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ERROR ESTIMATES FOR AN EFFECTIVE MODEL FOR THE INTERACTION BETWEEN A THIN FLUID FILM AND AN ELASTIC PLATE

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Abstract. The non-steady flow of an incompressible fluid in a thin rectangle domain with an elastic plate as the upper part of the boundary is studied. The flow is modeled by the Stokes equations and governed by a pressure drop and an external force. Error estimates are obtained for the approximation by an effective model derived by studying the limiting case when the thickness of the fluid domain tends to zero.

1. Introduction

Models involving lubrication by a fluid and interaction with a solid body are widely used in various fields of science (biology, medicine, geophysics, oceanography, etc.) and have many applications in technology development. Such models belong to the group of fluid-structure interaction (FSI) models, and due to their importance and diverse applications, they have been treated intensively in the last decades (e.g. see [1, 5, 6, 9–11] and references within). Mathematics deals with well-posedness, the existence and uniqueness of solutions, numerical modeling, the determination of approximate solutions and estimation of their errors, etc. In the case of the presence of a very small (or large) model parameter (e.g. domain dimension or a characteristic value of the fluid or structure), the problem can be treated by asymptotic analysis. This requires the development of an effective, simpler model whose solution is an approximation of the exact solution. The use of such a simplified model is justified by theorems on convergence and error estimates.

The problem of interaction between a thin layer of incompressible fluid and a thin elastic plate located on a part of the boundary of the fluid domain is studied by a full three-dimensional model for both the fluid and the plate [2] and by their simplified dimensional reduction (e.g. see [4, 9–11]).

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Different boundary conditions were considered and effective models were derived, mainly including a six-dimensional parabolic equation [2, 4, 9]. Our work began by considering fluid and elastic plate models from engineering literature [7]. The flow of the fluid and the displacement of the plate are governed by external forces and the pressure drop in the fluid-filled channel. Due to the assumption of small displacement, the model is linearized, which would correspond to the modeling of high-frequency oscillations with small displacement of elastic structures.

Earlier, the existence and uniqueness of the solution for the model were proved [3] and an effective model was derived and justified by the convergence theorem [4]. In this work, our goal is to estimate the error for the approximation with the solution of the effective model. This is usually done by using the error for the test function in the weak formulation. For this purpose, the regularity of the solution of the effective model is considered. In addition, a corrector for the error test function is constructed to fulfill the boundary conditions.

In Section 2, a mathematical model is formulated for the problem under consideration. Since the existence and uniqueness of the solution and its estimates are the subject of previous research, only an overview of the results is given to provide insight into the problem in an easily readable outline. In Section 3, the effective model is derived. The model construction is presented only in a brief overview, and the regularity of the solution is given. In Section 4, the approximation error for the first-order approximation in a suitable solution spaces is proved.

2. Problem formulation

We consider the interaction between an incompressible, viscous fluid filling a thin two-dimensional channel and an elastic plate located at the top of the channel. For a small parameter $\varepsilon > 0$ we introduce the fluid domain $\Omega^\varepsilon = (0,1) \times (0,\varepsilon)$ and its boundary parts $\Gamma_{\mathrm{b}} = \overline{\Omega}^\varepsilon \cap \{x_2 = 0\}$, $\Gamma^\varepsilon_{\mathrm{in}} = \overline{\Omega}^\varepsilon \cap \{x_1 = 0\}$, $\Gamma^\varepsilon_{\mathrm{out}} = \overline{\Omega}^\varepsilon \cap \{x_1 = 1\}$ and $\Gamma^\varepsilon_{\mathrm{e}} = \overline{\Omega}^\varepsilon \cap \{x_2 = \varepsilon\}$ at the bottom, at the entrance, at the exit and at the top of the channel, respectively. The interaction between the fluid and the elastic plate and the external force cause a transverse displacement of the plate, which leads to changes in the fluid domain. We assume that the plate is thin and not very elastic, i.e. that its bending stiffness $B^\varepsilon = B/\varepsilon$ is of the order of $1/\varepsilon$, and that the deformation of the fluid domain is consequently small enough to consider the fluid flow in an initial state. Considering that the bending stiffness depends on Young's modulus of elasticity, which is very large, this assumption makes sense from a physical point of view. The interaction is observed in the time interval $(0, T^\varepsilon)$. Our main goal is to estimate the error of the first-order approximation of the plate displacement and fluid velocity obtained by an asymptotic analysis. This requires a longer time period for thinner channels, and in our previous research we set the value of $T^\varepsilon = T/\varepsilon^2$, where T is independent of ε. We note that the same result as in this paper can be obtained for any combination of the considered product $B^\varepsilon T^\varepsilon$ of order $1/\varepsilon^3$.

Fluid flow is described by

$$\rho_{\mathrm{f}}\partial_t \boldsymbol{u}^\varepsilon - \operatorname{div}\sigma_{\mathrm{f}}^\varepsilon = \boldsymbol{g}^\varepsilon \quad \text{in } \Omega^\varepsilon \times (0,T^\varepsilon), \tag{2.1}$$

$$\operatorname{div}\boldsymbol{u}^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon \times (0,T^\varepsilon), \tag{2.2}$$

$$\boldsymbol{u}^\varepsilon = 0 \quad \text{on } \Gamma_{\mathrm{b}} \times (0,T^\varepsilon), \tag{2.3}$$

$$v^\varepsilon = 0 \text{ and } p^\varepsilon = 0 \quad \text{on } \Gamma_{\mathrm{in}}^\varepsilon \times (0,T^\varepsilon), \tag{2.4}$$

$$v^\varepsilon = 0 \text{ and } p^\varepsilon = A^\varepsilon \quad \text{on } \Gamma_{\mathrm{out}}^\varepsilon \times (0,T^\varepsilon), \tag{2.5}$$

$$\boldsymbol{u}^\varepsilon(\cdot,0) = 0 \quad \text{in } \Omega^\varepsilon, \tag{2.6}$$

where ρ_{f} and μ are the fluid density and viscosity, $\boldsymbol{u}^\varepsilon = (u^\varepsilon, v^\varepsilon)$ and p^ε are the fluid velocity and pressure perturbation from the initial value, respectively and $\sigma_{\mathrm{f}}^\varepsilon = -p^\varepsilon\mathbf{I} + \mu(\nabla\boldsymbol{u}^\varepsilon + (\nabla\boldsymbol{u}^\varepsilon)^T)$ is the stress tensor of the fluid. The external force acting on the fluid is denoted by $\boldsymbol{g}^\varepsilon$ and $A^\varepsilon = A^\varepsilon(t)$ is a time-dependent pressure drop between the inlet and outlet regions of the boundary. A non-slip boundary condition is prescribed at the bottom, the fluid enters the channel and leaves it orthogonal to the boundary and at the beginning, the entire structure is in equilibrium and the initial velocity is zero.

By h^ε we denote the transverse displacement of the elastic plate with respect to the flat initial configuration. The continuity of the velocity along the elastic plate leads to the following boundary condition for the fluid:

$$\boldsymbol{u}^\varepsilon = (0, \partial_t h^\varepsilon) \text{ on } \Gamma_{\mathrm{e}}^\varepsilon \times (0,T^\varepsilon). \tag{2.7}$$

Plate displacement is described by

$$\rho_{\mathrm{s}} b \partial_t^2 h^\varepsilon + B^\varepsilon \partial_{x_1}^4 h^\varepsilon = f^\varepsilon - \boldsymbol{n}\cdot\sigma_{\mathrm{f}}^\varepsilon|_{x_2=\varepsilon}\,\boldsymbol{n} \quad \text{in } (0,1)\times(0,T^\varepsilon), \tag{2.8}$$

$$h^\varepsilon(0,\cdot) = h^\varepsilon(1,\cdot) = \partial_{x_1}h^\varepsilon(0,\cdot) = \partial_{x_1}h^\varepsilon(1,\cdot) = 0 \quad \text{in } (0,T^\varepsilon), \tag{2.9}$$

$$h^\varepsilon(\cdot,0) = 0 \quad \text{in } (0,1), \tag{2.10}$$

where $\boldsymbol{n}$ is the outward unit normal on the fluid boundary, ρ_s, b, B^ε are the density, thickness and bending stiffness of the plate. The external force acting on the plate is denoted by f^ε. The plate equation may include a term with the second-order spatial derivative of the plate displacement, representing the contribution of the in-plane tension to the out-of-plane forces. From a physical point of view, this term can be neglected and does not affect the mathematical analysis of the problem, since all estimates and the definitions of the appropriate spaces are based on the leading fourth-order derivative. Therefore, we have omitted it. Considering the initial condition (2.6) for the fluid velocity and the kinematic boundary condition (2.7), we complete our system with the following initial condition

$$\partial_t h^\varepsilon(\cdot,0) = 0 \text{ in } (0,1). \tag{2.11}$$

2.1. *Zero pressure drop.* To obtain a weak formulation, we multiply the fluid equation (2.1) by the test function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ satisfying the boundary conditions: $\boldsymbol{\varphi} = 0$ at the bottom Γ_b of the channel, $\varphi_2 = 0$ at the inlet and outlet part $\Gamma^\varepsilon_{\text{in}} \cup \Gamma^\varepsilon_{\text{out}}$ and $\varphi_1 = 0$ at the top $\Gamma^\varepsilon_{\text{e}}$. As usual, partial integration is applied, and considering that the force of the fluid on the plate is described by the stress tensor, we easily obtain

$$\begin{aligned}
&- \rho_{\text{f}} \int_0^{T^\varepsilon} \int_{\Omega^\varepsilon} \boldsymbol{u}^\varepsilon \cdot \partial_t \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}\mathrm{d}t + 2\mu \int_0^{T^\varepsilon} \int_{\Omega^\varepsilon} e(\boldsymbol{u}^\varepsilon) : e(\boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&- \rho_{\text{s}} b \int_0^{T^\varepsilon} \int_{\Gamma^\varepsilon_{\text{e}}} \partial_t h^\varepsilon \partial_t \varphi_2 \, \mathrm{d}x_1 \mathrm{d}t + \frac{B}{\varepsilon} \int_0^{T^\varepsilon} \int_{\Gamma^\varepsilon_{\text{e}}} \partial^2_{x_1} h^\varepsilon \partial^2_{x_1} \varphi_2 \, \mathrm{d}x_1 \mathrm{d}t \\
&- \int_0^{T^\varepsilon} \int_{\Omega^\varepsilon} p^\varepsilon \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}\mathrm{d}t = \int_0^{T^\varepsilon} \int_{\Gamma^\varepsilon_{\text{e}}} f^\varepsilon \varphi_2 \, \mathrm{d}x_1 \mathrm{d}t \\
&+ \int_0^{T^\varepsilon} \int_{\Omega^\varepsilon} \boldsymbol{g}^\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}\mathrm{d}t - \int_0^{T^\varepsilon} \int_{\Gamma^\varepsilon_{\text{out}}} A^\varepsilon \varphi_1 \, \mathrm{d}x_2 \mathrm{d}t
\end{aligned} \tag{2.12}$$

where $e(\boldsymbol{u}^\varepsilon) = \frac{1}{2}(\nabla \boldsymbol{u}^\varepsilon + (\nabla \boldsymbol{u}^\varepsilon)^T)$ is symmetrized gradient. Fluid flow described by pressure drop and external force is common. However, due to the coupling of the fluid with the plate and the external force on the plate, it can be assumed that the pressure drop is A^ε=0, since the pressure drop can be represented as part of the external forces. More precisely:

$$\begin{aligned}
&- \int_{\Gamma^\varepsilon_{\text{out}}} A^\varepsilon \varphi_1 \, \mathrm{d}x_2 = \int_0^\varepsilon A^\varepsilon \int_0^1 \partial_{x_1} (\varphi_1 x_1) \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega^\varepsilon} A^\varepsilon x_1 \partial_{x_1} \varphi_1 \, \mathrm{d}\boldsymbol{x} + \int_{\Omega^\varepsilon} A^\varepsilon \varphi_1 \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega^\varepsilon} A^\varepsilon x_1 \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} - \int_{\Omega^\varepsilon} A^\varepsilon x_1 \partial_{x_2} \varphi_2 \, \mathrm{d}\boldsymbol{x} + \int_{\Omega^\varepsilon} A^\varepsilon \varphi_1 \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega^\varepsilon} A^\varepsilon x_1 \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} - \int_0^1 A^\varepsilon x_1 \big(\varphi_2(x_1, \varepsilon, \cdot) - \varphi_2(x_1, 0, \cdot)\big) \mathrm{d}x_1 + \int_{\Omega^\varepsilon} A^\varepsilon \varphi_1 \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega^\varepsilon} A^\varepsilon x_1 \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} - A^\varepsilon \int_{\Gamma^\varepsilon_{\text{e}}} x_1 \varphi_2 \, \mathrm{d}x_1 + \int_{\Omega^\varepsilon} A^\varepsilon \varphi_1 \, \mathrm{d}\boldsymbol{x}
\end{aligned}$$

The first term can be considered as a linear pressure increase through the horizontal variable. The second term, also for the linear decrease, changes the external force f^ε on the plate, while the third term can be understood as an addition to the horizontal component of the external force $\boldsymbol{g}^\varepsilon$. Although this change does not affect the existence of the solution, it is clear that it affects the solution estimates. It should be noted that the external forces and pressure drop also depend on ε. Their dependence on ε will be defined later, and it is such that the order of the external forces leads to the expected order of the solution estimates. Therefore, this linear pressure increase by the factor A^ε does not affect the order of the estimate at the end. From now on we assume that $A^\varepsilon = 0$, and despite this change we continue to use the symbols for external forces introduced earlier.

2.2. *Assumptions.* In this subsection we will define the dependence of the external forces on ε as well as the spaces to which they belong. Let f^ε and $\boldsymbol{g}^\varepsilon$ be of the form:

$$f^\varepsilon(x_1, t) = f(x_1, \varepsilon^2 t)$$
$$\boldsymbol{g}^\varepsilon(x_1, x_2, t) = \boldsymbol{g}(x_1, x_2/\varepsilon, \varepsilon^2 t),$$

where f and $\boldsymbol{g}$ are defined respectively on $(0,1) \times (0,T)$ and $\Omega \times (0,T)$, $\Omega = (0,1) \times (0,1)$. We assume

$$f \in H^2(0,T; L^2(0,1)) \cap L^2(0,T; H^2(0,1)),$$
$$\boldsymbol{g} = (g_1, g_2) \in H^2(0,T; L^2(\Omega)^2), \text{ such that } g_1 \in L^2(0,T; H^2(\Omega))$$

and initial zero forces: $f(\cdot, 0) = 0$ in $(0,1)$, $\boldsymbol{g}(\cdot, 0) = 0$, in Ω.

Since the selection of spaces is part of our earlier work, we will only give a brief overview of the assumptions. The existence and uniqueness of the fluid velocity and the plate displacement can be proved by the Galerkin approximation, assuming only the L^2 regularity of the forces. Because of the prescribed pressure drop, it is not sufficient to prove the existence of the pressure up to a function of time. Furthermore, since in the fluid structure problem the force of the fluid acting on the plate is described by the pressure, the pressure is uniquely determined. The time regularity of the forces of order H^1 is sufficient for a unique existence result for the pressure and also automatically guarantees the time smoothness of velocity and displacement. See [3] for details.

The additional regularity in time together with the zero initial values of the forces make it possible to study the time derivatives of velocity, plate and pressure also in the context of convergence. This is possible because these time derivatives are exactly the solutions for the case when the external forces are $\partial_t f^\varepsilon$ and $\partial_t \boldsymbol{g}^\varepsilon$ and the boundary conditions remain unchanged.

The reduced problem, as mentioned earlier, involves a sixth-order equation. Assuming regularity in spatial variables, this equation has a solution in the L^2 sense, i.e., the reduced displacement has H^6 regularity in space. See [4] for details.

If the reader wishes to avoid details from previous work, one can assume a C^∞ regularity for which this linear problem has a smooth solution.

REMARK 2.1. By setting initial values the appearance of the boundary layer in time is avoided. The values do not need to vanish, but they must be small enough not to affect the order of the solution. Similar assumptions can be found in [8], where it is assumed that the external influence vanishes for small values of time.

2.3. *Weak formulation.* In this subsection variational formulation of the problem (2.1)-(2.11) is presented. We introduce the following functional spaces for the fluid velocity

$$\mathcal{V}^\varepsilon = \{\boldsymbol{u}^\varepsilon \in H^1(\Omega^\varepsilon)^2 : \operatorname{div} \boldsymbol{u}^\varepsilon = 0,\ \boldsymbol{u}^\varepsilon = 0 \text{ on } \Gamma_\mathrm{b},\ \boldsymbol{u}^\varepsilon \times \boldsymbol{n} = 0 \text{ on } \Gamma^\varepsilon_{\mathrm{in/out}} \cup \Gamma^\varepsilon_\mathrm{e}\},$$

$$V^\varepsilon = L^\infty(0, T^\varepsilon; L^2(\Omega^\varepsilon)^2) \cap L^2(0, T; \mathcal{V}^\varepsilon),$$

and for the plate displacement

$$D^\varepsilon = L^2(0, T^\varepsilon; H^2_0(0,1)) \cap H^1(0, T^\varepsilon; L^2(0,1)).$$

Taking into account that the trace on $\Gamma^\varepsilon_\mathrm{e}$ of a test function for variational formulation (2.12) is in $H^2(0,1)$, we introduce the set $\mathcal{H}^\varepsilon$ of vector functions $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in H^1(\Omega^\varepsilon)^2$ such that $\boldsymbol{\varphi} = 0$ on Γ_b, $\boldsymbol{\varphi} \times \boldsymbol{n} = 0$ on $\Gamma^\varepsilon_{\mathrm{in/out}} \cup \Gamma^\varepsilon_\mathrm{e}$ and $\varphi_2 \in H^2_0(0,1)$ on $\Gamma^\varepsilon_\mathrm{e}$. For each ε we define the solution of the ε-problem as follows:

Definition 2.2. *Vector function $(\boldsymbol{u}^\varepsilon, h^\varepsilon, p^\varepsilon) \in V^\varepsilon \times D^\varepsilon \times L^2(\Omega^\varepsilon \times (0, T^\varepsilon))$ is a weak solution of the ε-problem if*

1. *$h^\varepsilon(\cdot, 0) = 0$ in $(0,1)$,*
2. *$\boldsymbol{u}^\varepsilon = (0, \partial_t h^\varepsilon)$ on $\Gamma^\varepsilon_\mathrm{e} \times (0, T^\varepsilon)$,*
3. *for all $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in H^1(0, T; \mathcal{H}^\varepsilon)$ such that $\boldsymbol{\varphi}(\cdot, T^\varepsilon) = 0$ in Ω^ε variational formulation (2.12) holds.*

3. Derivation of the effective model

In this Section, we derive a weak formulation of the effective model, repeat some of the results on boundary conditions from previous studies, and supplement them with the regularity results needed for error estimates.

3.1. *Rescaled model.* To determine the asymptotic approximation of the solution, it is necessary to define the problem on the fixed domain $\Omega \times (0, T)$, $\Omega = (0,1) \times (0,1)$. We denote the boundary parts of Ω analogously to the original domain and omit the symbol ε. We introduce rescaled functions

$$\begin{aligned}
&\boldsymbol{u}(\varepsilon)(y_1, y_2, \tau) = (u(\varepsilon), v(\varepsilon))(y_1, y_2, \tau) = \boldsymbol{u}^\varepsilon(y_1, \varepsilon y_2, \tau/\varepsilon^2)\\
&p(\varepsilon)(y_1, y_2, \tau) = p^\varepsilon(y_1, \varepsilon y_2, \tau/\varepsilon^2)\\
&h(\varepsilon)(y_1, \tau) = h^\varepsilon(y_1, \tau/\varepsilon^2)
\end{aligned}$$

defined on $\Omega \times (0, T)$ and $(0,1) \times (0, T)$. The introduced scaling leads to changes in the derivative with respect to the second variable and to time, while the derivative after the first variable remains unchanged. In the sequel we use the following notation for rescaled differential operators applied to vector and scalar functions

$$\nabla^\varepsilon \boldsymbol{u} = \begin{bmatrix} \partial_{y_1} u & \partial_{y_2} u/\varepsilon \\ \partial_{y_1} v & \partial_{y_2} v/\varepsilon \end{bmatrix}, \quad \nabla^\varepsilon p = \begin{bmatrix} \partial_{y_1} p \\ \partial_{y_2} p/\varepsilon \end{bmatrix}.$$

The rescaled symmetric part of the gradient and the rescaled divergence are denoted by

$$e^\varepsilon(\boldsymbol{u}) = \frac{1}{2}\left(\nabla^\varepsilon \boldsymbol{u} + (\nabla^\varepsilon \boldsymbol{u})^\tau\right),$$

$$\operatorname{div}^\varepsilon \boldsymbol{u} = \partial_{y_1} u + \frac{1}{\varepsilon}\partial_{y_2} v.$$

To define the space $\mathcal{V}(\varepsilon)$ of rescaled fluid velocity, we rely on earlier definition of the velocity space for the case $\varepsilon = 1$, changing the incompressibility condition to $\operatorname{div}^\varepsilon \boldsymbol{u} = 0$. More precisely

$$\mathcal{V}(\varepsilon) = \left\{\boldsymbol{u} \in H^1(\Omega)^2 : \operatorname{div}^\varepsilon \boldsymbol{u} = 0, u = 0 \text{ on } \Gamma_{\mathrm{b}},\ \boldsymbol{u} \times \boldsymbol{n} = 0 \text{ on } \Gamma_{\mathrm{in/out}} \cup \Gamma_{\mathrm{e}}\right\}.$$

Similarly, using $\varepsilon = 1$, we set the test function space $\mathcal{H}^1$ as an analogue of $\mathcal{H}^\varepsilon$ for the fixed domain. Taking into account that $\mathrm{d}x_2 = \varepsilon \mathrm{d}y_2$ and $\mathrm{d}t = \mathrm{d}\tau/\varepsilon^2$, we can derive from Definition 2.2 the definition of the weak solution of the rescaled problem.

Definition 3.1. *Vector function $(\boldsymbol{u}(\varepsilon), h^\varepsilon, p(\varepsilon))$ that belongs to*

$$\left(L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;\mathcal{V}(\varepsilon))\right) \times D^1 \times L^2(\Omega \times (0,T))$$

is a weak solution of the rescaled ε-problem on $[0,T)$ if the following hold

1. $h(\varepsilon)(\cdot,0) = 0$ *in* $(0,1)$,
2. $\boldsymbol{u}(\varepsilon) = (0, \varepsilon^2\partial_\tau h(\varepsilon))$ *on* $\Gamma_{\mathrm{e}} \times (0,T)$,
3. *for all* $\boldsymbol{\varphi} \in H^1(0,T;\mathcal{H}^1)$ *such that* $\boldsymbol{\varphi}(\cdot,T) = 0$ *in* Ω *the following variational equation is satisfied*

(3.1)

$$\begin{aligned}
&-\rho_{\mathrm{f}}\varepsilon \int_0^T \int_\Omega \boldsymbol{u}(\varepsilon)\cdot\partial_\tau\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau + \frac{2\mu}{\varepsilon}\int_0^T\int_\Omega e^\varepsilon(\boldsymbol{u}(\varepsilon)) : e^\varepsilon(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau \\
&-\rho_{\mathrm{s}} b\varepsilon^2 \int_0^T\int_{\Gamma_{\mathrm{e}}} \partial_\tau h(\varepsilon)\partial_\tau\varphi_2\,\mathrm{d}y_1\mathrm{d}\tau + \frac{B}{\varepsilon^3}\int_0^T\int_{\Gamma_{\mathrm{e}}} \partial_{y_1}^2 h(\varepsilon)\partial_{y_1}^2\varphi_2\,\mathrm{d}y_1\mathrm{d}\tau \\
&-\frac{1}{\varepsilon}\int_0^T\int_\Omega p(\varepsilon)\operatorname{div}^\varepsilon\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau = \frac{1}{\varepsilon^2}\int_0^T\int_{\Gamma_{\mathrm{e}}} f\varphi_2\,\mathrm{d}y_1\mathrm{d}\tau + \frac{1}{\varepsilon}\int_0^T\int_\Omega \boldsymbol{g}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau.
\end{aligned}$$

By standard procedure and careful calculations, it is possible to obtain an estimate for the fluid velocity and plate displacement using Gronwall's lemma. Strictly speaking, the fluid velocity cannot be a test function since its trace on the top of the channel is not smooth enough. (In this step, we do not use the regularity of the solution in time, because we will use the same conclusion for the time derivatives of the solution and in that case the solution is truly not smooth enough to serve as a test function.) However, considering that

$\partial_\tau \partial_{y_1}^2 h \cdot \partial_{y_1}^2 h = 1/2 \cdot \partial_\tau (\partial_{y_1}^2 h)^2$ in the sense of distribution and that partial integration with respect to time can be applied due to the regularity, this difficulty can be bypassed. This leads to

$$\begin{aligned} &\|\partial_{y_2} \boldsymbol{u}(\varepsilon)\|^2_{L^2(\Omega\times(0,T))^2} + \varepsilon^2 \|\partial_{y_1} \boldsymbol{u}(\varepsilon)\|^2_{L^2(\Omega\times(0,T))^2} \leq C\varepsilon^4 \\ &\left\|\partial^2_{y_1} h(\varepsilon)\right\|^2_{L^\infty(0,T;L^2(0,1))} \leq C\varepsilon^2, \end{aligned} \tag{3.2}$$

where C is constant independent of ε. From now on we will denote all constants independent of ε by C. Since the forces acting on the system are smooth in time and initially equal to zero, it is possible to perform evaluations for the time derivative of the velocity and plate displacement.

3.2. *Pressure estimates.* The estimate for the pressure p is obtained by choosing the test function as an appropriate linear combination of $L^2(\Omega)$ functions related to the problem

$$\begin{aligned} \operatorname{div} \tilde{\boldsymbol{\varphi}} &= q \quad \text{in } \Omega, \\ \tilde{\boldsymbol{\varphi}} &= 0 \quad \text{on } \Gamma_{\rm b} \cup \Gamma_{\rm e}, \\ \tilde{\varphi}_2 &= 0 \quad \text{on } \Gamma_{\rm in} \cup \Gamma_{\rm out}, \\ \|\tilde{\boldsymbol{\varphi}}\|_{H^1(\Omega)^2} &\leq C \|q\|_{L^2(\Omega)}. \end{aligned}$$

Referring to the density of the chosen linear combination in $L^2(\Omega \times (0,T)$, we obtain estimates

$$\begin{aligned} &\|p(\varepsilon)\|_{L^2(\Omega\times(0,T))} \leq C, \\ &\|\partial_x p(\varepsilon)\|_{L^2(0,T;H^{-1}(\Omega))} + \frac{1}{\varepsilon} \|\partial_y p(\varepsilon)\|_{L^2(0,T;H^{-1}(\Omega))} \leq C. \end{aligned}$$

3.3. *Convergence results.* The estimates for the solution of the rescaled ε-problem imply the following claim.

THEOREM 3.2. *There exists a subsequence, denoted in the same way, of sequence $(\boldsymbol{u}(\varepsilon), h(\varepsilon), p(\varepsilon))$ of solutions of the rescaled ε-problems, such that*

$$\begin{aligned} &\frac{1}{\varepsilon^2}\boldsymbol{u}(\varepsilon) \rightharpoonup \boldsymbol{u} = (u,v) \ \textit{weakly in } L^2(\Omega\times(0,T))^2, \\ &\frac{1}{\varepsilon^2}\partial_{y_2}\boldsymbol{u}(\varepsilon) \rightharpoonup \partial_{y_2}\boldsymbol{u} \ \textit{weakly in } L^2(\Omega\times(0,T))^2, \\ &p(\varepsilon) \rightharpoonup p \ \ \textit{weakly in } L^2(\Omega\times(0,T)), \\ &\frac{h(\varepsilon)}{\varepsilon} \overset{*}{\rightharpoonup} h \ \textit{weak} * \textit{in } L^\infty(0,T;H^2_0(0,1)). \end{aligned}$$

3.4. *Boundary conditions.* Rescaled functions $\boldsymbol{u}(\varepsilon)$ and $h(\varepsilon)$ satisfy boundary conditions on $\Gamma_{\rm b}$ and the edges of the plate. Due to the convergence of $\partial_{y_2}\boldsymbol{u}(\varepsilon)$

and $\partial^2_{y_1} h(\varepsilon)$ in $L^2(\Omega \times (0,T))^2$ and $L^\infty(0,T; L^2(0,1))$ respectively, limiting functions fulfill analogous boundary conditions, namely

$$\begin{aligned}&\boldsymbol{u}|_{\Gamma_{\mathrm{b}}} = 0 \quad \text{in } (0,T),\\ &h(0,\cdot) = \partial_{y_1} h(0,\cdot) = h(1,\cdot) = \partial_{y_1} h(1,\cdot) = 0 \quad \text{in } (0,T).\end{aligned}$$

Moreover, $u|_{\Gamma_{\mathrm{e}}} = 0 \quad \text{in } (0,T)$.

3.5. *Transverse velocity component.* Due to the divergence-free condition $\mathrm{div}_\varepsilon\, \boldsymbol{u}(\varepsilon) = 0$, for almost all t and all $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega \frac{\partial_y v(\varepsilon)}{\varepsilon^2} \varphi \, \mathrm{d}\boldsymbol{y} = \varepsilon \int_\Omega \frac{u(\varepsilon)}{\varepsilon^2} \partial_x \varphi \, \mathrm{d}\boldsymbol{y} \to 0 \quad \text{as } \varepsilon \to 0$$

which leads to the conclusion $\partial_y v = 0$, and combined with the boundary condition on Γ_{b} gives $v = 0$.

3.6. *Incompressibility condition.* Similarly, the test function $\boldsymbol{\varphi} = (0, \varepsilon^2 \psi \varphi_2)$ is used, where $\psi \in C^1(0,T)$, $\psi(T) = 0$ and $\varphi_2 \in C_0^\infty(\Omega)$, for the variational formulation (3.1) of the rescaled problem

$$\begin{aligned}\int_0^T \int_\Omega \psi p(\varepsilon) \partial_{y_2} \varphi_2 \, \mathrm{d}\boldsymbol{y}\mathrm{d}\tau &= -\rho_{\mathrm{f}} \varepsilon^3 \int_0^T \int_\Omega \psi' v(\varepsilon) \varphi_2 \, \mathrm{d}\boldsymbol{y}\mathrm{d}\tau\\ &+ 2\mu\varepsilon \int_0^T \int_\Omega \psi \frac{1}{2} \left(\frac{1}{\varepsilon} \partial_{y_2} u(\varepsilon) + \partial_{y_1} v(\varepsilon) \right) \partial_{y_1} \varphi_2 \, \mathrm{d}\boldsymbol{y}\mathrm{d}\tau\\ &+ 2\mu\varepsilon \int_0^T \int_\Omega \psi \frac{1}{\varepsilon} \partial_{y_2} v(\varepsilon) \cdot \frac{1}{\varepsilon} \partial_{y_2} \varphi_2 \, \mathrm{d}\boldsymbol{y}\mathrm{d}t - \varepsilon \int_0^T \int_\Omega \psi g_2 \varphi_2 \, \mathrm{d}\boldsymbol{y}\mathrm{d}\tau.\end{aligned}$$

Since the right-hand side converges to 0 as $\varepsilon \to 0$, it follows

$$\int_0^T \int_\Omega \psi p \partial_{y_2} \varphi_2 \, \mathrm{d}\boldsymbol{y}\mathrm{d}\tau = 0$$

for any ψ and φ_2, so we derive $\partial_{y_2} p = 0$, i.e. p does not depend on y_2.

Furthermore, from the divergence-free condition and the continuity of the velocity at the top boundary, it is possible to derive an estimate and convergence that guarantees the fulfillment of the initial condition for the plate. The same conclusion can be drawn, as stated in Subsection 2.2, using the estimates for $\partial_\tau h(\varepsilon)$ in terms of $\partial_\tau f$ and $\partial_\tau g$. We use the latter approach and conclude $h(\cdot, 0) = 0$ in $(0,1)$ and $h \in H^1(0,T; H_0^2(0,1)$.

Therefore from $\mathrm{div}^\varepsilon\, \boldsymbol{u}(\varepsilon) = 0$, and $v(\varepsilon)|_{\Gamma_{\mathrm{e}}} = \varepsilon^2 \partial_\tau h(\varepsilon)$, for any $\psi \in C_0^\infty(0,T)$ and $\varphi = \varphi(y_1) \in C_0^\infty(0,1)$, the following holds

$$-\int_0^T \int_\Omega \psi u(\varepsilon) \partial_{y_1} \varphi \, \mathrm{d}\boldsymbol{y}\mathrm{d}\tau + \varepsilon^2 \int_0^T \int_0^1 \frac{\partial_\tau h(\varepsilon)}{\varepsilon} \psi \varphi \, \mathrm{d}y_1 \mathrm{d}\tau = 0.$$

After integration by parts, the passing to the limit $\varepsilon \to 0$ yields

$$-\int_0^T \int_\Omega \psi u \partial_{y_1} \varphi \, \mathrm{d}\boldsymbol{y}\mathrm{d}\tau - \int_0^T \int_0^1 h \psi' \varphi \, \mathrm{d}y_1 \mathrm{d}\tau = 0.$$

Integration by parts with respect to y_1 and τ now yields

$$\frac{\partial}{\partial y_1}\int_0^1 u\,\mathrm{d}y_2 + \partial_\tau h = 0. \tag{3.3}$$

Even if we did not assume regularity in time, this equality would hold, but in the sense of distributions. But thanks to this additional assumption and consequently the smoothness of h, we get

$$\frac{\partial}{\partial y_1}\int_0^1 u\,\mathrm{d}y_2 \in L^2(0,T;H_0^2(0,1)). \tag{3.4}$$

3.7. *Effective model.* It is not difficult to see that, by passing to limit $\varepsilon \to 0$ in the weak formulation of the rescaled problem, we obtain that for all $\boldsymbol{\varphi} \in H^1(0,T;\mathcal{H}^1)$, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$, such that $\boldsymbol{\varphi}(\cdot, T) = 0$ in Ω (the test function used for the rescaled weak formulation is $(\varphi_1, \varepsilon\varphi_2)$) the following variational formulation holds

$$\begin{aligned}
&\mu\int_0^T\int_\Omega \partial_{y_2}u\,\partial_{y_2}\varphi_1\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau + B\int_0^T\int_{\Gamma_\mathrm{e}}\partial_{y_1}^2 h\partial_{y_1}^2\varphi_2\,\mathrm{d}y_1\mathrm{d}\tau\\
&-\int_0^T\int_\Omega p\,\mathrm{div}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau = \int_0^T\int_{\Gamma_\mathrm{e}} f\varphi_2\,\mathrm{d}y_1\mathrm{d}\tau + \int_0^T\int_\Omega g_1\varphi_1\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau.
\end{aligned} \tag{3.5}$$

This weak formulation, together with the previously stated boundary and initial conditions for u and h and the modified law of incompressibility (3.3) forms a linear problem that has a solution (limit of a sequence of solutions of rescaled problems) and this solution is unique [4].

3.8. *Pressure boundary conditions.* By choosing an appropriate test function for the variational formulation of the effective problem, it is possible to prove a higher regularity of pressure, which also gives meaning to the boundary conditions for the pressure at the entrance and at the exit of the channel. Let $\varphi = \varphi(y_1)$ be an arbitrary function in $C^\infty(0,1)$ and $\psi \in C^\infty(0,T)$. By substituting into the variational equation (3.5) the test function $\varphi_1(y_1,y_2,\tau) = y_2(1-y_2)\psi(\tau)\varphi(y_1)$, $\varphi_2 = 0$, remembering that p does not depend on y_2, we get

$$\begin{aligned}
&\mu\int_0^T\int_\Omega(1-2y_2)\psi\varphi\,\partial_{y_2}u\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau - \frac{1}{6}\int_0^T\int_0^1\psi\varphi'\,p\,\mathrm{d}y_1\mathrm{d}\tau\\
&=\int_0^T\int_\Omega y_2(1-y_2)\psi\varphi g_1\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau.
\end{aligned}$$

Integration by parts leads to

$$\begin{aligned}(3.6)\qquad &2\mu\int_0^T\int_0^1\psi\varphi\int_0^1 u\,\mathrm{d}y_2\mathrm{d}y_1\mathrm{d}\tau-\frac{1}{6}\int_0^T\int_0^1\psi\varphi'\,p\,\mathrm{d}y_1\mathrm{d}\tau\\ &=\int_0^T\int_\Omega y_2(1-y_2)\psi\varphi g_1\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau.\end{aligned}$$

Reducing the space of the test function for φ to $C_c^\infty(0,1)$, leads to, in the sense of the distribution, equation

$$2\mu\int_0^1 u\,\mathrm{d}y_2+\frac{1}{6}\partial_{y_1}p=\int_0^1 y_2(1-y_2)g_1\mathrm{d}y_2. \tag{3.7}$$

From (3.4) and the assumptions 2.2 we conclude $\partial_{y_1}p\in L^2(0,T;H^2(0,1))$. Considering again not necessarily compactly supported test functions, we obtain from (3.6) and (3.7) that for all $\varphi=\varphi(y_1)\in C^\infty(0,1)$ holds

$$-\frac{1}{6}\int_0^T\psi\big(p(1,\cdot)\varphi(1)-p(0,\cdot)\varphi(0)\big)\,\mathrm{d}\tau=0,$$

leading to boundary conditions $p=0$ on Γ_{in} and Γ_{out}.

4. Error estimates

We obtain the effective velocity $(u,v)=(u,0)$ as the limit of the solutions of the rescaled problem and as the solution of the effective model. The divergence of this velocity is not 0, and the question arises how good this approximation is for the velocity of an incompressible fluid and how to interpret the incompressibility in the effective model appropriately. To obtain the incompressibility in the approximation, we need a higher order term (in epsilon) for the transverse velocity. Therefore, we correct this velocity component with v_1, which we construct as follows. From the variational formulation (3.5) of the effective model, considering the pressure regularity, it follows that in $L^2(0,T;H^2(\Omega))$ holds

$$\mu\partial_{y_2}^2u=\partial_{y_1}p-g_1.$$

By integrating equation twice with respect to y_2, we get

$$\mu u(y_1,y_2,\tau)=\partial_{y_1}p(y_1,\tau)\frac{y_2^2}{2}-\int_0^{y_2}\int_0^\eta g_1(y_1,\xi,t)\,\mathrm{d}\xi\mathrm{d}\eta+C_1(y_1,\tau)y_2+C_2(y_1,\tau).$$

If we change the order of integration, we get

$$\mu u(y_1,y_2,\tau)=\partial_{y_1}p(y_1,\tau)\frac{y_2^2}{2}-\int_0^{y_2}(y_2-\xi)g_1(y_1,\xi,t)\,\mathrm{d}\xi+C_1(y_1,\tau)y_2+C_2(y_1,\tau).$$

Due to the boundary condition $u|_{\Gamma_{\mathrm{b}}}=0$ we have $C_2(y_1,\tau)=0$, while C_1 is determined from the boundary condition $u|_{\Gamma_{\mathrm{e}}}=0$:

$$C_1(y_1,\tau)=-\frac{\partial_{y_1}p(y_1,\tau)}{2}+\int_0^1(1-\xi)g_1(y_1,\xi,\tau)\,\mathrm{d}\xi. \tag{4.1}$$

Finally, the longitudinal component of the fluid velocity has the following form

$$
\begin{aligned}
u(y_1, y_2, \tau) &= \frac{y_2(y_2-1)}{2\mu}\partial_{y_1} p(y_1, \tau) \\
&- \frac{1}{\mu}\int_0^{y_2}(y_2-\xi)g_1(y_1,\xi,\tau)\,\mathrm{d}\xi + \frac{y_2}{\mu}\int_0^1(1-\xi)g_1(y_1,\xi,\tau)\,\mathrm{d}\xi.
\end{aligned}
\tag{4.2}
$$

Note that $u \in L^2((0,T); H^2(\Omega))$. Incompressibility, interpreted as $\operatorname{div}(u, v_1) = 0$, and the boundary condition $v_1|_{\Gamma_b} = 0$ imply that the transverse component of velocity is defined by

$$
\begin{aligned}
v_1(y_1, y_2, \tau) &= -\int_0^{y_2}\partial_{y_1} u(y_1,\eta,\tau)\,\mathrm{d}\eta = -\left(\frac{y_2^3}{3} - \frac{y_2^2}{2}\right)\frac{\partial_{y_1}^2 p(y_1,\tau)}{2\mu} \\
&+ \frac{1}{\mu}\int_0^{y_2}\int_0^{\eta}(\eta-\xi)\partial_{y_1} g_1(y_1,\xi,t)\,\mathrm{d}\xi\mathrm{d}\eta - \int_0^{y_2}\frac{\eta}{\mu}\int_0^1(1-\xi)\partial_{y_1} g_1(y_1,\xi,\tau)\,\mathrm{d}\xi\mathrm{d}\eta
\end{aligned}
$$

transformed by changing the order of integration to

$$
\begin{aligned}
v_1(y_1, y_2, \tau) &= -\left(\frac{y_2^3}{3} - \frac{y_2^2}{2}\right)\frac{\partial_{y_1}^2 p(y_1,\tau)}{2\mu} \\
&+ \int_0^{y_2}\frac{(y_2-\xi)^2}{2\mu}\partial_{y_1} g_1(y_1,\xi,\tau)\,\mathrm{d}\xi - \frac{y_2^2}{2\mu}\int_0^1(1-\xi)\partial_{y_1} g_1(y_1,\xi,\tau)\,\mathrm{d}\xi.
\end{aligned}
$$

Note that because of the (3.3) boundary condition $v_1|_{\Gamma_e} = \partial_\tau h$ holds on the top boundary. It also holds $v_1 \in L^2((0,T); H^1(\Omega))$ and furthermore v_1 has the second derivative with respect to y_2 in $L^2(\Omega \times (0,T))$.

In order to estimate the approximation error, we define

$$
\begin{aligned}
\boldsymbol{u}_{\text{error}}(\varepsilon) &= \left(\frac{u(\varepsilon)}{\varepsilon^2}, \frac{v(\varepsilon)}{\varepsilon^3}\right) - (u, v_1) \\
p_{\text{error}}(\varepsilon) &= p(\varepsilon) - p \\
h_{\text{error}}(\varepsilon) &= h(\varepsilon)/\varepsilon - h.
\end{aligned}
$$

Let $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ be an arbitrary function from $H^1(0,T;\mathcal{H}^1)$. Using the test functions $(\varepsilon\varphi_1, \varepsilon^2\varphi_2)$ and $\boldsymbol{\varphi}$ for weak formulations of the rescaled problem (3.1) and the effective model (3.5), respectively, and subtracting them we obtain

(4.3)

$$
\begin{aligned}
&\mu\int_0^T\int_\Omega \partial_{y_2}u_{\mathrm{error}}(\varepsilon)\partial_{y_2}\varphi_1\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau + B\int_0^T\int_{\Gamma_{\mathrm{e}}}\partial_{y_1}^2 h_{\mathrm{error}}(\varepsilon)\partial_{y_1}^2\varphi_2\,\mathrm{d}y_1\mathrm{d}\tau\\
&-\int_0^T\int_\Omega p_{\mathrm{error}}(\varepsilon)\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau = \varepsilon\int_0^T\int_\Omega g_2\varphi_2\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau\\
&+\rho_{\mathrm{f}}\varepsilon^2\int_0^T\int_\Omega \boldsymbol{u}(\varepsilon)\cdot\partial_\tau(\varphi_1,\varepsilon\varphi_2)\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau + \rho_{\mathrm{s}}b\varepsilon^4\int_0^T\int_{\Gamma_{\mathrm{e}}}\partial_\tau h(\varepsilon)\partial_\tau\varphi_2\,\mathrm{d}x_1\mathrm{d}\tau\\
&-\mu\int_0^T\int_\Omega\Big(2\partial_{y_1}u(\varepsilon)\partial_{y_1}\varphi_1+\partial_{y_2}u(\varepsilon)\partial_{y_1}\varphi_2+\frac{1}{\varepsilon}\partial_{y_1}v(\varepsilon)\partial_{y_2}\varphi_1\\
&\quad+\varepsilon\partial_{y_1}v(\varepsilon)\partial_{y_1}\varphi_2+\frac{2}{\varepsilon}\partial_{y_2}v(\varepsilon)\partial_{y_2}\varphi_2\Big)\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau.
\end{aligned}
$$

It would be ideal to use $\boldsymbol{u}^\varepsilon_{\mathrm{error}}$ for the test function. This function is smooth enough, but the problem is that the second velocity component does not meet the boundary conditions at the inlet and outlet.

4.1. *Correction for the test function.* Let us define

$$
\begin{aligned}
v_{\mathrm{in}}(y_2,\tau) &= v_1(0,y_2,\tau)\\
v_{\mathrm{out}}(y_2,\tau) &= v_1(1,y_2,\tau).
\end{aligned}
$$

Due to the boundary condition $v_1|_{\Gamma_{\mathrm{e}}} = \partial_\tau h$ we have $v_{\mathrm{in}}(1,\tau) = v_{\mathrm{out}}(1,\tau) = 0$ and due to $\partial_{y_2}v_1 = \partial_{y_1}u$ and the choice of C_1 in (4.1) we have $\partial_{y_2}v_{\mathrm{in}}(1,\tau) = \partial_{y_2}v_{\mathrm{out}}(1,\tau) = 0$. It is easy to see that the same boundary conditions are met for $y_2 = 0$.

For $0<\varepsilon\ll 1$ we define $\mu_\varepsilon : [0,1]\to[0,1]$ in $C^\infty(0,1)$ such that:

$$
\mu_\varepsilon(y_1) = \begin{cases} 1-y_1, & y_1<\varepsilon\\ 0, & y_1>2\varepsilon.\end{cases}
$$

Then for

$$
\tilde{\boldsymbol{\varphi}}(\varepsilon) = (\mu_\varepsilon(y_1)\partial_{y_2}v_{\mathrm{in}}, -\mu_\varepsilon'(y_1)v_{\mathrm{in}}) + (-\mu_\varepsilon(1-y_1)\partial_{y_2}v_{\mathrm{out}}, -\mu_\varepsilon'(1-y_1)v_{\mathrm{out}})
$$

we have

$$
\begin{aligned}
\operatorname{div}\tilde{\boldsymbol{\varphi}}(\varepsilon) &= 0 \quad \text{in } \Omega,\\
\tilde{\boldsymbol{\varphi}}(\varepsilon) &= 0 \quad \text{on } \Gamma_{\mathrm{b}}\cup\Gamma_{\mathrm{e}},\\
\tilde{\boldsymbol{\varphi}}(\varepsilon) &= (0,v_1) \quad \text{on } \Gamma_{\mathrm{in}}\cup\Gamma_{\mathrm{out}}
\end{aligned}
$$

and obviously

$$
\|\tilde{\boldsymbol{\varphi}}(\varepsilon)\|_{L^2(\Omega)^2} + \|\partial_{y_2}\tilde{\boldsymbol{\varphi}}(\varepsilon)\|_{L^2(\Omega)^2} \le C\varepsilon. \tag{4.4}
$$

After partial integration by time of the terms in (4.3) containing the time derivative of the solution, we can use $\boldsymbol{\varphi}_{\mathrm{error}}(\varepsilon) = \boldsymbol{u}_{\mathrm{error}}(\varepsilon) - \tilde{\boldsymbol{\varphi}}(\varepsilon)$ as a test function. We note that the trace of this function at the top boundary is $(0, \partial_\tau h_{\text{error}}(\varepsilon))$.

4.2. *Error estimates on the fixed domain.* We substitute the selected test function in (4.3). Let us highlight some general points about the derivation of estimates:

1. (u, v_1) is a function defined by the effective model whose regularity we have proved. This function has a norm bounded by a constant that depends on the given functions.
2. The corrector $\tilde{\boldsymbol{\varphi}}(\varepsilon)$ has a norm bound (4.4).
3. Sometimes v_1 is combined with the corrector $\tilde{\varphi}_2(\varepsilon)$ and the norm of their sum is bounded by C.
4. On the right-hand side, the norms of the different terms will appear with a negative sign, so we will estimate them with zero.
5. We will use estimates (3.2). Let's recall that the assumption 2.2 enable us to estimate time derivatives (in terms of the time derivatives of the given functions) in the same order of ε.

Let us write out the terms of the last integral in (4.3) with the chosen test function. For the first term in the integral, due to the incompressibility, the following applies:

$$
\begin{aligned}
&-\int_0^T \int_\Omega 2\partial_{y_1} u(\varepsilon)\partial_{y_1}\varphi_{\text{error}1}(\varepsilon)\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau = -\int_0^T \int_\Omega \frac{2}{\varepsilon}\partial_{y_2} v(\varepsilon)\partial_{y_2}\varphi_{\text{error}2}(\varepsilon)\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau \\
&= -\frac{2}{\varepsilon}\int_0^T \int_\Omega \partial_{y_2} v(\varepsilon)\partial_{y_2}\left(\frac{v(\varepsilon)}{\varepsilon^3} - v_1 - \tilde{\varphi}_2(\varepsilon)\right)\mathrm{d}\boldsymbol{y}\mathrm{d}\tau \\
&= -2\varepsilon^2 \int_0^T \int_\Omega \partial_{y_2}\varphi_{\text{error}2}(\varepsilon)\partial_{y_2}\varphi_{\text{error}2}(\varepsilon) + \partial_{y_2}(v_1 + \tilde{\varphi}_2(\varepsilon))\partial_{y_2}\varphi_{\text{error}2}(\varepsilon)\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau \\
&\leq 2\varepsilon^2\left(-\|\partial_{y_2}\varphi_{\text{error}2}(\varepsilon)\|^2_{L^2(\Omega\times(0,T))} + C\|\partial_{y_2}\varphi_{\text{error}2}(\varepsilon)\|_{L^2(\Omega\times(0,T))}\right) \\
&\leq 2\varepsilon^2\left(-\|\partial_{y_2}\varphi_{\text{error}2}(\varepsilon)\|^2_{L^2(\Omega\times(0,T))} + \frac{C^2}{4} + \|\partial_{y_2}\varphi_{\text{error}2}(\varepsilon)\|^2_{L^2(\Omega\times(0,T))}\right) \\
&\leq C\varepsilon^2.
\end{aligned}
$$

Bearing in mind that the limit of transverse velocities is 0 and that this statement comes from the condition of incompressibility, it is to be expected that we must use incompressibility in deriving the estimates. Let us derive additional calculations for the smooth function ψ, which reaches value 0 on Γ_{in} and Γ_{out} :

$$
\begin{aligned}
&\int_\Omega \partial_{y_2} u(\varepsilon)\partial_{y_1}\psi\,\mathrm{d}\boldsymbol{y} = -\int_\Omega u(\varepsilon)\partial_{y_2}\partial_{y_1}\psi\,\mathrm{d}\boldsymbol{y} \\
&= \int_\Omega \partial_{y_1} u(\varepsilon)\partial_{y_2}\psi\,\mathrm{d}\boldsymbol{y} = -\frac{1}{\varepsilon}\int_\Omega \partial_{y_2} v(\varepsilon)\partial_{y_2}\psi\,\mathrm{d}\boldsymbol{y}.
\end{aligned}
$$

The equality is true by density for all H^1 functions vanishing on Γ_{in} and Γ_{out}. Similarly, for all smooth functions ψ vanishing on Γ_{b} and Γ_{e} we have:

$$\int_\Omega \partial_{y_1} v(\varepsilon)\partial_{y_2}\psi \,\mathrm{d}\boldsymbol{y} = -\int_\Omega v(\varepsilon)\partial_{y_1}\partial_{y_2}\psi \,\mathrm{d}\boldsymbol{y} = \int_\Omega \partial_{y_2} v(\varepsilon)\partial_{y_1}\psi \,\mathrm{d}\boldsymbol{y} \tag{4.5}$$

and again we can impose the density argument.

Since the test function vanishes as described, the sum of the second and the third term of the integral results in

$$\begin{aligned}
&-\int_0^T \int_\Omega \partial_{y_2} u(\varepsilon)\partial_{y_1}\varphi_{\text{error}2}(\varepsilon) + \frac{1}{\varepsilon}\partial_{y_1} v(\varepsilon)\partial_{y_2}\varphi_{\text{error}1}(\varepsilon)\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau\\
&= -\int_0^T \int_\Omega -\frac{1}{\varepsilon}\partial_{y_2} v(\varepsilon)\partial_{y_2}\varphi_{\text{error}2}(\varepsilon) + \frac{1}{\varepsilon}\partial_{y_2} v(\varepsilon)\partial_{y_1}\varphi_{\text{error}1}(\varepsilon)\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau\\
&= \int_0^T \int_\Omega \frac{1}{\varepsilon}\partial_{y_2} v(\varepsilon)\partial_{y_2}\varphi_{\text{error}2}(\varepsilon) + \frac{1}{\varepsilon}\partial_{y_2} v(\varepsilon)\partial_{y_2}\varphi_{\text{error}2}(\varepsilon)\,\mathrm{d}\boldsymbol{y}\mathrm{d}\tau
\end{aligned}$$

which sums up with the last term to get 0. Finally, for the fourth term we have

$$-\int_0^T \int_\Omega \varepsilon\partial_{y_1} v(\varepsilon)\partial_{y_1}\left(\frac{v(\varepsilon)}{\varepsilon^3} - v_1 - \tilde{\varphi}_2(\varepsilon)\right)\mathrm{d}\boldsymbol{y}\mathrm{d}\tau \le 0 + \varepsilon\cdot C\varepsilon\cdot C \le C\varepsilon^2.$$

Now, one can easily see that from (4.3) we can obtain an estimate

$$\begin{aligned}
&\mu\left\|\partial_{y_2} u_{\text{error}}(\varepsilon)\right\|^2_{L^2(\Omega\times(0,T))} + B\left\|\partial^2_{y_1} h_{\text{error}}(\varepsilon)(\cdot,T)\right\|^2_{L^2(\Gamma_e)}\\
&\le \mu\left\|\partial_{y_2} u_{\text{error}}(\varepsilon)\right\|_{L^2(\Omega\times(0,T))}\cdot\left\|\partial_{y_2}\tilde{\varphi}_1(\varepsilon)\right\|_{L^2(\Omega\times(0,T))}\\
&\quad + \varepsilon\left\|g_2\right\|_{L^2(\Omega\times(0,T))}\left\|\varphi_{\text{error}2}(\varepsilon)\right\|_{L^2(\Omega\times(0,T))} + C\varepsilon^2\\
&\le \frac{\mu}{4}\left\|\partial_{y_2} u_{\text{error}}(\varepsilon)\right\|^2_{L^2(\Omega\times(0,T))} + \frac{1}{\mu}\left\|\partial_{y_2}\tilde{\varphi}_1(\varepsilon)\right\|^2_{L^2(\Omega\times(0,T))}\\
&\quad + C\varepsilon\left(\left\|u_{\text{error}2}(\varepsilon)\right\|_{L^2(\Omega\times(0,T))} + \left\|\tilde{\varphi}_2(\varepsilon)\right\|_{L^2(\Omega\times(0,T))}\right) + C\varepsilon^2\\
&\le \frac{\mu}{2}\left\|\partial_{y_2} u_{\text{error}}(\varepsilon)\right\|^2_{L^2(\Omega\times(0,T))} + C\varepsilon^2.
\end{aligned}$$

It is easy to see that the test function may include a time step function by which the displacement at time t would be estimated instead of T. This proves the assertion of the next theorem.

THEOREM 4.1. *For the problem presented, the following error estimates apply*

$$\left\|\partial_{y_2} u_{\text{error}}(\varepsilon)\right\|_{L^2(\Omega\times(0,T))} + \left\|\partial^2_{y_1} h_{\text{error}}(\varepsilon)\right\|_{L^\infty(0,T;L^2(\Gamma_e))} \le \varepsilon.$$

5. Discussion

In order not to perform simple and straightforward calculations or to repeat ideas already applied, no error estimates for pressure and transverse velocity are given. It should be noted that errors in the same order can be determined and that it is possible to derive them by following the idea presented in subsection 3.2 for pressure, and for velocity $v_{\text{error}}(\varepsilon)$ only by a slightly different estimate in the first term of the last integral in (4.3).

Error estimation can also be performed for higher-order approximations. If the force g_2 does not depend on y_1, neither do the pressure approximations. If the force g_2 depends on y_1, so do the second and higher-order pressure approximations, which changes some calculations, but not the final conclusion on error estimation for higher order approximations.

From what has been presented, it is easy to derive estimates of the approximation error for the original domain $\Omega^\varepsilon \times (0, T^\varepsilon)$. Apart from the derivations, it is, of course, also possible to find estimates for the velocity and the displacement itself by using the Poincaré inequality.

In summary, expected estimates are derived that rigorously justify the effective model. The model itself can be simplified from the variational formulation to a parabolic equation of sixth degree. The regularity presented here justifies the computation, usually called formal (since the regularity is not known in advance), and it is easy to perform.

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Ocjene greške za efektivni model interakcije tankog sloja fluida i elastične ploče

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Sažetak. Promatramo nestacionarni tok inkompresibilnog fluida u tankoj pravokutnoj domeni s elastičnom pločom kao gornjim rubom. Tok je modeliran Stokesovim jednadžbama pod djelovanjem padom tlaka i vanjske sile. Ocjene greške izvedene su za aproksimaciju efektivnim modelom koji je dobiven proučavanjem rješenja problema kada debljina domene fluida teži k nuli.

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