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# OPTIMAL CONTROL OF A FRICTIONAL CONTACT PROBLEM FOR LOCKING MATERIALS

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Abstract. In this paper, we consider a bilateral contact with Tresca's friction law between a locking material and a rigid foundation. The goal is to study an optimal control problem which consists of leading the stress tensor as close as possible to a given target, by acting with a control on the boundary of the body. We state an optimal control problem that admits at least one solution. We also introduce the penalized and regularized optimal control problem for which we study the convergence when the penalization and regularization parameter tends to zero.

## 1. Introduction

A locking material is a material which is characterized by the fact that it is deformed under the effect of an external force, but the deformation cannot continue when it reaches a certain value "$M_L$". After that, for any external force, the material cannot be deformed. The material is elastic if the deformation remains bounded. It returns back to its initial shape if we stop to exercise any external force on it. Locking materials are part of a class of hyperelastic materials in which the strain tensor is constrained to stay in a given convex set. The study of elastic materials with locking effects was first introduced in [19, 20, 21]. There, the constitutive law of such materials was derived and different mechanical interpretations have been presented. The theoretical study of variational problems of locking materials was introduced in [6, 7]. Optimal control governed by variational inequalities has been studied in several articles, see for instance [2, 3, 4, 8, 9, 11, 13, 16, 17, 22]. Recall that the optimal control of contact problems for elastic materials was studied in [1, 5, 12, 14, 15, 26, 27] and the references therein.

In this paper, we study the optimal control of a contact problem for nonlinear elastic locking materials. The contact is assumed to be static and it is

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described by Tresca's friction law between a locking body and a rigid foundation. Our control problem concerns the acting of a surface load in a part of the boundary in order to approach a given target by the stress tensor. First, we derive a variational formulation of the problem and establish the existence and uniqueness result (Problem $P_2$). Next, we define the optimal control problem related to this model (Problem C1), which consists of minimizing a cost functional. We prove the existence of a solution of problem C1, and we define a penalized and regularized problem (Problem $P_\delta$), whose solution converges to the solution of Problem $P_2$. Finally, we introduce a penalized and regularized optimal control problem (Problem C2), and we prove its convergence to the optimal control problem C1 when $\delta$ tends to zero.

The rest of this document is structured as follows. In Section 2 we introduce some notation, describe the mechanical problem, and prove its weak solvability, Theorem 2.1. In Section 3, we state the optimal control C1 and prove that it has at least one solution, Theorem 3.1. In Section 4, we prove that the solution of the penalized and regularized problem converges strongly to the solution of Problem $P_2$ (Theorem 4.3). In Section 5, we prove a convergence result of the penalized and regularized optimal control problem C2, Theorem 5.2.

## 2. The contact problem and its weak solvability

We consider a locking body which initially occupies a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with a sufficiently smooth boundary $\partial\Omega = \Gamma$ partitioned into three disjoint measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that $meas\,(\Gamma_1) > 0$. The body is clamped on $\Gamma_1$ and then the displacement vanishes there. It is acted upon by a volume force of density $\varphi_0$ in $\Omega$ and a surface traction of density $\varphi$ on $\Gamma_2$. On $\Gamma_3$ the body is in bilateral contact following Tresca's friction law with a rigid foundation.Thus, the classical formulation of the mechanical problem is written as follows.

**Problem $P_1$.** Find a displacement field $u : \Omega \to \mathbb{R}^d$ such that

$$div\sigma\,(u) + \varphi_0 = 0 \text{ in } \Omega, \tag{2.1}$$

$$\sigma\,(u) \in \mathcal{F}\varepsilon\,(u) + \partial I_B\,(\varepsilon\,(u)) \text{ in } \Omega, \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1, \tag{2.3}$$

$$\sigma\,(u)\,\nu = \varphi \quad \text{on } \Gamma_2, \tag{2.4}$$

$$\left.\begin{array}{l} u_\nu = 0 \\ |\sigma_\tau\,(u)| \le g \\ |\sigma_\tau\,(u)| < g \Longrightarrow u_\tau = 0 \\ |\sigma_\tau\,(u)| = g \Longrightarrow \exists\lambda \ge 0 \text{ such that } \sigma_\tau\,(u) = -\lambda u_\tau \end{array}\right\} \text{ on } \Gamma_3. \tag{2.5}$$

Here, we denote by $\sigma = \sigma(u)$ the stress field and $\varepsilon(u)$ the strain tensor. Equation (2.1) represents the equilibrium equation. Equation (2.2) represents the elastic constitutive law where $\mathcal{F}$ is a given nonlinear function and $I_B$ is the indicator function of the set defined by

$$B = \{\xi \in S_d;\ |\xi| \leq M_L\}$$

such that

$$\begin{cases} I_B(\xi) = 0, & \text{if } \xi \in B, \\ I_B(\xi) = +\infty, & \text{if } \xi \notin B \end{cases} \quad \text{for } \xi \in S_d.$$

where $S_d$ is the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$. Recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $S_d$ are given by

$$\begin{aligned} u.v &= u_i v_i, \quad |v| = (v.v)^{\frac{1}{2}} \quad \forall u, v \in \mathbb{R}^d, \\ \sigma.\tau &= \sigma_{ij}\tau_{ij}, \ |\tau| = (\tau.\tau)^{\frac{1}{2}} \quad \forall \sigma, \tau \in S_d, \end{aligned}$$

Here and below, the indices $i$ and $j$ run between 1 and $d$, and the summation convention over repeated indices is adopted.

Equations (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which $\nu$ denotes the unit outward normal vector on $\Gamma$ and $\sigma(u)\nu$ represents the normal stress vector. Finally, (2.5) represents the bilateral contact with Tresca's friction, law where $g$ is a given friction bound.

Now, to proceed with the variational formulation, we need the following function spaces:

$$H = \left(L^2(\Omega)\right)^d, \ H_1 = \left(H^1(\Omega)\right)^d, \ Q = \left\{\tau = (\tau_{ij});\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\}.$$

Note that $H$ and $Q$ are real Hilbert spaces endowed with the respective canonical inner products:

$$(u, v)_H = \int_\Omega u_i v_i dx, \quad (\sigma, \tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \text{ where } \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) \text{ and } u_{i,j} = \frac{\partial u_i}{\partial x_j};$$

$div\sigma = (\sigma_{ij,j})$ is the divergence of $\sigma$. For every element $v \in H_1$, we denote by $v_\nu$ and $v_\tau$ the normal and the tangential components of $v$ on the boundary $\Gamma$ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

Also, for a regular function (say $C^1$) $\sigma \in Q$, we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu$$

and we recall that the following Green's formula holds:

$$(\sigma, \varepsilon(v))_Q + (div\sigma, v)_H = \int_\Gamma \sigma\nu.vda \quad \forall v \in H_1,$$

where $da$ is the surface measure element.

Let $V$ be the closed subspace of $H_1$ defined by

$$V = \{v \in H_1;\ v = 0 \text{ on } \Gamma_1,\, v_\nu = 0 \text{ on } \Gamma_3\},$$

and the closed convex subset of $V$

$$K = \{v \in V;\, \varepsilon(v(x)) \in B \ a.e.\ x \in \Omega\}.$$

Also, we define by $\langle .,. \rangle$ the duality pairing between $V'$ and $V$. Next, since $meas(\Gamma_1) > 0$, the following Korn's inequality holds [10],

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.6}$$

where the constant $c_\Omega > 0$ depends only on $\Omega$ and $\Gamma_1$. We equip $V$ with the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q$$

and $\|.\|_V$ is the associated norm. It follows from Korn's inequality (2.6) that the norms $\|.\|_{H_1}$and $\|.\|_V$ are equivalent on $V$. Then $(V, \|.\|_V)$ is a real Hilbert space. Moreover, by Sobolev's trace theorem, there exists $d_\Omega > 0$ which depends only on the domain $\Omega$, $\Gamma_1$ and $\Gamma_3$ such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.7}$$

We assume that the body forces and surface tractions have the regularity

$$\varphi_0 \in H, \qquad \varphi \in \left(L^2(\Gamma_2)\right)^d \tag{2.8}$$

and we define the functional $j : V \to \mathbb{R}_+$ by

$$j(v) = \int_{\Gamma_3} g\,|v_\tau|\,da,$$

where $g$ is assumed to satisfy

$$g \in L^\infty(\Gamma_3) \text{ and } g \geq 0 \ a.e. \text{ on } \Gamma_3. \tag{2.9}$$

Next, in the study of Problem $P_1$ we assume that the nonlinear elasticity operator $\mathcal{F}$ satisfies

$$(2.10)\quad \begin{cases} (a)\ \mathcal{F}:\Omega\times S_d\to S_d; \\ (b) \text{ there exists } M>0 \text{ such that} \\ \left|\mathcal{F}(x,\varepsilon_1)-\mathcal{F}(x,\varepsilon_2)\right|\le M\left|\varepsilon_1-\varepsilon_2\right|,\ \forall\varepsilon_1,\varepsilon_2\in S_d, \\ \qquad a.e.\ x\ \in\Omega; \\ (c) \text{ there exists } m>0 \text{ such that} \\ \qquad \left(\mathcal{F}(x,\varepsilon_1)-\mathcal{F}(x,\varepsilon_2)\right).\left(\varepsilon_1-\varepsilon_2\right)\ge m\left|\varepsilon_1-\varepsilon_2\right|^2, \\ \qquad \forall\varepsilon_1,\varepsilon_2\in S_d,\ a.e.\ x\ \in\Omega; \\ (d) \text{ the mapping } x\to\mathcal{F}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for any } \varepsilon\in S_d; \\ (e)\ \mathcal{F}(x,0)=0 \text{ for } a.e.\ x\in\Omega. \end{cases}$$

Examples of nonlinear constitutive law that satisfy (2.10) can be found in [18, 25].

Now, we derive the variational formulation of Problem $P_1$. To this end, let $(u,\sigma(u))$ be a pair of smooth functions which satisfies (2.1) – (2.5). Let $v\in V$. Multiplying the equilibrium equation (2.1) by $v-u$ and using the Green formula, we deduce that

$$(\sigma(u),\varepsilon(v)-\varepsilon(u))_Q=(\varphi_0,v-u)_H+\int_\Gamma\sigma(u)\,\nu.(v-u)\,da.$$

Using the boundary conditions (2.3) and (2.4), we have

$$(2.11)\quad \begin{aligned}&(\sigma(u),\varepsilon(v)-\varepsilon(u))_Q\\ &=(\varphi_0,v-u)_H+(\varphi,v-u)_{(L^2(\Gamma_2))^d}+\textstyle\int_{\Gamma_3}\sigma_\tau(u).(v_\tau-u_\tau)\,da.\end{aligned}$$

On the other hand, condition (2.5) implies

$$(2.12)\quad \int_{\Gamma_3}\sigma_\tau(u).(v_\tau-u_\tau)\,da\ge\int_{\Gamma_3}g\left(|u_\tau|-|v_\tau|\right)da.$$

From the constitutive law (2.2), we have

$$\sigma(u)=\mathcal{F}\varepsilon(u)+\varsigma(u)\ \text{ and }\ \varsigma(u)\in\partial I_B(\varepsilon(u))\ \text{ in }\Omega.$$

The latter, for $v,u\in K$, implies

$$\varsigma(u).(\varepsilon(v)-\varepsilon(u))\le I_B(\varepsilon(v))-I_B(\varepsilon(u))=0 \text{ in }\Omega.$$

Hence, we obtain

$$(2.13)\quad (\sigma(u),\varepsilon(v)-\varepsilon(u))_Q\le(\mathcal{F}\varepsilon(u),\varepsilon(v)-\varepsilon(u))_Q.$$

We define the operator $A:V\to V$ by

$$(2.14)\quad (Au,v)_V=(\mathcal{F}\varepsilon(u),\varepsilon(v))_Q,\ \forall u,v\in V.$$

Inserting (2.13) and (2.12) in (2.11) and taking into account (2.14), we obtain the following variational formulation of Problem $P_1$.

**Problem** $P_2$**.** Find $u \in K$ such that

$$(Au, v-u)_V + j(v) - j(u) \geq (\varphi_0, v-u)_H + (\varphi, v-u)_{(L^2(\Gamma_2))^d} \quad \forall v \in K. \tag{2.15}$$

THEOREM 2.1. *Assume* (2.8), (2.9) *and* (2.10). *Then, there exists a unique solution of Problem* $P_2$.

PROOF. By (2.10), the operator $A$ is Lipschitz continuous and strongly monotone; using (2.9), the functional $j$ is proper, convex and lower semicontinuous. Then, by using (2.8), since $K$ is a non-empty closed convex, it follows (see [23]) that the inequality (2.15) has a unique solution. ∎

## 3. The optimal control problem

We now suppose that $\varphi_0 \in H$ is fixed and consider the following state variational problem.

**Problem Q1.** For $\varphi \in \left(L^2(\Gamma_2)\right)^d$ (called control), find $u \in K$ such that

$$(Au, v-u)_V + j(v) - j(u) \geq (\varphi_0, v-u)_H + (\varphi, v-u)_{(L^2(\Gamma_2))^d} \quad \forall v \in K. \tag{3.1}$$

Following the existence and uniqueness of Problem $P_2$, we deduce that for every control $\varphi \in \left(L^2(\Gamma_2)\right)^d$, the state variational problem Q1 has a unique solution $u \in K$.

For $\alpha, \beta > 0$ and $u_d \in K$, we define the cost functional

$$\mathcal{L} : V \times \left(L^2(\Gamma_2)\right)^d \to \mathbb{R}_+,$$

by

$$\mathcal{L}(u, \varphi) = \alpha \|u - u_d\|_V^2 + \beta \|\varphi\|_{(L^2(\Gamma_2))^d}^2, \tag{3.2}$$

We have that $\sigma_d = \sigma(u_d) = \mathcal{F}\varepsilon(u_d)$, then for $u \in K$, we have $\sigma(u) = \mathcal{F}\varepsilon(u)$, and $\|\sigma(u) - \sigma(u_d)\|_Q \leq M \|u - u_d\|_V$ ; so $\sigma(u)$ is a close to $\sigma(u_d)$.

Next, we define the set of admissible pairs $U_{ad}$ as

$$U_{ad} = \left\{(u, \varphi) \in K \times \left(L^2(\Gamma_2)\right)^d, \text{ such that } (3.1) \text{ is satisfied}\right\},$$

and we consider the optimal control problem below.

**Problem C1.** Find $(u^*, \varphi^*) \in U_{ad}$ such that

$$\mathcal{L}(u^*, \varphi^*) = \min_{(u,\varphi) \in U_{ad}} \mathcal{L}(u, \varphi).$$

THEOREM 3.1. *Assume* (2.8), (2.9) *and* (2.10) ($c$). *Then Problem C1 has at least one solution.*

PROOF. Take $v = 0_V$ in (3.1), using (2.7) and (2.10) $(c)$, we deduce that the solution $u$ of Problem Q1 is bounded in $V$ as

$$\|u\|_V \leq \frac{c_0}{m}\left(\|\varphi_0\|_H + d_\Omega \|\varphi\|_{(L^2(\Gamma_2))^d}\right),$$

where $c_0 > 0$. This inequality implies that

$$0 \leq \inf_{(u,\varphi)\in U_{ad}} \mathcal{L}(u,\varphi) < +\infty.$$

Then, there exists a sequence $(u^n, \varphi^n) \subset U_{ad}$ such that

$$\mathcal{L}(u^n,\varphi^n) \to \inf_{(u,\varphi)\in U_{ad}} \mathcal{L}(u,\varphi) \text{ as } n \to +\infty.$$

The sequence $(u^n, \varphi^n)$ is bounded in $V \times \left(L^2(\Gamma_2)\right)^d$, so there exists an element

$$(u^*, \varphi^*) \in V \times \left(L^2(\Gamma_2)\right)^d$$

such that passing to a subsequence still denoted by $(u^n, \varphi^n)$, we have

$$\text{(3.3)} \qquad \begin{cases} (a) \ u^n \to u^* \text{ weakly in } V, \quad \text{as } n \to +\infty, \\ (b) \ \varphi^n \to \varphi^* \text{ weakly in } \left(L^2(\Gamma_2)\right)^d \text{ as } n \to +\infty,. \end{cases}$$

But for the remainder of the proof, we had to show that

$$\text{(3.4)} \qquad u^n \to u^* \text{ strongly } in\ V\ as\ n \to +\infty.$$

In fact, as $(u^n, \varphi^n) \in U_{ad}$, $u^n$ satisfies the inequality:

$$\text{(3.5)} \qquad \begin{aligned} &(Au^n, v - u^n)_V + j(v) - j(u^n) \\ &\qquad \geq (\varphi_0, v - u^n)_H + (\varphi^n, v - u^n)_{(L^2(\Gamma_2))^d} \quad \forall v \in K. \end{aligned}$$

Using (2.10) $(c)$ and (3.5), we deduce that

$$\text{(3.6)} \qquad \begin{cases} m\|u^n - u^*\|_V^2 \leq (Au^n - Au^*, u^n - u^*)_V \\ \leq (Au^n, u^n - u^*)_V - (Au^*, u^n - u^*)_V \\ \leq (Au^*, u^* - u^n)_V + j(u^*) - j(u^n) \\ \ + (\varphi_0, u^n - u^*)_H + (\varphi^n, u^n - u^*)_{(L^2(\Gamma_2))^d}. \end{cases}$$

Now from (3.3) $(a)$, we have that $(Au^*, u^n - u^*)_V \to 0$ as $n \to +\infty$. Next, using that $u^n \to u^*$ weakly in $V$ implies that $u^n \to u^*$ strongly in $\left(L^2(\Gamma_2)\right)^d$ and as $(\varphi^n)$ is bounded in $\left(L^2(\Gamma_2)\right)^d$, then

$$j(u^*) - j(u^n) + (\varphi_0, u^n - u^*)_H + (\varphi^n, u^n - u^*)_{(L^2(\Gamma_2))^d} \to 0 \text{ as } n \to +\infty.$$

This suggests that the last member on the right side of the last inequality tends to be zero. Hence, from (3.6), we get (3.4). On the other hand, $K$ is closed convex of $V$, then $u^* \in K$.Moreover, using (3.3) $(b)$, (3.4) and passing to the limit as $n \to +\infty$ in (3.5), we get that $(u^*, \varphi^*) \in U_{ad}$ and then this is a solution to problem C1. ∎

## 4. The penalized and regularized problem

In this part we consider the regularized problem that can be exploited numerically. The interest is to approximate the nondifferentiable term by a sequence of differentiable ones. This regularization is obtained by substituting the functional $j$ with a regularized function $j_\delta : V \to \mathbb{R}$ defined by

$$j_\delta(v) = \int_{\Gamma_3} g\sqrt{v_\tau^2 + \delta^2}\,da,$$

where $\delta > 0$ is a small parameter.

Recall (see [1]) that the functional $j_\delta$ is proper, convex, lower semicontinuous and satisfies $j_\delta \in C^2(V)$. Indeed, denoting by $Dj_\delta(u)$ the differential of $j_\delta$ at the point $u$, we have

$$(Dj_\delta(u), v)_V = \int_{\Gamma_3} g\frac{u_\tau v_\tau}{\sqrt{u_\tau^2 + \delta^2}}da \quad \forall v \in V$$

and

$$\left(D^2 j_\delta(u, v), w\right)_V = \int_{\Gamma_3} g\frac{\delta^2 v_\tau w_\tau}{(u_\tau^2 + \delta^2)\sqrt{u_\tau^2 + \delta^2}}da \quad \forall v, w \in V.$$

Next, we denote

$$P_\delta(\tau) = \frac{2}{\delta}\left[(|\tau| - 1)_+\right]\frac{\tau}{|\tau|},\ \forall \tau \in S_d, \tag{4.1}$$

where $(|\tau| - 1)_+ = \max\{0, |\tau| - 1\}$.

We define the operator $A_\delta$ as

$$(A_\delta u, v)_V = (P_\delta(\varepsilon(u)), \varepsilon(v))_Q,\ \forall u, v \in V.$$

Then, we have the lemma below.

Lemma 4.1. *Operator $A_\delta$ verifies the following properties:*
(1) *$A_\delta$ is monotone:*

$$(A_\delta u - A_\delta v, u - v)_V \geq 0,\ \forall u, v \in V. \tag{4.2}$$

(2) *$A_\delta$ is Lipschitz continuous:*

$$|(A_\delta u - A_\delta v, w)_V| \leq \frac{4}{\delta}\|u - v\|_V \|w\|_V,\ \forall u, v, w \in V. \tag{4.3}$$

Proof. The mapping $\Psi_\delta : S_d \to \mathbb{R}_+;\ \xi \longmapsto \frac{1}{\delta}\left((|\xi| - M_L)_+\right)^2$ is a convex function and continuously differentiable and

$$\begin{aligned}(\Psi'_\delta(\xi); \tau) &= \lim_{\lambda\to 0}\frac{\Psi_\delta(\xi + \lambda\tau) - \Psi_\delta(\xi)}{\lambda}\\ &= \frac{2}{\delta}(|\xi| - M_L)_+\frac{\xi}{|\xi|}.\tau = P_\delta(\xi).\tau,\ \forall \xi, \tau \in S_d.\end{aligned}$$

Then, the mapping $G : V \to \mathbb{R};\ u \longmapsto \Psi_\delta\left(\varepsilon\left(u\right)\right)$ is also a convex function and continuously differentiable and

$$\left\langle G^{'}\left(u\right), v\right\rangle = (\Psi_\delta^{'}\left(\varepsilon\left(u\right)\right); \varepsilon\left(v\right)) = \tfrac{2}{\delta}\left(\left|\varepsilon\left(u\right)\right| - M_L\right)_+ \tfrac{\varepsilon(u)}{|\varepsilon(u)|}.\varepsilon\left(v\right)$$
$$= P_\delta\left(\varepsilon\left(u\right)\right).\varepsilon\left(v\right),\ \forall u, v \in V.$$

The property of convexity of $G$ implies that $G^{'}$is monotone, then

$$\left\langle G^{'}\left(u\right) - G^{'}\left(v\right), u - v\right\rangle \geq 0,\ \forall u, v \in V.$$

By integrating the two sides of the previous inequality on $\Omega$, one gets inequality (4.2). Now, to prove (2), we have

$$(A_\delta u - A_\delta v, w)_V =$$
$$\frac{2}{\delta}\int_\Omega \left(\frac{(\left|\varepsilon\left(u\right)\right| - M_L)_+\varepsilon\left(u\right)\left|\varepsilon\left(v\right)\right| - (\left|\varepsilon\left(u\right)\right| - M_L)_+\varepsilon\left(v\right)\left|\varepsilon\left(u\right)\right|}{\left|\varepsilon\left(u\right)\right|\left|\varepsilon\left(v\right)\right|}\right)\varepsilon\left(w\right)dx.$$

We see that there are three cases. The first case: if $\left|\varepsilon\left(u\right)\right| \leq M_L$, $\left|\varepsilon\left(v\right)\right| \leq M_L$, then $(T_\delta u - T_\delta v, w)_V = 0$. $(A_\delta u - A_\delta v, w)_V = 0$. The second case: if $\left|\varepsilon\left(u\right)\right| > M_L$, $\left|\varepsilon\left(v\right)\right| \leq M_L$, then

$$\left|(A_\delta u - A_\delta v, w)_V\right| = \tfrac{2}{\delta}\left|\int_\Omega \left(\left|\varepsilon\left(u\right)\right| - M_L\right)\tfrac{\varepsilon(u)}{|\varepsilon(u)|}\varepsilon\left(w\right)dx\right|$$
$$\leq \tfrac{2}{\delta}\int_\Omega \left|\varepsilon\left(u\right) - \varepsilon\left(v\right)\right|\left|\varepsilon\left(w\right)\right|dx$$
$$\leq \tfrac{2}{\delta}\left\|u - v\right\|_V \left\|w\right\|_V.$$

The third case: if $\left|\varepsilon\left(u\right)\right| > M_L$, $\left|\varepsilon\left(v\right)\right| > M_L$, we have

$$\left|(A_\delta u - A_\delta v, w)_V\right| = \tfrac{2}{\delta}\left|\int_\Omega ((\varepsilon\left(u\right) - \varepsilon\left(v\right)) - M_L(\tfrac{\varepsilon(u)}{|\varepsilon(u)|} - \tfrac{\varepsilon(v)}{|\varepsilon(v)|}))\varepsilon\left(w\right)dx\right|$$
$$\leq \tfrac{2}{\delta}\int_\Omega \left|\varepsilon\left(u\right) - \varepsilon\left(v\right)\right|\left|\varepsilon\left(w\right)\right|dx + \tfrac{2}{\delta}\left|\int_\Omega M_L\left(\tfrac{\varepsilon(v)}{|\varepsilon(v)|} - \tfrac{\varepsilon(u)}{|\varepsilon(u)|}\right)\varepsilon\left(w\right)dx\right|$$
$$\leq \tfrac{2}{\delta}\int_\Omega \left|\varepsilon\left(u\right) - \varepsilon\left(v\right)\right|\left|\varepsilon\left(w\right)\right|dx + \tfrac{2}{\delta}\int_\Omega \left|M_L\left(\tfrac{\varepsilon(v)}{|\varepsilon(v)|} - \tfrac{\varepsilon(u)}{|\varepsilon(u)|}\right)\right|\left|\varepsilon\left(w\right)\right|dx$$
$$\leq \tfrac{4}{\delta}\int_\Omega \left|\varepsilon\left(u\right) - \varepsilon\left(v\right)\right|\left|\varepsilon\left(w\right)\right|dx.$$
$$\leq \tfrac{4}{\delta}\left\|u - v\right\|_V \left\|w\right\|_V.$$

Then, it follows that in all the cases, (4.3) is satisfied. Hence, we end the proof of Lemma 4.1. ∎

We now consider the penalized and regularized problem below.

**Problem** $P_\delta$**.** Find $u^\delta \in V$ such that

$$\begin{aligned}&\left(Au^\delta, v\right)_V + \left(Dj_\delta\left(u^\delta\right), v\right)_V + \left(P_\delta\left(\varepsilon\left(u^\delta\right)\right), \varepsilon\left(v\right)\right)_Q \\ &= \left(\varphi_0, v\right)_H + \left(\varphi, v\right)_{(L^2(\Gamma_2))^d}\quad \forall v \in V.\end{aligned} \tag{4.4}$$

THEOREM 4.2. *Let* (2.8), (2.9), (2.10), (4.2) *and* (4.3) *hold. Then, Problem* $P_\delta$ *has a unique solution.*

PROOF. We define the operator $B_\delta : V \to V$ by

$$(B_\delta u, v)_V = (Au, v)_V + (Dj_\delta (u), v)_V + (P_\delta (\varepsilon (u)), \varepsilon (v))_Q , \forall u, v \in V.$$

Using (2.8), (2.10) (*b*) and (4.3), we have that for all $u, v, w \in V$

$$|(B_\delta u - B_\delta v, w)_V| \leq (M + \frac{4}{\delta}) \|u - v\|_V \|w\|_V + |(Dj_\delta (u) - Dj_\delta (v), w)_V| .$$

On the other hand, we have

$$(Dj_\delta (u) - Dj_\delta (v), w)_V =$$
$$\int_{\Gamma_3} \frac{g\delta^2 (u_\tau - v_\tau) w_\tau}{\left((v_\tau + \theta (u_\tau - v_\tau))^2 + \delta^2\right) \sqrt{\left((v_\tau + \theta (u_\tau - v_\tau))^2 + \delta^2\right)}} da,$$
$$\text{where } \theta \in (0, 1) .$$

We have

$$|(Dj_\delta (u) - Dj_\delta (v), w)_V| \leq \frac{\|g\|_{L^\infty(\Gamma_3)} \|u_\tau - v_\tau\|_{(L^2(\Gamma_3))^d} \|w_\tau\|_{(L^2(\Gamma_3))^d}}{\delta}.$$

Then, using (2.8), we deduce

$$|(Dj_\delta (u) - Dj_\delta (v), w)_V| \leq \frac{d_\Omega^2 \|g\|_{L^\infty(\Gamma_3)} \|u - v\|_V \|w\|_V}{\delta}.$$

Hence

$$\|B_\delta u - B_\delta v\|_V \leq \left(M + \frac{4 + d_\Omega^2 \|g\|_{L^\infty(\Gamma_3)}}{\delta}\right) \|u - v\|_V .$$

Then, the operator $B_\delta$ is Lipschitz continuous. Now, we prove that $B_\delta$ is strongly monotone. Indeed, using (2.10) (*c*) and (4.2), we have for all $u, v \in V$

$$(B_\delta u - B_\delta v, u - v)_V \geq m \|u - v\|_V^2 +$$
$$\int_{\Gamma_3} \frac{g\delta^2 (u_\tau - v_\tau)^2}{\left((v_\tau + \theta (u_\tau - v_\tau))^2 + \delta^2\right) \sqrt{\left((v_\tau + \theta (u_\tau - v_\tau))^2 + \delta^2\right)}} da$$
$$\geq m \|u - v\|_V^2 .$$

Moreover, using (2.7) , we deduce that Problem $P_\delta$ has a unique solution. □

The next convergence is demonstrated below.

THEOREM 4.3. *The following convergence holds:*

(4.5) $$u^\delta \to u \text{ strongly in } V \text{ as } \delta \to 0,$$

*where* $u$ *is a solution of Problem* $P_2$.

PROOF. We take $v = u^\delta$ in (4.4), then as $\left(Dj_\delta\left(u^\delta\right), u^\delta\right)_V \geq 0$, $\left(P_\delta\left(\varepsilon\left(u^\delta\right)\right), \varepsilon\left(u^\delta\right)\right)_Q \geq 0$, we deduce that

$$\left(Au^\delta, u^\delta\right)_V \leq \left(\varphi_0, u^\delta\right)_H + \left(\varphi, u^\delta\right)_{(L^2(\Gamma_2))^d}.$$

Thus, by (2.7) and (2.10) $(c)$, we get

$$\left\|u^\delta\right\|_V \leq \frac{c_0}{m}\left(\|\varphi_0\|_H + d_\Omega \|\varphi\|_{((L^2(\Gamma_2))^d)}\right). \tag{4.6}$$

This estimate implies that there exists an element $\bar{u}$ such that

$$u^\delta \to \bar{u} \text{ weakly in } V. \tag{4.7}$$

Moreover, we take $v \in K$ in (4.4), thus we deduce

$$\begin{aligned}&\left(Au^\delta, v - u^\delta\right)_V + \left(Dj_\delta\left(u^\delta\right), v - u^\delta\right)_V + \left(P_\delta\left(\varepsilon\left(u^\delta\right)\right), \varepsilon(v) - \varepsilon\left(u^\delta\right)\right)_Q \\ &= \left(\varphi_0, v - u^\delta\right)_H + \left(\varphi, v - u^\delta\right)_{(L^2(\Gamma_2))^d} \quad \forall v \in K.\end{aligned}$$

As we have

$$\left(P_\delta\left(\varepsilon\left(u^\delta\right)\right) - P_\delta\left(\varepsilon(v)\right), \varepsilon(v) - \varepsilon\left(u^\delta\right)\right)_Q \leq 0,$$

thus

$$\left(P_\delta\left(\varepsilon\left(u^\delta\right)\right), \varepsilon(v) - \varepsilon\left(u^\delta\right)\right)_Q \leq \left(P_\delta\left(\varepsilon(v)\right), \varepsilon(v) - \varepsilon\left(u^\delta\right)\right)_Q = 0.$$

On the other hand, we have

$$\left(Dj_\delta\left(u^\delta\right), v - u^\delta\right)_V \leq j_\delta(v) - j_\delta\left(u^\delta\right),$$

Then, it follows that

$$\left(Au^\delta, v - u^\delta\right)_V + j_\delta(v) - j_\delta\left(u^\delta\right) \geq \left(\varphi_0, v - u^\delta\right)_H + \left(\varphi, v - u^\delta\right)_{(L^2(\Gamma_2))^d}. \tag{4.8}$$

Taking now $v = \bar{u}$ in (4.8), we see that

$$\left(Au^\delta, u^\delta - \bar{u}\right)_V \leq j_\delta(\bar{u}) - j_\delta\left(u^\delta\right) + \left(\varphi_0, u^\delta - \bar{u}\right)_H + \left(\varphi, u^\delta - \bar{u}\right)_{(L^2(\Gamma_2))^d}.$$

As the right hand side of the above inequality tends to zero when $\delta \to 0$, then we get

$$\lim_{\delta\to 0}\sup\left(Au^\delta, u^\delta - \bar{u}\right)_V \leq 0.$$

Thus, using the pseudomonotonicity of $A$, we deduce that

$$(A\bar{u}, \bar{u} - v)_V \leq \lim_{\delta\to 0}\inf\left(Au^\delta, u^\delta - v\right)_V, \forall v \in K.$$

Keeping in mind (4.7) and passing to the limit as $\delta \to 0$ in (4.8), we get

$$j_\delta(v) = \int_{\Gamma_3} g\sqrt{v_\tau^2 + \delta^2}\,da \to j(v),$$

$$j_\delta\left(u^\delta\right) = \int_{\Gamma_3} g\left(\sqrt{(u_\tau^\delta)^2 + \delta^2} - |\bar{u}_\tau|\right)da + \int_{\Gamma_3} g\,|\bar{u}_\tau|\,da \to \int_{\Gamma_3} g\,|\bar{u}_\tau|\,da = j(\bar{u}),$$

since

$$\begin{aligned}\left|\int_{\Gamma_3} g(\sqrt{(u_\tau^\delta)^2+\delta^2}-|\bar{u}_\tau|)da\right| &\leq \|g\|_{L^\infty(\Gamma_3)}\left\|u_\tau^\delta-\bar{u}_\tau\right\|_{(L^2(\Gamma_2))^d} \\ +\|g\|_{L^\infty(\Gamma_3)}\,\delta mes\,(\Gamma_3) &\to 0.\end{aligned}$$

Also, using (4.6) and $u^\delta \to u$ strongly in $\left(L^2\left(\Gamma_2\right)\right)^d$, we have

$$\left(\varphi_0, v-u^\delta\right)_H+\left(\varphi, v-u^\delta\right)_{(L^2(\Gamma_2))^d} \to (\varphi_0, v-\bar{u})_H+(\varphi, v-\bar{u})_{(L^2(\Gamma_2))^d},$$

thus we get

$$(A\bar{u}, v-\bar{u})_V+j\left(v\right)-j\left(\bar{u}\right) \geq (\varphi_0, v-\bar{u})_H+(\varphi, v-\bar{u})_{(L^2(\Gamma_2))^d}.$$

Now we claim to prove that $\overline{u} \in K$. Indeed, take $v=u^\delta$ in (4.4), we get

$$\left(P_\delta\left(\varepsilon\left(u^\delta\right)\right), \varepsilon\left(u^\delta\right)\right)_Q \leq \left(\varphi_0, u^\delta\right)_H+\left(\varphi, u^\delta\right)_{(L^2(\Gamma_2))^d}.$$

This inequality with (4.6) implies that

$$\int_\Omega\left(\left|\varepsilon\left(u^\delta\right)\right|-M_L\right)_+\left|\varepsilon\left(u^\delta\right)\right|dx \leq \frac{\delta c_0}{m}\left(\|\varphi_0\|_H+d_\Omega\|\varphi\|_{((L^2(\Gamma_2))^d)}\right)^2. \tag{4.9}$$

Thus, from the inequality (4.9), we deduce that

$$\limsup_{\delta\to 0}\int_\Omega\left(\left|\varepsilon\left(u^\delta\right)\right|-M_L\right)_+\left|\varepsilon\left(u^\delta\right)\right|dx=0.$$

Then, by (4.6), passing to a subsequence still denoted by $\left(u^\delta\right)$, we have that

$$\left(\left|\varepsilon\left(\bar{u}\right)\right|-M_L\right)_+\left|\varepsilon\left(\overline{u}\right)\right| \leq \liminf_{\varepsilon\to 0}\left(\left|\varepsilon\left(u^\delta\right)\right|-M_L\right)_+\left|\varepsilon\left(u^\delta\right)\right|.$$

Thus, by Lemma's Fatou, we deduce

$$\begin{aligned}\int_\Omega\left(\left|\varepsilon\left(\bar{u}\right)\right|-M_L\right)_+\left|\varepsilon\left(\bar{u}\right)\right|dx &\leq \int_\Omega\liminf_{\delta\to 0}\left(\left|\varepsilon\left(u^\delta\right)\right|-M_L\right)_+\left|\varepsilon\left(u^\delta\right)\right|dx \\ &\leq \liminf_{\delta\to 0}\int_\Omega\left(\left|\varepsilon\left(u^\delta\right)\right|-M_L\right)_+\left|\varepsilon\left(u^\delta\right)\right|dx \\ &\leq \limsup_{\delta\to 0}\int_\Omega\left(\left|\varepsilon\left(u^\delta\right)\right|-M_L\right)_+\left|\varepsilon\left(u^\delta\right)\right|dx=0.\end{aligned}$$

Thus, we get

$$\int_\Omega\left(\left|\varepsilon\left(\bar{u}\right)\right|-M_L\right)_+\left|\varepsilon\left(\bar{u}\right)\right|dx=0.$$

Then, we deduce that $\left(\left|\varepsilon\left(\bar{u}\right)\right|-M_L\right)_+=0$ *a.e.* in $\Omega$, i.e. $\left|\varepsilon\left(\bar{u}\right)\right| \leq M_L$ *a.e.* in $\Omega$, i.e. $\bar{u} \in K$. Hence, we deduce that $\bar{u}$ is a solution of Problem $P_2$. Then by the uniqueness part of Theorem 2.1, we obtain that $\bar{u}=u$.

Now, we claim to prove the strong convergence. Indeed, we have by using (4.8) :

$$\begin{aligned} m\left\|u_\delta-u\right\|_V^2 &\leq \left(Au_\delta-Au,u_\delta-u\right)_V \\ &=\left(Au_\delta,u_\delta-u\right)_V-\left(Au,u_\delta-u\right)_V \\ &\leq \left(Au,u-u_\delta\right)_V+j_\delta\left(u\right)-j_\delta\left(u_\delta\right) \\ &\quad+\left(\varphi_0,u_\delta-u\right)_H+\left(\varphi_0,u_\delta-u\right)_{(L^2(\Gamma_2))^d}. \end{aligned} \tag{4.10}$$

Then, using (4.6), and that $u^\delta\to u$ strongly in $\left(L^2\left(\Gamma\right)\right)^d$, we have, as $\delta\to 0$,

$$\left(Au,u-u_\delta\right)_V+j_\delta\left(u\right)-j_\delta\left(u_\delta\right)+\left(\varphi_0,u_\delta-u\right)_H+\left(\varphi_0,u_\delta-u\right)_{(L^2(\Gamma_2))^d}\to 0,$$

then, from (4.10), we obtain (4.5). □

## 5. The penalized and regularized optimal control problem

For $\delta>0$ and a fixed $\varphi_0\in H$, we introduce the following penalized and regularized state problem.

**Problem Q2.** For $\varphi\in\left(L^2\left(\Gamma_2\right)\right)^d$ (called control), find $u^\delta\in V$ such that

$$\begin{aligned} &\left(Au^\delta,v\right)_V+\left(Dj_\delta\left(u^\delta\right),v\right)_V+\left(P_\delta\left(\varepsilon\left(u^\delta\right)\right),\varepsilon\left(v\right)\right)_Q \\ &=\left(\varphi_0,v\right)_H+\left(\varphi,v\right)_{(L^2(\Gamma_2))^d}\quad\forall v\in V. \end{aligned} \tag{5.1}$$

According to Theorem 4.2, the state problem Q2 has a unique solution. Next, we define the set $U_{ad}^\delta$ as

$$U_{ad}^\delta=\left\{(u,\varphi)\in V\times\left(L^2\left(\Gamma_2\right)\right)^d,\text{ such that (5.11) is satisfied}\right\}.$$

Hence, we introduce the optimal control problem below.

**Problem C2.** Find $\left(\tilde{u}^\delta,\tilde{\varphi}^\delta\right)\in U_{ad}^\delta$ such that

$$\mathcal{L}\left(\tilde{u}^\delta,\tilde{\varphi}^\delta\right)=\min_{(u,\varphi)\in U_{ad}^\delta}\mathcal{L}\left(u,\varphi\right).$$

We have the following result.

Theorem 5.1. *Let* (2.8)*,* (2.9) *and* (2.10) (*c*) *hold. Then, Problem C2 has at least one solution.*

Proof. We refer the reader to the arguments used in the proof of Theorem 3.1. □

Next, to show the convergence results concerning the solutions of Problems C1 and C2. For thus we have the theorem below.

Theorem 5.2. *We have*

$$\lim_{\delta\to 0}\mathcal{L}\left(\tilde{u}^\delta,\tilde{\varphi}^\delta\right)=\mathcal{L}\left(u^*,\varphi^*\right).$$

The proof of this theorem is carried out in several steps. First, we prove the following lemma.

Lemma 5.3. *Let* $\left(\tilde{u}^{\delta},\tilde{\varphi}^{\delta}\right)$ *be a solution of Problem C2. Then, there exists a solution* $(\tilde{u},\tilde{\varphi})$ *of Problem C1, such that passing to a subsequence still denoted* $\left(\tilde{u}^{\delta},\tilde{\varphi}^{\delta}\right)$*, we have as* $\delta\to 0$*, the following convergences:*

$$\left\{\begin{array}{l}(a)\ \ \tilde{u}^{\delta}\to\tilde{u}\ \textit{strongly in } V,\\ (b)\ \ \tilde{\varphi}^{\delta}\to\tilde{\varphi}\ \textit{weakly in}\ \left(L^{2}\left(\Gamma_{2}\right)\right)^{d}.\end{array}\right.\tag{5.2}$$

Proof. Let $u_0^{\delta}$ the solution of Problem Q2 obtained for $\varphi=0_{(L^2(\Gamma_2))^d}$. We have

$$\mathcal{L}\left(\tilde{u}^{\delta},\tilde{\varphi}^{\delta}\right)\leq\mathcal{L}\left(u_0^{\delta},0_{(L^2(\Gamma_2))^d}\right)\leq c\left(\left\|\varphi_0\right\|_H^2+\left\|u_d\right\|_V^2\right),$$

where $c>0$. Thus, it follows that the sequence $\left(\tilde{u}^{\delta},\tilde{\varphi}^{\delta}\right)$ is bounded in $V\times\left(L^2\left(\Gamma_2\right)\right)^d$. Then, there exists an element

$$(\tilde{u},\tilde{\varphi})\in V\times\left(L^2\left(\Gamma_2\right)\right)^d$$

such that passing to a subsequence still denoted $\left(\tilde{u}^{\delta},\tilde{\varphi}^{\delta}\right)$, we have

$\tilde{u}^{\delta}\to\tilde{u}$ weakly in $V$,
$\tilde{\varphi}^{\delta}\to\tilde{\varphi}$ weakly in $\left(L^2\left(\Gamma_2\right)\right)^d$ (then (5.2) $(b)$ is proved).

Now, to prove that $\tilde{u}\in K$, it suffices to sketch the proof of Theorem 4.3. Moreover, using (2.10) $(c)$, we have

$$\left\{\begin{array}{l}m\left\|\tilde{u}^{\delta}-\tilde{u}\right\|_V^2\leq\left(A\tilde{u}-A\tilde{u}^{\delta},\tilde{u}-\tilde{u}^{\delta}\right)_V\\ \leq\left(A\tilde{u},\tilde{u}-\tilde{u}^{\delta}\right)_V+j_\delta\left(\tilde{u}\right)-j_\delta\left(\tilde{u}^{\delta}\right)+\left(P_\delta\left(\varepsilon\left(\tilde{u}^{\delta}\right)\right)-P_\delta\left(\varepsilon\left(\tilde{u}\right)\right),\varepsilon(\tilde{u}-\tilde{u}^{\delta})\right)_Q\\ +\left(P_\delta\left(\varepsilon\left(\tilde{u}\right)\right),\varepsilon(\tilde{u}-\tilde{u}^{\delta})\right)_Q+\left(\varphi_0,\tilde{u}-\tilde{u}^{\delta}\right)_H+\left(\tilde{\varphi}^{\delta},\tilde{u}-\tilde{u}^{\delta}\right)_{(L^2(\Gamma_2))^d}.\end{array}\right.\tag{5.3}$$

Keeping in mind (4.2), we have

$$\left(P_\delta\left(\varepsilon\left(\tilde{u}^{\delta}\right)\right)-P_\delta\left(\varepsilon\left(\tilde{u}\right)\right),\varepsilon(\tilde{u}-\tilde{u}^{\delta})\right)_Q\leq 0,$$

and since $\tilde{u}\in K$,

$$\left(P_\delta\left(\varepsilon\left(\tilde{u}\right)\right),\varepsilon(\tilde{u}-\tilde{u}^{\delta})\right)_Q=0.$$

Then, (5.3) implies

$$\begin{array}{l}m\left\|\tilde{u}^{\delta}-\tilde{u}\right\|_V^2\\ \quad\leq\left(A\tilde{u},\tilde{u}-\tilde{u}^{\delta}\right)_V+j_\delta\left(\tilde{u}\right)-j_\delta\left(\tilde{u}^{\delta}\right)+\left(\varphi_0,\tilde{u}-\tilde{u}^{\delta}\right)_H+\left(\tilde{\varphi}^{\delta},\tilde{u}-\tilde{u}^{\delta}\right)_{(L^2(\Gamma_2))^d}.\end{array}\tag{5.4}$$

Hence as $\tilde{u}^{\delta}\to\tilde{u}$ weakly in $V$, when $\delta\to 0$, implies that $\tilde{u}^{\delta}\to\tilde{u}$ strongly in $\left(L^2\left(\Gamma_2\right)\right)^d$, then $j_\delta\left(\tilde{u}\right)-j_\delta\left(u^{\delta}\right)\to 0$. Hence we deduce that the right hand side of the previous inequality (5.4) tends to zero. Then, we obtain (5.2) $(a)$.

Now, we still have to prove that $(\tilde{u}, \tilde{\varphi}) \in U_{ad}$. Indeed, from the inequality (4.4), we deduce by (4.2)

$$\begin{aligned} &\left(A\tilde{u}^{\delta}, v - \tilde{u}^{\delta}\right)_V + j_\delta(v) - j_\delta\left(\tilde{u}^{\delta}\right) \\ &\qquad \geq \left(\varphi_0, v - \tilde{u}^{\delta}\right)_H + \left(\tilde{\varphi}^{\delta}, v - \tilde{u}^{\delta}\right)_{(L^2(\Gamma_2))^d}, \ \forall v \in K. \end{aligned} \tag{5.5}$$

Then, using (5.2) $(a)$, it follows that the following convergences hold:

$$\begin{aligned} &\lim_{\delta\to 0}\left(A\tilde{u}^{\delta}, v - \tilde{u}^{\delta}\right)_V = \lim_{\delta\to 0}\left(\left(A\tilde{u}^{\delta} - Au, v - \tilde{u}^{\delta}\right)_V + \left(A\tilde{u}, v - \tilde{u}^{\delta}\right)_V\right) \\ &= \lim_{\delta\to 0}\left(A\tilde{u}^{\delta} - Au, v - \tilde{u}^{\delta}\right)_V + \lim_{\delta\to 0}\left(A\tilde{u}, v - \tilde{u}^{\delta}\right)_V = \left(A\tilde{u}, v - \tilde{u}\right)_V, \\ &\lim_{\delta\to 0}\left(j_\delta(v) - j_\delta\left(\tilde{u}^{\delta}\right)\right) = j(v) - j(\tilde{u}), \\ &\lim_{\delta\to 0}\left(\left(\varphi_0, v - \tilde{u}^{\delta}\right)_H + \left(\tilde{\varphi}^{\delta}, v - \tilde{u}^{\delta}\right)_{(L^2(\Gamma_2))^d}\right) = \left(\varphi_0, v - \tilde{u}\right)_H + \left(\tilde{\varphi}, v - \tilde{u}\right)_{(L^2(\Gamma_2))^d}. \end{aligned}$$

Hence, using these convergences and passing to the limit as $\delta \to 0$ in (5.5), we deduce that $(\tilde{u}, \tilde{\varphi})$ satisfies (3.1). Now, □

We shall complete the proof of Theorem 5.2 by proving the following lemma.

LEMMA 5.4. *We have that*

$$\mathcal{L}(\tilde{u}, \tilde{\varphi}) = \mathcal{L}(u^*, \varphi^*).$$

PROOF. Consider the sequence $\left(u^{\delta}\right)$ such that for each $\delta > 0$, $u^{\delta}$ is the unique solution of Problem Q2 written for $\varphi^* \in \left(L^2(\Gamma_2)\right)^d$. Hence, for each $\delta > 0$, $\left(u^{\delta}, \varphi^*\right) \in U_{ad}^{\delta}$ and by Lemma 5.3, it follows that

$$\left(u^{\delta}, \varphi^*\right) \to \left(u^*, \varphi^*\right) \text{ strongly in } V \times \left(L^2(\Gamma_2)\right)^d \text{ as } \delta \to 0. \tag{5.6}$$

As the functional $\mathcal{L}$ is convex and continuous, then

$$\mathcal{L}(\tilde{u}, \tilde{\varphi}) \leq \lim_{\delta\to 0}\inf \mathcal{L}\left(\tilde{u}^{\delta}, \tilde{\varphi}^{\delta}\right). \tag{5.7}$$

We also have, as $\left(\tilde{u}^{\delta}, \tilde{\varphi}^{\delta}\right)$ is a solution of Problem C2

$$\lim_{\delta\to 0}\sup \mathcal{L}\left(\tilde{u}^{\delta}, \tilde{\varphi}^{\delta}\right) \leq \lim_{\delta\to 0}\sup \mathcal{L}\left(u^{\delta}, \varphi^*\right). \tag{5.8}$$

Using (5.6), we have

$$\lim_{\delta\to 0}\sup \mathcal{L}(u^{\delta}, \varphi^*) = \mathcal{L}(u^*, \varphi^*) \tag{5.9}$$

and as $(u^*, \varphi^*)$ is a solution of Problem C1, it follows

$$\mathcal{L}(u^*, \varphi^*) \leq \mathcal{L}(\tilde{u}, \tilde{\varphi}). \tag{5.10}$$

Also, from (5.8), we deduce

$$\lim_{\delta\to 0}\sup \mathcal{L}\left(\tilde{u}^{\delta}, \tilde{\varphi}^{\delta}\right) \leq \mathcal{L}(u^*, \varphi^*). \tag{5.11}$$

Hence, from (5.7) – (5.11), we obtain

$$\mathcal{L}(\tilde{u}, \tilde{\varphi}) = \mathcal{L}(u^*, \varphi^*).$$

Finally, we conclude that the solution of the penalized and regularized optimal control problem C2 converges to the solution of the optimal control problem C1, which completes the proof of Theorem 5.2. ∎

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## Optimalno upravljanje kontaktnog problema trenja za *locking* materijale

*Rachid Guettaf i Arezki Touzaline*

Sažetak. U ovom radu razmatramo bilateralni kontakt s Trescinim zakonom trenja između *locking* materijala i krutog temelja. Cilj je proučiti problem optimalnog upravljanja koji se sastoji od dovođenja tenzora naprezanja što je bliže moguće zadanom cilju, djelovanjem s kontrolom na granici tijela. U radu formuliramo problem optimalnog upravljanja koji ima barem jedno rješenje. Također uvodimo penalizirani i regularizirani problem optimalnog upravljanja za koji proučavamo konvergenciju kada penalizacija i parametar regularizacije teže k nuli.

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