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AUXILIARY PRINCIPLE TECHNIQUE FOR SOLVING TRIFUNCTION HARMONIC VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce and investigate some new classes of trifunction harmonic variational inequalities. Several important new problems are obtained as special cases. Some new harmonic Bregman distance functions are derived for the Shannon entropy and Burg entropy harmonic convex functions. The auxiliary principle technique involving the harmonic Bregman distance function is applied to suggest and analyze some hybrid inertial iterative methods for solving the trifunction harmonic variational inequality. The convergence analysis of these iterative methods is also considered under some mild conditions. Some special cases are also pointed out. Results proved in this paper can be viewed as a refinement and improvement of the known results.

1. Introduction

Variational inequalities theory introduced in 1964 has emerged as a powerful tool to investigate and study a wide class of unrelated problems arising in industrial, regional, physical, pure and applied sciences in a unified and general framework. The ideas and techniques of the variational inequalities are being applied in a variety of diverse areas and prove to be productive and innovative. Variational inequalities have been extended and generalized in several direction using novel and new techniques, see [11–15, 19–26, 28, 29, 31–33, 35, 36, 38–40, 43–46, 48–50, 52, 54, 55]. There are significant developments of variational inequalities related with multivalued, nonmonotone, nonconvex optimization and structural analysis. An important and useful generalization of variational inequalities is harmonic variational inequality. Anderson et al. [6] studied the properties of the harmonic functions. Al-Azemi et al. [2] explored the applications harmonic means in asian options and stock exchange. For the applications, generalizations and properties of the harmonic functions, see [32–34, 37, 39–43, 45]. The harmonic variational inequalities were introduced and investigated by Noor et. al [32, 33, 35, 38] by

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using the concept of the differentiable harmonic convex. In particular, if the functions is a locally Lipschitz continuous harmonic function, then the minimum of the harmonic function can be characterized by harmonic directional variational inequality.

Noor and Oettli [47] introduced and studied the problem of trifunction equilibrium problem. It have been shown that variational inequalities, fixed-point problems, Nash equilibrium problems and saddle-point problems can be studied in the framework of equilibrium problems. For applications, motivation, numerical techniques and other aspects of trifunction variational inequalities and related optimization problems, see [28,31] and the references therein. Motivated and inspired by these facts, we consider another class of harmonic variational inequalities, which is called the *trifunction harmonic variational inequality.* This class includes the harmonic variational inequalities, complementarity problems, absolute value harmonic variational inequalities and optimization problems as special cases.

Variational inequalities problems have witnessed an explosive growth in theoretical advances, algorithmic developments and applications across almost all disciplines of engineering, pure and applied sciences. Analysis of these problems requires a blend of techniques and ideas from convex analysis, functional analysis, numerical analysis and nonsmooth analysis. There are several methods for solving variational inequalities and equilibrium problems. Due to the nature of the harmonic variational inequalities, projection and resolvent methods can not be applied for solving these problems. In such cases, the auxiliary principle technique is being used to suggest and analyze some iterative methods for solving harmonic variational inequalities and their variants forms. Glowinski et. al [12] used this technique to study the existence problem for mixed variational inequalities, the origin of which can be traced back to Lions et al. [15]. Noor [19, 22, 24–26, 31] and Noor et al. [14, 32, 33, 35, 38, 39, 43, 45, 46] has used this technique to suggest and analyze some iterative methods for solving various classes of variational inequalities and equilibrium problems. We have also introduce the harmonic Bregman distance functions for differentiable harmonic functions. We apply the auxiliary principle principle technique involving the harmonic Bregman distance function to suggest some classes of hybrid inertial iterative schemes for trifunction harmonic variational inequalities. We also prove that the convergence of these methods pseudomonotonicity, which is a weaker conditions than monotonicity. As special cases, we obtain new iterative schemes for solving various classes of harmonic variational inequalities and related optimization problems. The comparison of these methods with other methods is a subject of future research. We have only investigated the theoretical aspects of the results. The development of efficient numerical aspects of these methods is a significant open problem.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let $\mathcal{K}$ be a nonempty closed convex set in $\mathcal{H}$. Let $f : \mathcal{H} \longrightarrow R$ be a locally Lipschitz continuous function. Let Ω be an open bounded subset of R^n.

First of all, we recall the following concepts and results from nonsmooth analysis [6] and nonconvex analysis [4, 8–10, 18, 51].

Definition 2.1. *Let f be locally Lipschitz continuous at a given point $x \in \mathcal{H}$ and v be any other vector in $\mathcal{H}$. The Clarke's generalized directional derivative of f at x in the direction v, denoted by $f^0(x, v)$, is defined as*

$$f^0(x, v) = \lim_{t \to 0^+} \sup_{h \to 0} \frac{f(x + h + tv) - f(x + h)}{t}.$$

The generalized gradient of f at x, denoted $\partial f(x)$, is defined to be subdifferential of the function $f^0(x; v)$ at 0. That is

$$\partial f(x) = \{w \in \mathcal{H} : \langle w, v\rangle \leq f^0(x; v), \quad \forall v \in \mathcal{H}.\}.$$

If f is convex on $\mathcal{K}$ and locally Lipschitz continuous at $x \in \mathcal{K}$, then $\partial f(x)$ coincides with the subdifferential $f'(x)$ of f at x in the sense of convex analysis, and $f^0(x; v)$ coincides with the directional derivative $f'(x; v)$ for each $v \in \mathcal{H}$, that is, $f^0(x; v) = \langle f'(x), v\rangle, \quad \forall v \in \mathcal{H}$.

Definition 2.2 ([4]). *The set $\mathcal{K}_h$ is said to be a harmonic convex set if*

$$\frac{uv}{v + \lambda(u - v)} \in \mathcal{K}_h, \quad \forall u, v \in \mathcal{K}_h, \quad \lambda \in [0, 1].$$

Definition 2.3 ([4]). *The function ϕ on the harmonic convex set $\mathcal{K}_h$ is said to be harmonic convex if*

$$\phi(\frac{uv}{v + \lambda(u - v)}) \leq (1 - \lambda)\phi(u) + \lambda\phi(v), \quad \forall u, v \in \mathcal{K}_h \quad \lambda \in [0, 1].$$

The function ϕ is said to be harmonic concave function if and only if $-\phi$ is harmonic convex function.

We recall that the minimum of a locally Lipschitz continuous harmonic convex function on the harmonic convex set $\mathcal{K}_h$ can be characterized by an harmonic variational inequality.

Theorem 2.4. *Let ϕ be a locally Lipschitz continuous harmonic convex function on the harmonic convex set $\mathcal{K}_h$. Then $u \in \mathcal{C}_h$ is a minimum of ϕ if and only if $u \in \mathcal{K}_h$ satisfies the inequality*

$$\phi'(u; \frac{uv}{u - v}) \geq 0, \quad \forall v \in \mathcal{K}_h. \tag{2.1}$$

The inequality of type (2.1) is called the bifunction harmonic variational inequality.

PROOF. Let $u \in \mathcal{K}_h$ is a minimum of locally Lipschitz continuous harmonic convex function ϕ. Then

$$\phi(u) \leq \phi(v), \quad \forall v \in \mathcal{K}_h. \tag{2.2}$$

Since $\mathcal{K}_h$ is a harmonic convex set, so $\forall u, v \in \mathcal{K}_h, \quad v_\lambda = \frac{uv}{u+\lambda(u-v)} \in \mathcal{K}_h$. Replacing v by v_λ in (2.2) and diving by λ and taking limit as $\lambda \to 0$, we have

$$0 \leq \lim_{\lambda \to 0} \frac{\phi(\frac{uv}{u+\lambda(u-v)}) - \phi(u)}{\lambda} = \phi'(u; \frac{uv}{u-v})$$

the required result (2.1). Conversely, let the function ϕ be locally Lipschitz continuous harmonic convex function on the harmonic convex set $\mathcal{K}_h$. Then

$$\frac{uv}{v+\lambda(u-v)} \leq (1-\lambda)\phi(u) + \lambda\phi(v) = \phi(u) + \lambda(\phi(v) - \phi(u)),$$

which implies that

$$\phi(v) - \phi(u) \geq \lim_{\lambda \to 0} \frac{\phi(\frac{uv}{v+\lambda(u-v)}) - \phi(u)}{\lambda} = \phi'(u; \frac{uv}{u-v}) \geq 0, \quad \text{using} \quad (2.1).$$

Consequently, it follows that

$$\phi(u) \leq \phi(v), \quad \forall v \in \mathcal{K}_h.$$

This shows that $u \in \mathcal{K}_h$ is the minimum of a locally Lipschitz continuous harmonic convex function. □

We would like to mention that Theorem 2.4 implies that harmonic optimization programming problem can be studied via the harmonic variational inequality (2.1).

For strongly harmonic convex functions f, we define the harmonic Bregman distance function as

$$\begin{aligned} B(v,u) &= f(v) - f(u) - \langle f'(u), \frac{uv}{u-v} \rangle \\ &\geq \alpha \|v-u\|^2, \forall u, v \in \mathcal{K}_h. \end{aligned} \tag{2.3}$$

It is important to emphasize that various types of function f give different harmonic Bregman distance function. We give the following important examples of some practical important types of harmonic convex function f and their corresponding Bregman distance functions.

Examples

1. For convex function $f(v) = \|v\|^2$,

$$B(v,u) = \|v-u\|^2, \quad \forall u, v \in \mathcal{H},$$

   is the squared Euclidean Bregman distance function (SE).

2. If the Shannon entropy [53] $f(v) = \sum_{i=1}^{n} v_i \log v_i$ is a differentiable harmonic convex function, then its corresponding harmonic Bregman distance function is given as

$$B(v,u) = \sum_{i=1}^{n} \Big(v_i\{\log v_i - \log u_i\} + (v_i - u_i + \frac{v_i u_i}{u_i - v_i}) \log u_i - \frac{v_i u_i}{u_i - v_i}) \Big)$$
$$= \sum_{i=1}^{n} \Big(v_i \log(\frac{v_i}{u_i}) + (v_i - u_i + \frac{v_i u_i}{u_i - v_i}) \log u_i - \frac{v_i u_i}{u_i - v_i}) \Big).$$

This distance is called the harmonic Kullback-Leibler distance (KL), which may play a very important tool in several areas of applied mathematics such as information, data analysis and machine learning.

3. If the Burg entropy $f(v) = -\sum_{I=1}^{n} \log v_i$ is a differentiable harmonic functions, then its corresponding harmonic Bregman distance function given as

$$B(v,u) = \sum_{i=1}^{n} \Big(-\log \frac{v_i}{u_i} + \frac{v_i}{u_i - v_i} \Big),$$

is called the harmonic Itakura-Saito distance (IS) and appears to be new one. It is not symmetric, that is, $B(v,u) \neq B(u,v)$. One of the advantages of the Itakura-Saito divergence is its scale invariance which means that $B(\lambda v, \lambda u) = B(u,v)$, for any number λ. This makes it a very suitable measure for the comparison of audio spectra. One can explore the applications of this Bregman distance harmonic function in data analysis, the information theory, and machine learning. These harmonic Bregman distance functions inspire further research and applications in various branches of risk analysis, transportation and other related optimization programming problems.

Remark 2.5. It is a challenging problem to explore the applications of Bregman distance function for other types of nonconvex functions such as biconvex, k-convex functions, preinvex functions and harmonic biconvex functions.

Motivated by these facts, we now introduce the trifunction harmonic variational inequalities and discuss some special cases. To be more precise, for given nonlinear trifunctions $F(.,.,.), \Phi(.,.,.) : \mathcal{K}_h \times \mathcal{K}_h \times \mathcal{K}_h \longrightarrow \mathcal{R}$ and nonlinear continuous operators $T, A : \mathcal{K}_h \longrightarrow \mathcal{H}$, consider the problem of finding $u \in \mathcal{K}_h$ such that

$$F(u, Tu, \frac{uv}{u-v}) + \Phi(u, A(u), \frac{uv}{u-v}) \geq 0, \quad \forall v \in \mathcal{K}_h. \tag{2.4}$$

The problem (2.4) is called the trifunction harmonic variational inequality.

Special Cases.

(I). If $\Phi(u, A(u), \frac{uv}{u-v}) = \Phi(u, A(u); \frac{uv}{u-v}), \quad \forall v \in \mathcal{K}_h$, then (2.4) reduces to finding $u \in \mathcal{K}_h$ such that

$$F(u, Tu, \frac{uv}{u-v}) + \Phi(u, A(u); \frac{uv}{u-v}) \geq 0, \quad \forall v \in \mathcal{K}_h. \tag{2.5}$$

(II). For $\Phi(u, A(u), \frac{uv}{u-v}) = f^0(u; \frac{uv}{u-v}), \quad \forall v \in \mathcal{K}_h$, the problem (2.4) collapses to finding $u \in \mathcal{K}_h$ such that

$$F(u, Tu, \frac{uv}{u-v}) + f^0(u; \frac{uv}{u-v}) \geq 0, \quad \forall v \in \mathcal{K}_h. \tag{2.6}$$

Here $f^0(u; \frac{uv}{u-v})$ denotes the generalized directional derivative of the function $f(u)$ at u in the direction $\frac{uv}{u-v}$. Such type of nonlinear functions f arise in the structural analysis, see [49, 50]. Problem of type (2.6) is called the *trifunction harmonic hemivariational inequality.* Panagiotopoulos [49] studied the hemivariational inequalities to formulate variational principles connected to energy functions which are neither convex nor smooth. It is has been shown that the technique of hemivariational inequalities is very efficient to describe the behaviour of complex structure arising in structural and industrial engineering sciences, see [49, 50, 52] and the references therein.

(III). Let f be a differentiable harmonic convex function and

$$\Phi(u, A(u), \frac{uv}{u-v}) = \Phi(u, f'(u), \frac{uv}{u-v}).$$

Then problem (2.6) collapses to finding $u \in \mathcal{K}_h$ such that

$$F(u, Tu, \frac{uv}{u-v}) + \Phi(u, f'(u), \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in \mathcal{K}_h, \tag{2.7}$$

which is known as the mildly nonlinear trifunction harmonic variational inequality and appears to be a new one.

(IV). If $\Phi(u, A(u), \frac{uv}{u-v}) = \Phi(u, A|u|; \frac{uv}{u-v}), \quad \forall v \in \mathcal{K}_h$, then (2.4) reduces to finding $u \in \mathcal{K}_h$ such that

$$F(u, Tu, \frac{uv}{u-v}) + \Phi(u, A|u|, \frac{uv}{u-v}) \geq 0, \quad \forall v \in \mathcal{K}_h, \tag{2.8}$$

is called the absolute value trifunction harmonic variational inequality. Mangasarian et al. [16] studied the systems of absolute value equations and proved its equivalence with complementarity problems.. It is worth mentioning that the systems of absolute equations can be obtained as a special case of the variational inequalities and complementarity problems introduced and studied by Noor [19, 20].

(V). If $F(u, Tu, \frac{uv}{u-v}) = B(Tu, \frac{uv}{u-v}),\ \Phi(u, A(u), \frac{uv}{u-v}) = W(A(u), \frac{uv}{u-v})$, where $B(.,.)$ and $W(.,.)$ are continuous bifunctions, then problem (2.4) is equivalent to finding $u \in \mathcal{K}_h$ such that

$$B(Tu, \frac{uv}{u-v}) + W(A(u), \frac{uv}{u-v}) \geq 0, \quad \forall v \in \mathcal{K}_h, \tag{2.9}$$

which is called the bifunction harmonic variational inequality, introduced and studied by AlShejari et al. [1].

(VI). If $F(u, Tu, \frac{uv}{u-v}) = \langle Tu, \frac{uv}{u-v}\rangle$ and $\Phi(u, A(u), \frac{uv}{u-v}) = \langle A(u), \frac{uv}{u-v}\rangle$, then problem (2.4) is equivalent to finding $u \in \mathcal{K}_h$ such that

$$\langle Tu, \frac{uv}{u-v}\rangle + \langle A(u), \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in \mathcal{K}_h, \tag{2.10}$$

which is known as the harmonic variational inequality.

(VII). For $\Phi(u, A(u), \frac{uv}{u-v}) = 0$, then problem (2.4) reduces to finding $u \in \mathcal{K}_h$ such that

$$F(u, Tu, \frac{uv}{u-v}) \quad \geq 0, \quad \forall v \in \mathcal{K}_h, \tag{2.11}$$

which is called the trifunction harmonic variational inequality. In brief, for suitable and appropriate choice of the trifunction, one can obtain several classes of harmonic variational inequalities. This clear shows that the problem (2.4) is more general and flexible and includes the previous ones as special cases.

Definition 2.6. *The trifunction $F(.,.,.)$ and the operator T is said to be:*
(a) jointly monotone *with respect to* $\Phi(.,.,.)$, *if*

$$F(u, Tu, \frac{uv}{u-v}) + F(v, Tv, \frac{uv}{v-u}) \leq 0, \quad \forall u, v \in \mathcal{K}_h.$$

(b) jointly pseudomonotone *with respect to* $\Phi(.,.,.)$, *if*

$$\begin{aligned} & F(u, Tu, \frac{uv}{u-v}) + \Phi(u, A(u), \frac{uv}{u-v}) \geq 0 \\ & \Longrightarrow \\ & F(v, Tv, \frac{uv}{v-u}) + \Phi(v, A(v), \frac{uv}{v-u}) \geq 0, \quad \forall u, v \in \mathcal{K}_h. \end{aligned}$$

(c) partially relaxed strongly jointly monotone *with respect to* $\Phi(.,.,.)$, *if there exists a constant* $\gamma > 0$ *such that*

$$F(u, Tu, \frac{uv}{u-v}) + F(v, Tv, \frac{vz}{z-v}) \leq \gamma\|u - z\|^2, \quad \forall u, v, z \in \mathcal{K}_h.$$

Note that for $z = u$ partially relaxed strongly jointly monotonicity reduces to jointly monotonicity. This shows that partially relaxed strongly jointly monotonicity implies jointly monotonicity, but the converse is not true

Lemma 2.7. $\forall u, v \in \mathcal{H}$,

$$2\langle u, v\rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2. \tag{2.12}$$

3. Iterative methods and convergence analysis

In this section, we suggest and analyze some iterative methods for solving trifunction harmonic variational inequality (2.4) using the auxiliary principle technique of Glowinski et al. [12] involving Bregman distance harmonic function.

For a given $u \in \mathcal{K}_h$ satisfying (2.4), consider the auxiliary problem of finding $w \in {}_h$ such that

$$\rho F(w, Tw, \frac{vw}{v-w}) + \langle E'(w) - E'(v), \frac{vw}{v-w}\rangle + \rho\Phi(w, A(w), \frac{vw}{v-w}) \geq 0, \quad \forall v \in \mathcal{K}_h, \tag{3.1}$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a differentiable harmonic convex function $E(u)$ at $u \in \mathcal{K}_h$. Since $E(u)$ is a strongly convex function, problem (3.1) has a unique solution. We note that, if $w = u$, then clearly w is solution of the problem (2.4). This observation enables us to suggest and analyze the following iterative method for solving (2.4).

Algorithm 3.1. *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(u_{n+1}, Tu_{n+1}, \frac{vu_{n+1}}{v-u_{n+1}}) + \langle E'(u_{n+1}) - E'(u_n), \frac{vu_{n+1}}{v-u_{n+1}}\rangle + \rho\Phi(u_{n+1}, A(u_{n+1}), \frac{vu_{n+1}}{v-u_{n+1}}) \geq 0, \quad \forall v \in \mathcal{K}_h, \tag{3.2}$$

where $\rho > 0$ is a constant.

Algorithm 3.1 is called the proximal method for solving problem (2.4). In passing we remark that the proximal point method was suggested by Martinet [8] in the context of convex programming problems as regularization technique. For the recent developments and applications of the proximal point algorithms, see the references.

(I). For $\Phi(u, Au, \frac{uv}{u-v}) = \Phi(u, Au; \frac{uv}{u-v})$, Algorithm 3.1 collapses to:

Algorithm 3.2. *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(u_{n+1}, Tu_{n+1}, \frac{vu_{n+1}}{v-u_{n+1}}) + \langle E'(u_{n+1}) - E'(u_n), \frac{vu_{n+1}}{v-u_{n+1}}\rangle + \rho\Phi(u_{n+1}, A(u_{n+1}); \frac{vu_{n+1}}{v-u_{n+1}}) \geq 0, \quad \forall v \in \mathcal{K}_h, \tag{3.3}$$

for solving trifunction harmnoic hemivariational inequalities (2.5).

(II). If $F(u, Tu, \frac{uv}{u-v}) = B(Tu, \frac{uv}{u-v} v - u)$ and $\Phi(u, Au, \frac{uv}{u-v}) = W(Au, \frac{uv}{u-v})$ then Algorithm 3.1 collapses to the following method.

ALGORITHM 3.3. *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B(Tu_{n+1}, \frac{vu_{n+1}}{v - u_{n+1}}) + \langle E'(u_{n+1}) - E'(u_n), \frac{vu_{n+1}}{v - u_{n+1}} \rangle + \rho W(A(u_{n+1}), \frac{vu_{n+1}}{v - u_{n+1}}) \geq 0, \quad \forall v \in \mathcal{K}_h.$$

It is called the proximal point method for solving bifunction harmonic variational inequalities (2.6) and appears to be a new one.

(III). If $\Phi(u, A(u), \frac{uv}{u-v}) = 0$, then Algorithm 3.1 collapses to:

ALGORITHM 3.4. *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(u_{n+1}, Tu_{n+1}, \frac{vu_{n+1}}{v - u_{n+1}}) + \langle E'(u_{n+1}) - E'(u_n), \frac{vu_{n+1}}{v - u_{n+1}} \rangle \geq 0, \quad \forall v \in \mathcal{K}_h, \tag{3.4}$$

for solving trifunction harmonic variational inequality.

(IV). For $W(Tu, \frac{uv}{u-v}) = 0$, Algorithm 3.3 reduces to:

ALGORITHM 3.5. *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho B(Tu_{n+1}, \frac{vu_{n+1}}{v - u_{n+1}}) + \langle E'(u_{n+1}) - E'(u_n), \frac{vu_{n+1}}{v - u_{n+1}} \rangle \geq 0, \quad \forall v \in \mathcal{K}_h.$$

It is called the proximal point method for solving bifunction harmonic variational inequalities (2.6) and appears to be a new one.

In brief, for suitable and appropriate choice of the operators and the spaces, one can obtain a number of known and new algorithms for solving variational-like inequalities and related problems.

THEOREM 3.1. *Let $F(.,.,.)$ and the operator T be jointly pseudomonotone with respect to $\Phi(.,.,.)$. Let E be a locally Lipchitz continuous strongly harmonic convex function with module $\beta > 0$. Then the approximate solution u_{n+1} obtained from Algorithm 3.1 converges to a solution $u \in \mathcal{K}_h$ satisfying (2.4).*

PROOF. Let $u \in \mathcal{K}_h$ be a solution of (2.4). Then

$$F(u, Tu, \frac{uv}{u - v}) + \Phi(u, A(u), \frac{uv}{u - v}) \geq 0, \quad \forall v \in \mathcal{K}_h,$$

implies that

$$F(v,Tv,\frac{uv}{u-v})+\Phi(v,A(v),\frac{uv}{u-v})\geq 0,\quad \forall v\in\mathcal{K}_h, \tag{3.5}$$

since $F(.,.,.)$ is jointly pseudomonotone with respect to $\Phi(.,.,.)$.

Taking $v=u$ in (3.2) and $v=u_{n+1}$ in (3.5), we have

$$\begin{aligned}\rho F(u_{n+1},Tu_{n+1},\frac{uu_{n+1}}{u-u_{n+1}}) &+ \langle E'(u_{n+1})-E'(u_n),\frac{uu_{n+1}}{u-u_{n+1}}\rangle\\ &\geq -\rho\rho\Phi(u_{n+1},A(u_{n+1}),\frac{uu_{n+1}}{u-u_{n+1}}).\end{aligned} \tag{3.6}$$

and

$$-F(u_{n+1},Tu_{n+1},\frac{uu_{n+1}}{u-u_{n+1}})-\rho F(u_{n+1},Tu_{n+1},\frac{uu_{n+1}}{u-u_{n+1}})\geq 0. \tag{3.7}$$

We now consider the Bregman distance function

$$B(u,w)=E(u)-E(w)-\langle E'(w),\frac{uw}{w-u}\rangle\geq\beta\|u-w\|^2, \tag{3.8}$$

by using the strongly harmonic convexity of E.

Now combining (3.6) and (3.7), we have

$$\begin{aligned}B(u,u_n) &- B(u,u_{n+1})=E(u_{n+1})-E(u_n)-\langle E'(u_{n+1}),\frac{u_nu_{n+1}}{u_{n+1}-u_n}\rangle\\ &\quad+\langle E'(u_{n+1})-E'(u_n),\frac{uu_{n+1}}{u-u_{n+1}}\rangle\\ &\geq \beta\|u_{n+1}-u_n\|^2+\langle E'(u_{n+1})-E'(u_n),\frac{uu_{n+1}}{u-u_{n+1}}\rangle\\ &\geq \beta\|u_{n+1}-u_n\|^2-\rho F(u_{n+1},Tu_{n+1},\frac{uu_{n+1}}{u-u_{n+1}})\\ &\quad-\rho\Phi((u_{n+1},A(u_{n+1},\frac{uu_{n+1}}{u-u_{n+1}}),\\ &\geq \beta\|u_{n+1}-u_n\|^2,\end{aligned}$$

where we have used (3.7).

If $u_{n+1}=u_n$, then clearly u_n is a solution of the trifunction hemivariational inequality (2.4). Otherwise, it follows that $B(u,u_n)-B(u,u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0.$$

Now using the technique of Zhu and Marcotte [55], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the trifunction harmonic variational inequality (2.4). □

It is well-known that to implement the proximal point methods, one has to find the approximate solution implicitly, which is itself a difficult problem.

To overcome this drawback, we now consider another method for solving (2.4) using the auxiliary principle technique.

For a given $u \in \mathcal{K}_h$ satisfying (2.4), find $w \in \mathcal{K}_h$ such that

$$\begin{aligned} &\rho F(u, Tu, \frac{vw}{w-v}) + \langle E'(w) - E'(u), v - w\rangle \\ &\quad + \rho\Phi(u, A(u), \frac{wv}{w-v}) \geq 0, \quad \forall v \in \mathcal{K}_h, \end{aligned} \tag{3.9}$$

where $E'(u)$ is the differential of a strongly harmonic convex function $E(u)$ at $u \in \mathcal{K}_h$. Note that problems (3.1) and (3.9) are quite different problems.

It is clear that for $w = u$, w is a solution of (2.4). This fact allows us to suggest and analyze another iterative method for solving trifunction harmonic variational inequality (2.4).

Algorithm 3.6. *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\rho F(u_n, Tu_n, \frac{vu_{n+1}}{u_{n+1}-v}) + \langle E'(u_{n+1}) - E'(u_n), \frac{vu_{n+1}}{v-u_{n+1}}\rangle \\ &\quad + \rho\Phi(u_n, A(u_n), \frac{u_{n+1}v}{u_{n+1}-v}) \geq 0, \quad \forall v \in \mathcal{K}_h, \end{aligned} \tag{3.10}$$

which is called the implicit iterative method for solving the problem (2.4).

To implement Algorithm 3.6, we use the predictor corrector technique to suggest the following two step method for solving the problem (2.4).

Algorithm 3.7. *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho F(w_n, Tw_n, \frac{w_n v}{v-u_{n+1}}) &+ \langle E'(u_{n+1}) - E'(w_n), \frac{w_n v}{v-u_{n+1}}\rangle \\ &\geq -\rho\Phi(w_n, Aw_n, \frac{w_n v}{v-u_{n+1}}), \quad \forall v \in \mathcal{K}_h, \\ \mu F(u_n, Tu_n, \frac{vw_n}{v-w_n}) &+ \langle E'(w_n) - E'(u_n), \frac{vw_n}{v-w_n}\rangle \\ &\geq -\rho\Phi(u_n, Au_n, \frac{w_n v}{v-w_{n+1}}), \quad \forall v \in \mathcal{K}_h. \end{aligned}$$

Note that, for $F(.,.,.) = B(.,.)$ and $\Phi(.,.,.) = W(.,.),$, where $B(.,.), W(.,.)$ are bifunctions, Algorithm 3.7 reduces to:

ALGORITHM 3.8. *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho F(Tw_n, \frac{w_n v}{v-u_{n+1}}) \quad &+ \quad \langle E'(u_{n+1}) - E'(w_n), \frac{w_n v}{v-u_{n+1}}\rangle \\ &\geq -\rho\Phi(Aw_n, \frac{w_n v}{v-u_{n+1}}), \quad \forall v \in \mathcal{K}_h, \\ \mu F(Tu_n, \frac{vw_n}{v-w_n}) \quad &+ \quad \langle E'(w_n) - E'(u_n), \frac{vw_n}{v-w_n}\rangle \\ &\geq -\rho\Phi(Au_n, \frac{w_n v}{v-w_{n+1}}), \quad \forall v \in \mathcal{K}_h. \end{aligned}$$

which is called the predictor-corrector method for solving the bifunction harmonic variational inequality (2.6).

If $F(Tu, \frac{uv}{u-v}) = \langle Tu, \frac{uv}{u-v}\rangle$ and $W(Au, \frac{uv}{u-v}) = \langle Au, \frac{uv}{u-v}\rangle$, then Algorithm 3.8 collapses to the method for solving the harmonic variational inequalities (2.5).

ALGORITHM 3.9. *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho\langle Tw_n, \frac{w_n v}{v-u_{n+1}}\rangle \quad &+ \quad \langle E'(u_{n+1}) - E'(w_n), \frac{w_n v}{v-u_{n+1}}\rangle \\ &\geq -\rho\langle Aw_n, \frac{w_n v}{v-u_{n+1}}\rangle, \quad \forall v \in \mathcal{K}_h, \\ \mu\langle Tu_n, \frac{vw_n}{v-w_n}\rangle \quad &+ \quad \langle E'(w_n) - E'(u_n), \frac{vw_n}{v-w_n}\rangle \\ &\geq -\rho\langle Au_n, \frac{w_n v}{v-w_{n+1}}\rangle, \quad \forall v \in \mathcal{K}_h. \end{aligned}$$

If $\Phi(u, A(u), \frac{uv}{u-v}) = 0$, then Algorithm 3.6 reduces to the following iterative method for solving with trifunction harmonic variational (2.11).

ALGORITHM 3.10. *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\rho F(u_n, Tu_n, \frac{vu_{n+1}}{u_{n+1}-v}) \\ &+ \langle E'(u_{n+1}) - E'(u_n), \frac{vu_{n+1}}{v-u_{n+1}}\rangle \geq 0, \quad \forall v \in \mathcal{K}_h, \end{aligned} \tag{3.11}$$

Similarly for suitable and appropriate choice of the operators and the spaces, one can obtain various known and new algorithms for solving equilibrium problems and variational inequalities.

Using the technique of Theorem 3.1, one can consider the convergence criteria of Algorithm 3.6.

We again apply the approach of auxiliary principle without the Bregman distance function to suggest some implicit for approximate schemes for solving the problem (2.4).

For a given $u \in \mathcal{K}_h$ satisfying (2.4), consider the problem of finding $w \in \mathcal{K}_h$ such that

$$\rho F(w+\eta(u-w), T(w+\eta(u-w)), \frac{vw}{v-w} + \langle w-u, v-w\rangle \geq$$

$$- \rho\Phi(w+\eta(u-w), A(w), \frac{vw}{v-w}) \geq 0, \ \forall v \in \mathcal{K}_h, \tag{3.12}$$

where $\rho > 0, \eta \in [0,1]$ are constants.

Inequality of type (3.12) is called the auxiliary trifunction harmonic variational inequality.

If $w = u$, then w is a solution of (2.4). This simple observation enables us to suggest the following iterative method for solving (2.4).

ALGORITHM 3.11. *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$F(u_{n+1}+\eta(u_n-u_{n+1}), T(u_{n+1}+\eta(u_n-u_{n+1})), \frac{vu_{n+1}}{v-u_{n+1}})$$
$$+ \langle u_{n+1}-u_n, v-u_{n+1}\rangle \geq$$
$$- \rho\Phi(u_{n+1}+\eta(u_n-u_{n+1}), A(u_{n+1}+\eta(u_n-u_{n+1})), \frac{vu_{n+1}}{v-u_{n+1}}), \forall v \in \mathcal{K}_h.$$

Algorithm 3.11 is called the hybrid proximal point algorithm for solving trifunction harmonic variational inequalities(2.4).

Special Cases

We now consider some cases of Algorithm 3.11.

(I). For $\eta = 0$, Algorithm 3.11 reduces to:

ALGORITHM 3.12. *For a given $u_0 \in \mathcal{H}$ compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(u_{n+1}, T(u_{n+1}), \frac{vu_{n+1}}{v-u_{n+1}}) + \langle u_{n+1}-u_n, v-u_{n+1}\rangle$$

$$+ \Phi(u_{n+1}, A(u_{n+1}), \frac{vu_{n+1}}{v-u_{n+1}}) \geq 0, \forall v \in \mathcal{K}_h. \tag{3.13}$$

(II). If $\eta = 1$, then Algorithm 3.11 reduces to:

ALGORITHM 3.13. *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho F(u_n, Tu_n, \frac{u_n u_{n+1}}{v-u_{n+1}}) + \langle u_{n+1}-u_n, v-u_{n+1}\rangle$$
$$+ \rho\Phi(u_n, A(u_n), \frac{vu_{n+1}}{v-u_{n+1}}\rangle \geq 0, \forall v \in \mathcal{K}_h.$$

(III). If $\eta = \frac{1}{2}$, then Algorithm 3.11 collapses to:

ALGORITHM 3.14. *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(\frac{u_{n+1}+u_n}{2}, T(\frac{u_{n+1}+u_n}{2}), \frac{vu_{n+1}}{v-u_{n+1}}) + \langle u_{n+1}-u_n, v-u_{n+1}\rangle$$
$$+\rho\Phi(\frac{u_{n+1}+u_n}{2}, A(\frac{u_{n+1}+u_n}{2}), \frac{vu_{n+1}}{v-u_{n+1}}) \geq 0, \forall v \in \mathcal{K}_h,$$

which is called the mid-point proximal method for solving the problem (2.4).

(IV). If $\Phi(.,.;.) = 0$, then Algorithm 3.11 reduces to:

ALGORITHM 3.15. *For a given $u_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(u_{n+1}+\eta(u_n-u_{n+1}), T(u_{n+1}+\eta(u_n-u_{n+1})), \frac{vu_{n+1}}{v-u_{n+1}})$$
$$+\langle u_{n+1}-u_n, v-u_{n+1}\rangle \geq 0, \quad \forall v \in \mathcal{K}_h$$

for solving trifunction harmonic variational inequality.

We now consider the convergence criteria of Algorithm 3.12. The analysis is in the spirit of Theorem 3.1. We include its proof for the sake of completeness and to convey an idea of the technique involved.

THEOREM 3.2. *Let $F(.,.,.)$ be jointly pseudomonotone with respect to $\Phi(.,.,.)$. Then the approximate solution u_{n+1} obtained from Algorithm 3.12 converges to a solution $u \in \mathcal{K}_h$ satisfying (2.4), if*

$$\|u_{n+1}-u\|^2 \leq \|u_n-u\|^2 - \|u_{n+1}-u_n\|^2. \tag{3.14}$$

PROOF. Let $u \in \mathcal{K}_h$ be a solution of (2.4). Then

$$F(u, Tu, \frac{uv}{u-v}) + \Phi(u, A(u), \frac{uv}{u-v}) \geq 0, \quad \forall v \in \mathcal{K}_h,$$

implies that

$$F(v, Tv, \frac{uv}{u-v}) + \Phi(v, A(v), \frac{uv}{u-v}) \geq 0, \quad \forall v \in \mathcal{K}_h, \tag{3.15}$$

since $F(.,.,.)$ is jointly pseudomonotone with respect to $\Phi(.,.,.)$.

Taking $v = u$ in (3.14) and $v = u_{n+1}$ in (3.15), we have

$$F(u_{n+1}, T(u_{n+1}), \frac{uu_{n+1}}{u-u_{n+1}})$$
$$+\langle u_{n+1}-u_n, u-u_{n+1}\rangle \geq -\rho\Phi(u_{n+1}, A(u_{n+1}), \frac{uu_{n+1}}{u-u_{n+1}}), \forall v \in \mathcal{K}_h. \tag{3.16}$$

and

$$F(u_{n+1}, Tu_{n+1}, \frac{uu_{n+1}}{u-u_{n+1}}) + F(u_{n+1}, Tu_{n+1}, \frac{uu_{n+1}}{u-u_{n+1}}) \geq 0. \tag{3.17}$$

Combining (3.16) and (3.17), we obtain

$$\begin{aligned}&\langle u_{n+1}-u_n, u-u_{n+1}\rangle\\ &\geq -F(u_{n+1}, T(u_{n+1}), \frac{uu_{n+1}}{u-u_{n+1}})\\ &\quad -\Phi(u_{n+1}, A(u_{n+1}), \frac{uu_{n+1}}{u-u_{n+1}}) \geq 0.\end{aligned} \tag{3.18}$$

Setting $u = u - u_{n+1}$ and $v = u_{n+1} - u_n$ in (2.12), we obtain

$$\begin{aligned}&2\langle u_{n+1}-u_n, u-u_{n+1}\rangle\\ &= \|u-u_n\|^2 - \|u-u_{n+1}\|^2 - \|u_{n+1}-u_n\|^2.\end{aligned} \tag{3.19}$$

Combining (3.7), (3.18) and (3.19), we have

$$\|u_{n+1}-u\|^2 \leq \|u_n-u\|^2 - \|u_{n+1}-u_n\|^2,$$

the required result (3.14). □

Theorem 3.3. *Let $\mathcal{H}$ be a finite dimensional space and all the assumptions of Theorem 3.2 hold. Then the sequence $\{u_n\}_0^\infty$ given by Algorithm 3.12 converges to a solution u of (2.4).*

Proof. Let $u \in \mathcal{K}_h$ be a solution of (2.4). From (3.14), it follows that the sequence $\{||u-u_n||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\sum_{n=0}^{\infty} \|u_{n+1}-u_n\|^2 \leq \|u_0-u\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1}-u_n\| = 0. \tag{3.20}$$

Let $\hat{u}$ be the limit point of $\{u_n\}_0^\infty$; a subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_0^\infty$ converges to $\hat{u} \in \mathcal{K}_h$. Replacing w_n by u_{n_j} in (3.2), taking the limit $n_j \longrightarrow \infty$ and using (3.20), we have

$$F(\hat{u}, T\hat{u}, \frac{\hat{u}v}{v-\hat{u}}) + \Phi(\hat{u}, A(\hat{u}), \frac{\hat{u}v}{v-\hat{u}} \geq 0, \quad \forall v \in \mathcal{K}_h,$$

which implies that $\hat{u}$ solves the inequality (2.4) and

$$\|u_{n+1}-u\|^2 \leq \|u_n-u\|^2.$$

Thus, it follows from the above inequality that $\{u_n\}_0^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty}(u_n) = \hat{u}.$$

the required result. □

We again apply the technique of the auxiliary principle to suggest some hybrid inertial proximal point methods for solving the problem (2.4). It is worth mentioning that the inertial type methods were suggested by Polyak [52] to speed up the convergence criteria. For the applications of the inertial methods for solving the variational inclusions and inequalities, see [3,24,44,45] and the references.

For a given $u \in \mathcal{K}_h$ satisfying (2.4), consider the problem of finding $w \in \mathcal{K}_h$ such that

$$\begin{aligned} &\rho F(w+\eta(u-w), T(w+\eta(u-w)), \frac{vw}{v-w})) + \langle w-u+\alpha(u-u), v-w\rangle \\ &+ \Phi(w+\eta(u-w)), A(w+\eta(w-u)), \frac{vw}{v-w}\rangle \geq 0, \quad \forall v \in \mathcal{K}_h, \end{aligned} \tag{3.21}$$

where $\rho > 0, \alpha, \eta, \in [0,1]$ are constants.

Clearly, for $w = u$, w is a solution of (2.4). This fact motivated us to suggest the following inertial iterative method for solving (2.4), which is known as the hybrid inertial iterative method.

Algorithm 3.16. *For given $u_0, u_1 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\rho F(u_{n+1}+\eta(u_n-u_{n+1}, T(u_{n+1}+\eta(u_n-u_{n+1})), \frac{vu_{n+1}}{v-u_{n+1}}) \\ &+ \langle u_{n+1}-u_n+\alpha(u_n-u_{n-1}, v-u_{n+1}\rangle \geq \\ &-\rho\Phi(u_{n+1}+\eta(u_n-u_{n+1}, A((u_{n+1}+\eta(u_n-u_{n+1})), \frac{vu_{n+1}}{v-u_{n+1}}), \forall v \in \mathcal{K}_h. \end{aligned}$$

Note that for $\alpha = 1$, Algorithm 3.16 is exactly the Algorithm 3.12.

(V). If $\eta = 1$, then Algorithm 3.16 reduces to:

Algorithm 3.17. *For given $u_0, u_1 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\rho F(u_n, Tu_n, \frac{vu_{n+1}}{v-u_{n+1}}) + \langle u_{n+1}-u_n+\alpha(u_n-u_{n-1}, v-u_{n+1}\rangle \\ &\geq -\rho\Phi(u_n, A(u_n), \frac{vu_{n+1}}{v-u_{n+1}}), \quad \forall v \in \mathcal{K}_h. \end{aligned}$$

which is known as the explicit inertial iterative method.

(VI). For $\eta = \frac{1}{2}$, Algorithm 3.16 collapses to:

ALGORITHM 3.18. *For given $u_0, u_1 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\rho F(\frac{u_{n+1}+u_n}{2}, T(\frac{u_{n+1}+u_n}{2}), \frac{vu_{n+1}}{v-u_{n+1}})\\&\quad + \langle u_{n+1}-u_n+\alpha(u_n-u_{n-1}, v-u_{n+1}\rangle\\&\geq \ -\rho\Phi(\frac{u_{n+1}+u_n}{2}, A(\frac{u_{n+1}+u_n}{2}), \frac{vu_{n+1}}{v-u_{n+1}}), \quad \forall v \in \mathcal{K}_h.\end{aligned}$$

(VII). For $\eta = 0$, Algorithm 3.16 reduces to:

ALGORITHM 3.19. *For given $u_0, u_1 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\rho F(u_{n+1}, Tu_{n+1}, \frac{vu_{n+1}}{v-u_{n+1}}) + \langle u_{n+1}-u_n+\alpha(u_n-u_{n-1}, v-u_{n+1}\rangle\\&\geq -\rho\Phi(u_{n+1}, A(u_{n+1}), \frac{vu_{n+1}}{v-u_{n+1}}), \quad \forall v \in \mathcal{K}_h.\end{aligned}$$

which is known as the implicit inertial iterative method.

Using essentially the technique of Theorem 3.2, Theorem 3.3 and Noor [12], one can study the convergence analysis of Algorithm 3.19.

REMARK 3.4. For different and appropriate values of the parameters η, α, bifunctions $F(.,.), \Phi(.,.)$, operators T, A, harmonic convex set $\mathcal{K}_h$ and spaces, we can obtain a wide class of inertial type iterative methods for solving the harmonic variational inequalities and related optimization problems. This shows that proposed Algorithms are quite flexible, unified and general ones.

Using essentially the technique of Theorem 3.2 and Theorem 3.3, one can study the convergence analysis of Algorithm 3.16.

For different and appropriate values of the parameters, η, α, the operators T, A and spaces, one can obtain a wide class of inertial type iterative methods for solving the harmonic variational inequalities and related optimization problems.

Conclusion. We have considered and investigated some new classes of trifunction harmonic variational inequalities in this paper. It is shown that several important problems such as harmonic complementarity problems, system of harmonic absolute value problems and related problems can be obtained as special cases. The auxiliary principle technique involving the Bregman distance function is applied to suggest several hybrid inertial type methods for finding the approximate solution of trifunction harmonic variational inequalities. Convergence criteria of the proposed methods is investigated under suitable weaker conditions. We note that this technique is independent of the projection and the resolvent of the operator. Similar approximate schemes can be suggested for stochastic, random, fuzzy and quantum variational inequalities, which are an interesting and challenging problems for further research.

To our knowledge, no research has been carried in these fields. The comparison of the proposed methods with other techniques needs further efforts and is itself an open interesting problem. The interested readers are advised to explore this field further and discover novel and fascinating applications of the harmonic variational inequality theory in Banach and topological spaces. The general theory of harmonic variational inequalities is quite technical, so we shall content ourselves here to give the flavour of the main ideas involved. The framework chosen should be seen as a model setting for more general results. It is an interesting open problem to explore the applications of harmonic variational inequalities in various branches of pure and applied sciences and develop numerical implementable methods.

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Tehnika pomoćnog principa za rješavanje trifunkcijskih harmonijskih varijacijskih nejednadžbi

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Sažetak. U ovom članku uvodimo i istražujemo neke nove klase trifunkcijskih harmonijskih varijacijskih nejednakosti. Nekoliko važnih novih problema su dobiveni kao posebni slučajevi. Izvedene su neke nove harmonijske funkcije Bregmanove udaljenosti za Shannonovu entropijsku i Burgovu entropijsku harmonijsku konveksnu funkciju. Primijenjena je tehnika pomoćnog principa koja uključuje harmonijsku Bregmanovu funkciju udaljenosti kako bi se sugeriralo i analiziralo neke hibridne inercijske iterativne metode za rješavanje trifunkcijske harmonijske varijacijske nejednadžbe. Analiza konvergencije ovih iterativnih metoda također je razmatrana pod nekim blagim uvjetima. Neki posebni slučajevi su također istaknuti. Rezultati dokazani u ovom članku mogu se promatrati kao usavršavanje i poboljšanje poznatih rezultata.

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