_1 &= m_1, \\
d_2 &= -\tfrac{1}{2}(m_2 - m_1^2), \\
d_3 &= \tfrac{1}{2}\left(m_2 - m_1^2\right) b_0 + \tfrac{1}{6}\left(m_1^3 - 3m_2 m_1 + 2m_3\right), \\
d_4 &= \tfrac{1}{2}\left(m_2 - m_1^2\right) b_0^2 + \tfrac{1}{3}\left(-m_1^3 + 3m_2 m_1 - 2m_3\right) b_0 + 2\left(m_2 - m_1^2\right) b_1 \\
&\quad + \tfrac{1}{24}\left(m_1^4 - 6m_2 m_1^2 + 8m_3 m_1 + 3m_2^2 - 6m_4\right).
\end{aligned}$$

EXAMPLE 3.10. In the case of the geometric mean, $\varphi(x) = \log x$ and therefore all coefficients b_k in (3.6) are equal to zero. Coefficients d_k in the asymptotic expansion (1.2) of the geometric mean $G(x\mathbf{e} + \mathbf{a}) = M_\varphi(x\mathbf{e} + \mathbf{a})$ can be obtained from Theorem 3.8:

$$\begin{aligned}
d_0 &= 1, \\
d_k &= \frac{(-1)^{k+1}}{k} m_k + \frac{1}{k} \sum_{j=1}^{k-1} j L_j d_{k-j}, \quad k \in \mathbb{N}.
\end{aligned}$$

These coefficients are the same as in [11].

EXAMPLE 3.11. Digamma function has the following asymptotic expansion ([2]):

$$\psi(x) \sim \log x - \frac{1}{2x} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} x^{-2k},$$

where B_k denotes Bernoulli numbers. Then according to the Theorem 3.8 we have

$$\begin{aligned} M_\psi(x\mathbf{e}+\mathbf{a}) \sim x &+ m_1 + \tfrac{1}{2}(m_1^2 - m_2)x^{-1} \\ &+ \tfrac{1}{12}\big(m_1(m_1(2m_1+3) - 6m_2) - 3m_2 + 4m_3\big)x^{-2} \\ &+ \tfrac{1}{24}\big(m_1^4 + 4m_1^3 + (1-6m_2)m_1^2 + (8m_3 - 12m_2)m_1 \\ &\quad + m_2(3m_2 - 1) + 8m_3 - 6m_4\big)x^{-3} + \dots \end{aligned}$$

References

[1] U. Abel and M. Ivan, *A complete asymptotic expansion of power means*, J. Math. Anal. Appl. **325** (2007), 554–559.

[2] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards Applied Mathematics Series, vol 55, Washington, D.C., 1972.

[3] R. P. Boas and J. L. Brenner, *The asymptotic behavior of inhomogeneous means*, J. Math. Anal. Appl. **123** (1987), 262–264.

[4] T. Burić, *Asymptotic analysis of iterative power means*, J. Math. Anal. Appl. **433** (2016), 701–705.

[5] T. Burić and N. Elezović, *Asymptotic expansion of the arithmetic-geometric mean and related inequalities*, J. Math. Inequal. **9** (2015), 1181–1190.

[6] T. Burić and N. Elezović, *Computation and analysis of the asymptotic expansions of the compound means*, Appl. Math. Comput. **303** (2017), 48–54.

[7] T. Burić, N. Elezović and L. Mihoković, *Expectations of large data means*, J. Math. Inequal. **17** (2023), 403–418.

[8] C.-P. Chen, N. Elezović and L. Vukšić, *Asymptotic formulae associated with the Wallis power function and digamma function*, J. Class. Anal. **2** (2013), 151–166.

[9] N. Elezović, *Asymptotic expansions of gamma and related functions, binomial coefficients, inequalities and means*, J. Math. Inequal. **9** (2015), 1001–1054.

[10] N. Elezović, *Asymptotic inequalities and comparison of classical means*, J. Math. Inequal. **9** (2015), 177–196.

[11] N. Elezović and L. Mihoković, *Asymptotic behavior of power means*, Math. Inequal. Appl. **19** (2016), 1399–1412.

[12] N. Elezović and L. Vukšić, *Asymptotic expansions and comparison of bivariate parameter means*, Math. Inequal. Appl. **17** (2014), 1225–1244.

[13] N. Elezović and L. Vukšić, *Asymptotic expansions of bivariate classical means and related inequalities*, J. Math. Inequal. **8** (2014), 707–724.

[14] N. Elezović and L. Vukšić, *Asymptotic expansions of integral means and applications to the ratio of gamma functions*, Appl. Math. Comput. **235** (2014), 187–200.

[15] N. Elezović and L. Vukšić, *Neuman-Sándor mean, asymptotic expansions and related inequalities*, J. Math. Inequal. **9** (2015), 1337–1348.

[16] P. Gigante, *On the asymptotic behaviour of ϕ-means*, J. Math. Anal. Appl. **192** (1995), 915–919.

[17] L. Hoehn and I. Niven, *Averages on the move*, Math. Mag. **58** (1985), 151–156.

Asimptotsko ponašanje kvaziaritmetičkih sredina

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SAŽETAK. U ovom radu proučavamo asimptotsko ponašanje kvaziaritmetičkih sredina M_φ, za velike vrijednosti argumenata. Proširujemo i pojednostavljujemo poznate rezultate iz literature. Asimptotski razvoji ovih sredina izvode se pod vrlo slabim pretpostavkama o danoj funkciji φ. Koeficijenti u asimptotskim razvojima definirani su rekurzivnim formulama, a općeniti algoritmi za njihov izračun zatim su demonstrirani na nekim zanimljivim primjerima sredina.

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Received: 2.10.2023.
Revised: 14.10.2023.
Accepted: 14.11.2023.